Long waves in a relativistic pair plasma in a strong magnetic field

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The properties of low-frequency waves in a one-dimensional, relativistic electron-positron plasma in a strong external magnetic field typical of pulsar magnetospheres are discussed. Approximate dispersion relations are derived for a broad class of distribution functions that have an intrinsically relativistic spread in energies. The effects of the non-neutrality, associated with rotation, and of the relative motion of the plasma species are discussed briefly. In the plasma rest frame only three wave modes need be considered. The magnetosonic (t) mode becomes firehose unstable as the magnetic field weakens, and this occurs in the wind zone of the pulsar. The Alfvén (A) mode exists only below a maximum frequency, and is weakly damped only for sufficiently strong magnetic fields. The Langmuir-O mode is approximately longitudinal near its cutoff frequency, and approximately transverse at high frequencies. We argue that the emission zone is within $\sim 10^2$ pulsar radii, and that only t, A, and Langmuir waves may participate in the formation of the observed radio spectrum. [S1063-651X(98)07103-7]

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I. INTRODUCTION

Relativistic plasma plays an important role in a number of astrophysical objects such as active galactic nuclei, black hole magnetospheres, the primordial Universe, relativistic jets, cosmic rays, and others [1]. In particular, relativistic pair (electron-positron) plasma in a strong magnetic field plays a central role in the physics of pulsar magnetospheres and winds [2–4]. The observed radio emission ($\omega \sim 10^9-10^{11}\ \mathrm{s}^{-1}$) from pulsars, which are magnetized neutron stars, is generated in a relativistic pair plasma and must propagate through such plasma as it escapes [2,4]. The pair plasma is created in a two-stage process: primary particles are accelerated by an electric field parallel to the magnetic field near the poles (where the typical magnetic field is $\sim 10^{12}$ G) up to extremely high energies, and these produce a secondary, denser pair plasma via an avalanche or cascade process [5]. The number density, N_p, of the secondary pair plasma exceeds the Goldreich-Julian density N_{GJ} (which is required to maintain corotation) by the so-called multiplicity factor $M=N_p/N_{\mathrm{GJ}}\sim 10^2-10^6$ [2,4,5]. The pair plasma is intrinsically highly relativistic, its flow Lorentz factor γ_p being of the same order of magnitude as the typical spread (e.g., root mean square) Lorentz factor $\langle\gamma\rangle$, with $\gamma_p\sim\langle\gamma\rangle\sim 10-10^3$ [4].

The radio emission mechanism for pulsars is not adequately understood [4,6]. A plausible scenario is the excitation of waves due to a resonant kinetic plasma instability followed by nonlinear interaction between the waves to produce the spectrum of the escaping radiation. One version of this process is that suggested by one of us [7–9] and other versions have been reviewed elsewhere [4,6]. Whatever the details of the emission mechanism, the properties of the low-frequency waves in relativistic pair plasma in the pulsar magnetosphere are of central importance for understanding the underlying processes in the formation of the radio spectrum.

Waves in pulsar plasmas have been studied extensively over the past two decades. Early studies mainly concentrated on the relativistic plasma flow, assuming cold or only mildly relativistic distribution of electrons and positrons in the plasma rest frame (see, e.g., Ref. [10] and references therein). Kinetic analysis of the highly relativistic plasma concentrated mainly on longitudinal waves propagating along the magnetic field (see, e.g., [11]). A general expression for the dielectric tensor, except for the neglect of gyrotropic factors (see below), was derived by one of us [12] for oblique low-frequency waves in a plasma which is one-dimensional in the sense that the particles have motion only along the magnetic field lines. The dispersion relation for the oblique electromagnetic waves was obtained and linear polarization explained. Low-frequency waves were studied in detail by Arons and Barnard [13], where many of the results of the previous studies were rederived and generalized. The specific cases considered in detail in [13] were the cold plasma and waterbag distributions. These distributions are not sufficiently general to include all the possibly important effects in the application to pulsar plasmas. More recently, a relativistic thermal distribution was discussed by Polyakov [14], but this is also insufficiently general to contain all the possibly important features. In all these cases, the plasma is assumed to be one-dimensional, which is well justified for plasma in the superstrong pulsar magnetic fields. The astro-

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physical objective of such investigations is to understand the radio emission mechanism for pulsars, but neither the emission mechanism nor even the location of the emission region has been clearly identified (see, e.g., Ref. [15]).

The objective of the present paper is to determine the general properties (dispersion relations and polarization) of the low-frequency waves in pulsar plasmas making only the most general assumptions on the form of the distribution function. The plasma is assumed locally homogeneous, and although we examine the effects of the nonzero charge and current density on the wave properties, we argue that they can be ignored. The streaming motion of the plasma is removed by carrying out the analysis in the plasma rest frame, where the intrinsic spread in particle energies is assumed highly relativistic. Our aim is to derive compact expressions for the dispersion relations for the low-frequency modes, which apply in the regions of the magnetosphere where the observed radio emission is plausibly generated.

The paper is organized as follows. In Sec. II we discuss the plasma parameters used throughout the paper. In Sec. III we present an efficient method for the treatment of linear and nonlinear low-frequency waves, based on the direct expansion of the Vlasov equation in an inverse gyrofrequency power series, and use it in Sec. IV to derive the dielectric tensor for an arbitrary one-dimensional distribution function. In Sec. V we analyze the dispersion relation for different modes and establish the relation between the location of the region where the waves are considered (emission region) and mode features. In Sec. VI we summarize the results and discuss qualitative implications for the interpretation of pulsar radio emission.

II. PLASMA PARAMETERS

The pulsar plasma parameters in the source region are model dependent. There are different models for the generation of the secondary pairs, for the location of the radio emission region, and there are intrinsic variations from one pulsar to another, all of which introduce uncertainties into the estimates. We choose what we consider to be the most plausible parameters, but note that there is necessarily an uncertainty of several orders of magnitude in some estimates, most notably of the plasma density.

A standard model of the polar cap pair cascade implies that the pulsar rest frame density of the pair plasma is $N_p \approx MN_{\rm GJ}$ where $N_{\rm GJ}\approx B_0/Pec$ is the Goldreich-Julian density, and M is the multiplicity factor. For a pulsar with the polar magnetic field $B_0\approx 10^{12}$ G and period $P=1$ s, one finds $N_{\rm GJ}\approx 10^{11}$ cm^{-3}. The multiplicity factor is uncertain, with estimates in the range 10^2–10^6. We adopt $M=10^3$ for numerical estimates. The resulting plasma density is $N_p \approx 10^{14}$ cm^{-3}. This plasma is highly relativistic, flowing with a mean Lorentz factor of about $\gamma_p\approx 10^3$ and having a spread in Lorentz factors of about $\langle\gamma\rangle\approx 10^2$ (for the actual definition of this parameter see below). Thus, the plasma rest frame density near the pulsar surface is $N_r=N_p/\gamma_p \approx 10^{11}$ cm^{-3}.

The dipole magnetic field varies in the magnetosphere as $B=B_0(R_0/R)^{-3}$, where $R_0\approx 10^6$ cm is the radius of the neutron star. In most models of the pulsar radio emission [15] the emission zone is believed to be well inside the light cylinder $R_L=cP/2\pi$, beyond which corotation must break down. For a pulsar with the period $P=1$ s, the light cylinder is at the radius $R_L\approx 10^{10}$ cm$\approx 10^4 R_0$. The plasma density varies as $N_p \propto R^{-3}$ (as the magnetic field) in the region of interest ($R\lesssim R_L$), where γ_p and $\langle\gamma\rangle$ are independent of R.

The frequencies of interest are those in the observed radio range of 10^9–10^{11} s^{-1}, which translates into $\omega \sim 10^6$–10^8 s^{-1} in the plasma rest frame for $\gamma_p\approx 10^3$. The gyrofrequency $\Omega=eB/mc$ and the plasma frequency, defined here as $\omega_p=(4\pi N_r e^2/m)^{1/2}$ without any Lorentz factor, vary from $\Omega\approx 2\times 10^{19}$ s^{-1} and $\omega_p\approx 2\times 10^{10}$ s^{-1} near the polar cap, to $\Omega\approx 2\times 10^7$ s^{-1} and $\omega_p\approx 2\times 10^4$ s^{-1} at the light cylinder. If the emission zone is near 0.01 R_L [15], the corresponding frequencies are approximately $\Omega\approx 2\times 10^{13}$ s^{-1} and $\omega_p\approx 2\times 10^7$ s^{-1}. For more rapidly rotating pulsars these frequencies are higher.

III. GENERAL FORMALISM

The approach to the analysis of low-frequency long waves was described in detail in Ref. [16]. Here we briefly outline its modification for the case of relativistic plasma.

The ultrarelativistic pair plasma, typical for pulsar magnetospheres, should be described by the relativistic Vlasov equation

$$\frac{\partial}{\partial t} f_s + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} f_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\frac{\partial}{\partial \mathbf{u}} f_s = 0, \tag{1}$$

for each species s (electrons and positrons in our case), with $\mathbf{u}=\mathbf{p}/m$, $\mathbf{v}=\mathbf{u}/\gamma$, $\gamma^2=1+u^2$, and where we use units with $c=1$. The magnetic field $\mathbf{B}$ includes the constant external magnetic field chosen so that $\mathbf{B}_0=(0,0,B_0)$. Equation (1) applies in an arbitrary inertial frame, and we use it in the plasma rest frame.

In cylindrical coordinates with $\mathbf{u}=(u_\perp \cos\phi, u_\perp \sin\phi, u_z)$, the distribution function may be expressed as a Fourier series:

$$f_s = \sum_{n=-\infty}^{n=\infty} f_{s,n}(u_\perp, u_z)\exp(-in\phi). \tag{2}$$

Only the components $f_{s,0}$ and $f_{s,\sigma}$, $\sigma=\pm 1$ appear in the following expression for the current density:

$$j_z = \sum_s q_s \int v_z f_{s,0} u_\perp du_\perp du_z, \tag{3}$$

$$j_x = \sum_s \tfrac{1}{2}\sum_\sigma q_s \int v_\perp f_{s,\sigma} u_\perp du_\perp du_z, \tag{4}$$

$$j_y = \sum_s \tfrac{1}{2}\sum_\sigma i\sigma q_s \int v_\perp f_{s,\sigma} u_\perp du_\perp du_z, \tag{5}$$

and where Σ_s denotes summation over species. The dependence on s is omitted, but remains implicit, in the following equations.

Equation (1) is equivalent to the following infinite chain:

$$(L_n + in\tilde{\Omega})f_n + \sum_{\sigma=\pm 1} G_\sigma^{n-\sigma} f_{n-\sigma} = 0, \tag{6}$$

where the operators L_n and G_σ^n are defined by

$$L_n=\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}+\alpha E_z\frac{\partial}{\partial u_z}in\alpha\gamma^{-1}B_z, \tag{7}$$

$$G_\sigma^n=\frac{v_\perp}{2}\nabla_\sigma+\frac{\alpha}{2}(E_\sigma d_{\sigma n}+i\sigma B_\sigma r_{\sigma n}), \tag{8}$$

$$d_{\sigma n}=\frac{\partial}{\partial u_\perp}-\frac{\sigma n}{u_\perp},\quad n_{\sigma n}=v_z d_{\sigma n}-v_\perp\partial, \tag{9}$$

where $\alpha=q/m$, $\Omega=qB_0/m$, $\tilde{\Omega}=\Omega/\gamma$, $E_\sigma=E_x+i\sigma E_y$, $B_\sigma=B_x+i\sigma B_y$, and $\sigma=\pm1$, and with $\nabla_\sigma=\partial/\partial x+i\sigma\partial/\partial y$, and $\partial\equiv(\partial/\partial u_z)$.

In the general case, the infinite chain (6) is no simpler to solve than the original Vlasov equation (1). However, in the low-frequency, long-wavelength regime, there is a small parameter $\xi\sim\omega\gamma/\Omega\sim k\gamma/\Omega\ll1$ in which one may expand. This expansion is described in detail in Ref. [16]. It is done by simple substitution $\Omega\rightarrow\Omega/\xi$ (where now ξ is used as a formal smallness parameter, which is set equal to unity in the end), so that Eq. (6) for $|n|\geq1$ may be written

$$f_n=\frac{\xi}{in\tilde{\Omega}}\left[-L_nf_n-\sum_\sigma G_\sigma^{n-\sigma}f_{n-\sigma}\right]. \tag{10}$$

Since the equilibrium distribution is gyrotropic ($f_n\rightarrow0$ for $|n|\geq1$, if $\mathbf{E}\rightarrow0$, $\mathbf{B}\rightarrow0$, and $\boldsymbol{\nabla}\rightarrow0$), the distribution function can be represented as a following power series:

$$f_n=\sum_{m=|n|}^{\infty}\xi^m f_n^{(m)}, \tag{11}$$

where the lower summation limit is determined by taking into account Eq. (10). For our present purposes it is sufficient to restrict ourselves to the currents of order not higher than ξ^2. Since $f_n\sim O(\xi^{|n|})$, the chain (6) can be reduced to the following equations for f_0 and f_σ, $\sigma=\pm1$:

$$f_\sigma=-\frac{\xi}{i\sigma\tilde{\Omega}}G_\sigma^0f_0+\frac{\xi^2}{i\sigma\tilde{\Omega}}L_\sigma\frac{1}{i\sigma\tilde{\Omega}}G_\sigma^0f_0, \tag{12}$$

$$L_0f_0=\sum_\sigma G_{-\sigma}^\sigma\frac{\xi}{i\sigma\tilde{\Omega}}\left(1-\frac{\xi L_\sigma}{i\sigma\tilde{\Omega}}\right)G_\sigma^0f_0. \tag{13}$$

These equations form a closed set for plasma in a strong external magnetic field. A perturbative solution gives

$$L_0f_0^{(0)}=0, \tag{14}$$

$$L_0f_0^{(1)}=\sum_\sigma G_{-\sigma}^\sigma\frac{1}{i\sigma\tilde{\Omega}}G_\sigma^0f_0^{(0)}, \tag{15}$$

$$L_0f_0^{(2)}=\sum_\sigma G_{-\sigma}^\sigma\frac{1}{i\sigma\tilde{\Omega}}G_\sigma^0f_0^{(1)}-\sum_\sigma G_{-\sigma}^\sigma\frac{1}{i\sigma\tilde{\Omega}}L_\sigma\frac{1}{i\sigma\tilde{\Omega}}G_\sigma^0f_0^{(0)}. \tag{16}$$

Of course, the inverse operator L_0^{-1} should be properly defined to solve Eqs. (14)–(16).

The expansion procedure simplifies in the weak turbulence limit in which one can expand in a further small parameter $\eta\sim E/B_0\ll1$. As we are concerned with the linear response of the plasma we need retain only the zeroth and linear terms in this expansion. This leads to the expansion

$$f_0=F_0(u_\perp,u_z)+\eta\sum_{n=0}^{2}\xi^nf_0^{(n)}, \tag{17}$$

$$f_\sigma=\eta\sum_{n=1}^{2}\xi^nf_\sigma^{(n)}. \tag{18}$$

It is convenient to switch to Fourier space, assuming that all perturbations $\propto\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$, with $\mathbf{k}=(k_\perp,0,k_z)$. Omitting the lengthy algebra we find

$$f_0=F_0(u_\perp,u_z)-\frac{\alpha k_\perp v_\perp}{2\tilde{\Omega}\zeta}E_y\mu_0F_0+\frac{i\alpha k_\perp v_\perp}{2\tilde{\Omega}^2\omega}E_x\mu_0F_0+\frac{ik_\perp^2v_\perp v_z\alpha}{2\tilde{\Omega}^2\zeta}\mu_0F_0E_z+\frac{ik_\perp v_\perp\alpha}{2\tilde{\Omega}^2}\mu_0F_0E_x, \tag{19}$$

$$f_\sigma=-\frac{\alpha}{2i\sigma\tilde{\Omega}}[E_\sigma+E_z(k_\perp v_z/\zeta)]\mu_0F_0+\frac{i\alpha k_\perp^2v_\perp^2}{4i\sigma\tilde{\Omega}^2\zeta}E_y\mu_0F_0+\frac{i\alpha\zeta}{2\tilde{\Omega}^2}[E_\sigma+E_z(k_\perp v_z/\zeta)]\mu_0F_0, \tag{20}$$

where we use the relation $\mathbf{B}=\mathbf{k}\times\mathbf{E}/\omega$ and the notation $\zeta=\omega-k_zv_z$ and $\mu_0=[\zeta(\partial/\partial u_\perp)+k_zv_\perp(\partial/\partial u_z)]/\omega$. In the final expressions (19) and (20) the formal smallness parameter ξ is not necessary already and it is set to unity.

IV. GENERAL DIELECTRIC TENSOR AND DISPERSION EQUATION

The distribution functions f_0 and f_σ found from Eqs. (19) and (20) are used in Eqs. (3)–(5) to determine the conductivity tensor by writing $j_i=K_{ij}E_j$. The dielectric tensor then follows from

$$\epsilon_{ij}=\delta_{ij}+\frac{4\pi i}{\omega}K_{ij}. \tag{21}$$

We obtain

$$\epsilon_{zz}=1+\sum_s\frac{\omega_{ps}^2}{\omega}\int v_zu_\perp\zeta^{-1}\partial F_{s,0}du_\perp du_z-\sum_s\frac{\omega_{ps}^2k_\perp^2}{2\omega\Omega_s^2}\int u_\perp^2u_z^2\gamma^{-1}\zeta^{-1}\mu_0F_{s,0}du_\perp du_z, \tag{22}$$

$$\epsilon_{yz}=-\epsilon_{zy}=i\sum_s\frac{\omega_{ps}^2k_\perp}{2\omega\Omega_s}\int u_zu_\perp^2\zeta^{-1}\mu_0F_{s,0}du_\perp du_z, \tag{23}$$

$$\epsilon_{xz}=\epsilon_{zx}=-\sum_s \frac{\omega_{ps}^2 k_\perp}{2\omega\Omega_s^2}\int u_z u_\perp^2 \mu_0 F_{s,0} du_\perp du_z, \quad (24)$$

$$\epsilon_{xy}=-\epsilon_{yx}=i\sum_s \frac{\omega_{ps}^2}{2\omega\Omega_s}\int u_\perp^2 \mu_0 F_{,0} du_\perp du_z, \quad (25)$$

$$\epsilon_{xx}=1-\sum_s \frac{\omega_{ps}^2}{2\omega\Omega_s^2}\int \zeta\gamma u_\perp^2 \mu_0 F_{s,0} du_\perp du_z, \quad (26)$$

$$\epsilon_{yy}=\epsilon_{xx}+\sum_s \frac{\omega_{ps}^2 k_\perp^2}{4\omega\Omega_s^2}\int u_\perp^4 \gamma^{-1}\zeta^{-1}\mu_0 F_{s,0} du_\perp du_z, \quad (27)$$

where we restore subscript $s=\pm 1$ denoting summation over species ($s=1$ for positrons and $s=-1$ for electrons). Here $\omega_{ps}^2=4\pi e^2 N_{rs}/m$ and $\Omega_s=seB_0/m$, where we take into account $q_+=-q_-=e$, $m_+=m_-=m$. We now incorporate the plasma rest frame number density N_{rs} in the plasma frequency ω_{ps} and normalize the distribution function as follows:

$$\int F_{s,0} u_\perp du_\perp du_z=1. \quad (28)$$

The dispersion equation for the waves is

$$\det\| n^2\delta_{ij}-n_i n_j-\epsilon_{ij}\|=0, \quad (29)$$

where $n=|\mathbf{n}|$, with $\mathbf{n}=\mathbf{k}/\omega$ the refractive index, $\omega_p^2=4\pi q^2 N_r/m$.

In Eqs. (22)–(27) the distribution functions $F_{s,0}(u_\perp,u_z)$ are arbitrary. The above expressions can be partially integrated to give

$$\epsilon_{zz}=1+\sum_s \frac{\omega_{ps}^2}{\omega}\langle u_z\gamma^{-1}\zeta^{-1}\partial\rangle_s+\sum_s \frac{\omega_{ps}^2 k_\perp^2}{\omega^2\Omega_s^2}\langle u_z^2\gamma^{-1}\rangle_s$$
$$-\sum_s \frac{\omega_{ps}^2 k_\perp^2}{2\omega^2\Omega_s^2}\langle u_z^2 u_\perp^2\gamma^{-3}\rangle_s$$
$$-\sum_s \frac{\omega_{ps}^2 k_\perp^2 k_z}{2\omega^2\Omega_s^2}\langle u_z^2 u_\perp^2\gamma^{-2}\zeta^{-1}\partial\rangle_s, \quad (30)$$

$$\epsilon_{yz}=i\sum_s \frac{\omega_{ps}^2 k_\perp}{2\omega^2\Omega_s}[-2\omega\langle u_z\rangle_s+k_z\langle u_z u_\perp^2\gamma^{-1}\zeta^{-1}\partial\rangle_s], \quad (31)$$

$$\epsilon_{xz}=-\sum_s \frac{\omega_{ps}^2 k_\perp}{2\omega^2\Omega_s^2}[-2\omega\langle u_z\rangle_s+k_z\langle(2u_z^2-u_\perp^2)\gamma^{-1}\rangle_s], \quad (32)$$

$$\epsilon_{xy}=i\sum_s \frac{\omega_{ps}^2}{2\omega^2\Omega_s}[-2\omega+k_z\langle u_z\gamma^{-1}\rangle_s], \quad (33)$$

$$\epsilon_{xx}=1-\sum_s \frac{\omega_{ps}^2}{2\omega^2\Omega_s^2}[-2\omega^2\langle\gamma\rangle_s-\omega^2\langle u_\perp^2\gamma^{-1}\rangle_s$$
$$+4\omega k_z\langle u_z\rangle_s-k_z^2\langle(2u_z^2-u_\perp^2)\gamma^{-1}\rangle_s], \quad (34)$$

$$\epsilon_{yy}=\epsilon_{xx}+\sum_s \frac{\omega_{ps}^2 k_\perp^2}{4\omega^2\Omega_s^2}[-4\langle u_\perp^2\gamma^{-1}\rangle_s+\langle u_\perp^4\gamma^{-3}\rangle_s$$
$$+k_z\langle u_\perp^4\gamma^{-2}\zeta^{-1}\partial\rangle_s], \quad (35)$$

where

$$\langle\cdots\rangle_s\equiv\int u_\perp du_\perp du_z(\cdots)F_{s,0}. \quad (36)$$

In Eqs. (30)–(35), $F_{s,0}(u_\perp,u_z)$ remain arbitrary, and the derived general dielectric tensor describes the linear response of both anisotropic and isotropic plasmas. The distribution function of the electron-positron plasma in the pulsar magnetosphere is assumed one-dimensional $F_{s,0}\propto\delta(u_\perp)/u_\perp$, due to the perpendicular energy of relativistic electrons and positrons being radiated away. We assume $F_{s,0}=\tilde{F}_{s,0}(u_z)\delta(u_\perp)/u_\perp$ with normalization $\int\tilde{F}_{s,0}du_z=1$. For this one-dimensional distribution one has

$$\epsilon_{zz}=\epsilon_\parallel=1-\sum_s \frac{\omega_{ps}^2}{\omega^2}W_s(n_\parallel)+\sum_s \frac{\omega_{ps}^2 n_\perp^2}{\Omega^2}\langle u_z^2\gamma^{-1}\rangle_s, \quad (37)$$

$$\epsilon_{yz}=-iP=-i\sum_s \frac{\omega_{ps}^2 n_\perp}{\omega\Omega_s}\langle u_z\rangle_s, \quad (38)$$

$$\epsilon_{xz}=Q=\sum_s \frac{\omega_{ps}^2 n_\perp}{\Omega_s^2}(\langle u_z\rangle_s-n_\parallel\langle u_z^2\gamma^{-1}\rangle_s), \quad (39)$$

$$\epsilon_{xy}=-ig=-i\sum_s \frac{\omega_{ps}^2}{\omega\Omega_s}(1-n_\parallel\langle u_z\gamma^{-1}\rangle_s), \quad (40)$$

$$\epsilon_{xx}=\epsilon_{yy}=\epsilon_\perp=1+\sum_s \frac{\omega_{ps}^2}{\Omega_s^2}(\langle\gamma\rangle_s-2n_\parallel\langle u_z\rangle_s$$
$$+n_\parallel^2\langle u_z^2\gamma^{-1}\rangle_s), \quad (41)$$

where we introduce $n_\perp=k_\perp/\omega=n\sin\theta$, $n_\parallel=k_z/\omega=n\cos\theta$. The dispersion functions $W_s(n_\parallel)$ are defined by (for positive ω)

$$W_s(n_\parallel)=-\frac{1}{n_\parallel}\int_{-\infty}^{\infty}\frac{1}{1-n_\parallel v_z+i\tau}\frac{d\tilde{F}_{s,0}}{du_z}du_z, \quad (42)$$

where $i\tau$ ($\tau\rightarrow+0$) defines the contour of integration. The functions $W_s(n_\parallel)$ are defined so that in the cold plasma $W_s=1$. Alternative forms are

$$W_s(n_\parallel)=-\frac{1}{n_\parallel}\left[\mathcal{P}\int_{-\infty}^{\infty}\frac{1}{1-n_\parallel v}\frac{d\tilde{F}_{s,0}}{du_z}du_z\right.$$
$$\left.-i\pi\frac{\gamma_r^3}{|n_\parallel|}\frac{d\tilde{F}_{s,0}}{du_z}\bigg|_{u_z=u_r}\right], \quad (43)$$

$$W_s(n_\parallel)=-\frac{1}{n_\parallel}\left[\mathcal{P}\int_{-1}^{1}\frac{1}{1-n_\parallel v_z}\frac{d\tilde{F}_{s,0}}{dv_z}dv_z\right.$$
$$\left.-i\pi\frac{1}{|n_\parallel|}\frac{d\tilde{F}_{s,0}}{dv_z}\bigg|_{v_z=v_r}\right], \quad (44)$$

with $v_r=1/n_\parallel$, $\gamma_r=(1-v_r^2)^{-1/2}$, $u_r=\gamma_r v_r$, and $\mathcal{P}$ denotes the principal value integral.

The averaging procedure (36) becomes

$$\langle \cdots \rangle_s \int (\cdots) \widetilde{F}_{s,0} du_z . \tag{45}$$

The plasma rest frame is defined such that the net flow speed is zero. In the following we argue that the relative flow between the electrons and positrons is unimportant in determining the wave properties, and hence we effectively assume $\langle u_z \rangle_- = \langle u_z \rangle_+ = 0$ in the rest frame, where the subscripts denote electrons and positrons. However, $\langle u_z \rangle = 0$ does not guarantee that $\langle u_z \gamma^{-1} \rangle$ vanishes, except where the distribution function possesses a specific symmetry property. It is likely that the pulsar plasma distribution is noticeably asymmetric with respect to the outward and inward directions, because of the way it is generated in a pair cascade.

V. WAVE PROPERTIES

In this section we derive the properties of the waves for specific distribution functions, using these to infer approximate dispersion relations for a wider class of distributions.

A. Neglect of the gyrotropic terms

We start by arguing that the gyrotropic terms ϵ_{yz} and ϵ_{xy} may be neglected in the wave analysis. These terms are nonzero due to nonzero charge density $\rho \sim eN_{\rm GJ}$ and a parallel current density $J \sim eN_{\rm GJ}$ associated with the rotation of the magnetosphere [13], and each is smaller by a factor $\propto |N_- - N_+|/N \sim N_{\rm GJ}/N \sim 10^{-3} \ll 1$ than the nongyrotropic terms. Only the square of the gyrotropic terms enter the dispersion relation (29) for oblique propagation, and their effect would be significant, compared with the other terms in the dispersion relation, for $\omega \lesssim \omega_p (N_{\rm GJ}/N)/\langle \gamma \rangle^{1/2}$. Near the polar cap this inequality gives $\omega \lesssim 10^6$ s^{-1}. The plasma density decreases $\propto R^{-3}$ with increasing radius R, so that the frequencies where the gyrotropic terms are significant decrease with R. Assuming the source region to be at $R > 10R_0$ implies that the gyrotropic terms would be significant only at $\omega \lesssim 10^3$ s^{-1}, corresponding to an observational frequency $\sim 10^3 \gamma_r / 2\pi \approx 0.2$ MHz, which is well below the radio frequency range of interest. It follows that the gyrotropic terms are negligible in the dispersion relations. The gyrotropic terms imply an ellipticity of the polarization $\propto N_{\rm GJ}/N$, which is also negligible except in the limiting case of parallel propagation.

The case of parallel propagation requires separate consideration. The dispersion relation in the appropriate approximation becomes (retaining only the largest term)

$$n^2 = 1 \pm \frac{\omega_p^2}{\omega \Omega} \frac{|N_- - N_+|}{N}, \tag{46}$$

with $|N_- - N_+| \sim N_{\rm GJ}$. The right-hand side of Eq. (46) is insensitive to R so it suffices to estimate it near the pulsar surface. With the parameters of Sec. II one finds that the largest correction to the refractive index in the radio range is $\lesssim 10^{-8}$. This correction is not significant here.

In the following we neglect effects related to the nonzero charge and current densities, set $\widetilde{F}_{+,0} = \widetilde{F}_{-,0}$, and hence neglect the gyrotropic terms. However, the gyrotropic terms must become important for sufficiently low-frequency waves, specifically for waves for which the ratio of the rotation frequency of the star to the wave frequency is not negligible.

B. The dispersion equation

The neglect of the gyrotropic terms implies $P = g = 0$ in Eqs. (38) and (40), and for the same reason the terms involving $\langle v_z \rangle_s$ in Eq. (39) are neglected. We may also omit subscript s, using notation $\omega_{p+} = \omega_{p-} \equiv \omega_p$, $\Omega_+ = -\Omega_- \equiv \Omega$, and $W_+(n_\parallel) = W_-(n_\parallel) \equiv W(n_\parallel)$. The dispersion equation then factorizes into two independent dispersion relations for linearly polarized waves:

$$n^2 = \epsilon_\perp , \quad E_y \neq 0, \tag{47}$$

$$(n_\parallel^2 - \epsilon_\perp)(n_\perp^2 - \epsilon_\parallel) = (n_\perp n_\parallel + Q)^2, \quad E_y = 0, \tag{48}$$

where

$$\epsilon_\parallel = 1 - \frac{2\omega_p^2}{\omega^2} W(n_\parallel) + n_\perp^2 \Delta \lambda, \tag{49}$$

$$\epsilon_\perp = 1 + \Delta(\langle \gamma \rangle + n_\parallel^2 \lambda), \tag{50}$$

$$Q = -\Delta n_\perp n_\parallel \lambda, \tag{51}$$

$$\Delta = \frac{2\omega_p^2}{\Omega^2}, \quad \lambda = \langle u_z^2 \gamma^- \rangle. \tag{52}$$

Equation (47) corresponds to a strictly transverse wave mode, usually called the magnetosonic (t) mode, which name is used here. This mode was called the X mode in [13]. Equation (48) corresponds to waves which are neither strictly longitudinal nor strictly transverse in general, and it includes both the Langmuir and Alfvén modes as limiting cases. We discuss the waves described by Eqs. (47) and (48) separately.

C. Magnetosonic (t) waves

The dispersion relation (47) for t waves may be written in the form (cf. [9,13])

$$\omega_t^2 = k^2 v_A^2 (1 - \Delta\lambda \cos^2 \theta), \tag{53}$$

where $v_A = 1/(1 + \Delta\langle \gamma \rangle)^{1/2}$ is the relativistic Alfvén speed as defined by [17].

The t mode is subluminous ($\omega < k$), and this is a necessary condition for a resonant (Cherenkov) interaction with particles to be possible. Nevertheless, no resonant interaction is possible because the waves have $E_z = 0$, and the current associated with a particle is strictly along the z axis in the one-dimensional case. A resonant interaction becomes possible in principle when either the gyrotropic terms are included, as these lead to a nonzero longitudinal component of the polarization, or when the particles are not confined to

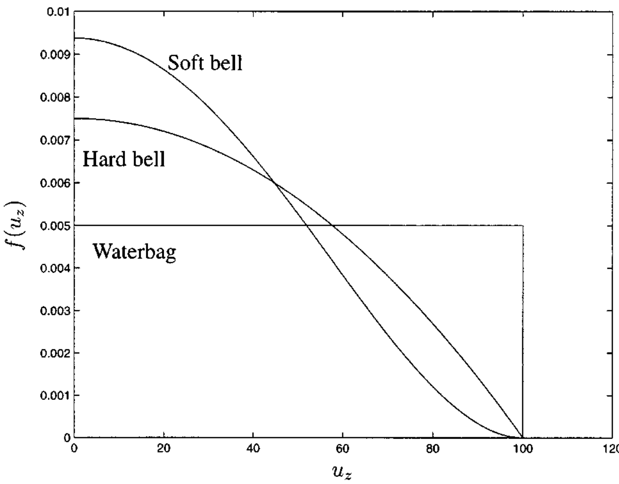

FIG. 1. The waterbag, hard bell, and soft bell distributions with $\gamma_m = 100$.

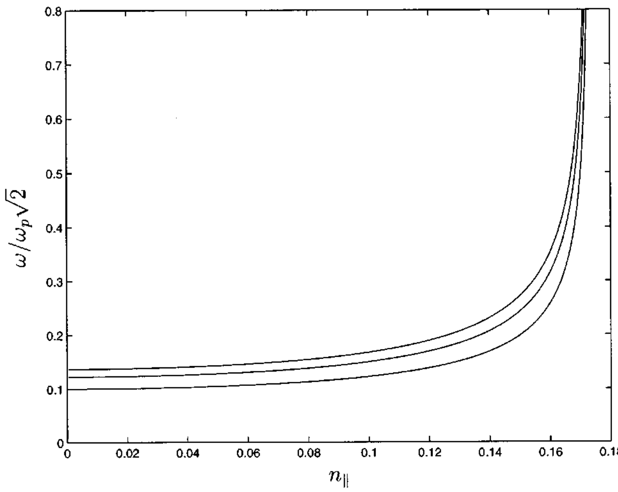

FIG. 2. Superluminous Langmuir-O mode for the three distributions. For given $n_\parallel$, the frequency is the lowest for the waterbag distribution and highest for the soft bell.

their lowest Landau orbital. In the first case the absorption coefficient should be proportional to $E_z/E_y \propto \Delta N_{\rm GJ}/N_p$, and hence is very weak. The second case requires that there be some mechanism to excite the particles out of their lowest Landau orbital, and the only effective mechanism is a resonant gyromagnetic interaction, which requires waves of much higher frequency than are of interest here. Such excitation through the anomalous Doppler resonance was discussed in [18]. Provided our assumption that the plasma is one dimensional remains valid, absorption (positive or negative) due to gyromagnetic interactions is not possible.

Equation (53) shows that the plasma becomes intrinsically (aperiodically) unstable when $\Delta\lambda > 1$. This is a special case of the firehose instability, which may occur in a hot anisotropic plasma [17]. Well within the light cylinder in a pulsar magnetosphere one has $\Delta \propto R^3$, and then the firehose instability develops for

$$R \gtrsim R_0 \left(\frac{\Omega_0^2}{\omega_{p0}^2 \langle \gamma \rangle} \right)^{1/3}, \qquad (54)$$

where the subscript 0 refers to the values near the pulsar surface. It is usually assumed that once the firehose instability develops, the distribution function isotropizes due to quasilinear interactions with the unstable waves. In principle, the effects of the quantization of the Landau levels needs to be taken into account here, because the conventional treatment in terms of a diffusion in pitch angle applies only in the nonquantum limit. However, for typical plasma parameters the firehose instability develops well beyond the light cylinder, in the wind zone, where $\Delta \propto R^{-2}$. Hence, it is not directly relevant to the present discussion.

The correction to the refractive index (and phase velocity) $\propto \Delta \langle \gamma \rangle$ is small for parameters of relevance here. The correction is only relevant if it is larger than other corrections, in particular, that due to vacuum polarization (e.g., Ref. [13]). The vacuum polarization gives a correction $\propto (\alpha_f/45\pi)(B/B_c)^2$, where $\alpha_f = 1/137$ is the fine structure constant, and the critical magnetic field is $B_c = 4.4 \times 10^{13}$ G. Near the pulsar surface this correction is $\sim 10^{-7}$ and drops to $\sim 10^{-13}$ at $R = 10R_0$, being proportional to R^{-6}. On the other hand, the plasma induced correction is $\Delta \langle \gamma \rangle \sim 10^{-16}$ near the pulsar surface, and increases as R^3 with the distance, reaching $\Delta \langle \gamma \rangle \sim 10^{-8}$ at $R = 10^3 R_0$. These numbers show that vacuum polarization effects are negligible and the infinite magnetic field approximation ($\Delta = 0$) is appropriate for $R \lesssim 10^3 R_0$. This conclusion also applies to the mixed (Alfvén-Langmuir) mode. Nevertheless, for completeness, we retain $\Delta \neq 0$.

D. Alfvén-Langmuir mode

The second dispersion relation (48) is more complicated. The identification of the modes is made by considering the case of parallel propagation. For $n_\perp = 0$, Eq. (48) factorizes into the dispersion relation $n^2 = \epsilon_\perp$ for the parallel Alfvén mode (which is degenerate with the parallel t wave) and $\epsilon_\parallel = 0$ for the parallel Langmuir wave [9,11]. On including a small obliquity, the relevant solutions of Eq. (48) are found to map continuously onto these parallel modes as the obliquity reduces to zero (in the long-wavelength limit $k \to 0$), and hence the classification into Alfvén and Langmuir waves remains well defined. However, the "Langmuir" mode does not necessarily remain even approximately longitudinal away from parallel propagation. The Langmuir mode evolves into a transverse electromagnetic mode, identified as the O mode by [13]. We also note that because all solutions of Eq. (48) have $E_z \neq 0$, the Cherenkov resonance, $\omega = k_z v_z$, allows Landau damping (or growth) for subluminous waves, where "subluminous" means $n_\parallel = k_z/\omega > 1$.

The existing nomenclature for these modes can be confusing. Our nomenclature is to refer to the mode which is degenerate with the t mode for parallel propagation as the Alfvén (A) mode. The A mode has a parallel phase velocity ($= 1/n_\parallel$) that is subluminous ($n_\parallel > 1$). The mode with a cutoff frequency has a parallel phase velocity that is superluminous, and is called the Langmuir-O (L-O) mode. The

L-O mode is referred to as the Langmuir mode only in the regime where the waves are approximately longitudinal.

We write Eq. (48) in the form

$$\frac{\omega^2}{2\omega_p^2}=\frac{n_\parallel^2-(1+\delta_1)}{n_\parallel^2-(1+\delta_2)\cos^2\theta}W(n_\parallel)\cos^2\theta, \tag{55}$$

where we retain first order terms in Δ in the small corrections $\delta_1=\Delta(\langle\gamma\rangle+\lambda)$ and $\delta_2=\Delta(\langle\gamma\rangle\cos^2\theta+\lambda)$. The form (55) contains the parallel A mode as a limiting case in which both the numerator and denominator vanish. To understand the behavior of the A and L-O modes in the general case, one needs to consider the signs of the factors $n_\parallel^2-(1+\delta_1)$, $n_\parallel^2-(1+\delta_2)\cos^2\theta$, and $W(n_\parallel)$. The plasma is transparent to one of these modes when $[n_\parallel^2-(1+\delta_1)][n_\parallel^2-(1+\delta_2)\cos^2\theta]W(n_\parallel)>0$. Since $W(n_\parallel<1)>0$, the nondamping L-O mode always exists for $0\leqslant n_\parallel^2<(1+\delta_2)\cos^2\theta$ (provided $\sin^2\theta>\delta_2$). The Langmuir end of this mode starts at cutoff $n_\parallel=0$, $\omega=\omega_p\sqrt{2\langle\gamma^{-3}\rangle}$. The O-mode end is nondispersive with the dispersion relation $\omega=k\sqrt{1+\delta_2}$.

The A-mode features depend on the details of the behavior of $W(n_\parallel)$. To investigate this mode we first consider several specific distributions for which $W(n_\parallel)$ can be found analytically.

E. Specific distribution functions

We consider several simple choices of distributions. For simplicity, only symmetric distributions $\tilde{F}_0(-u_z)=\tilde{F}_0(u_z)$ are investigated, and (with one exception) the distributions are assumed to have a high energy cutoff at $u_z=u_m$, with $\tilde{F}_0(u_z>u_m)=0$. The distributions discussed in detail are the waterbag, hard bell, and soft bell distributions illustrated in Fig. 1. The different shapes of the distribution functions, for given u_m, result in significantly different wave properties. These properties are illustrated in Figs. 2–5, which are discussed in detail below. Throughout this section we use the notation $\gamma_m=\sqrt{1+u_m^2}$, $v_m=u_m/\gamma_m$.

1. ''Waterbag'' distribution

First, consider the waterbag distribution (cf. Ref. [13]), which we take in the form $\tilde{F}_0=(1/2u_m)H(u_m^2-u_z^2)$, where $H(x)$ is the Heaviside function, $H(x)=1$ for $x>0$, and $H(x)=0$ otherwise. The dispersion function (44) becomes

$$W(n_\parallel)=\frac{1}{\gamma_m(1-n_\parallel^2v_m^2)}, \tag{56}$$

and Eq. (55) becomes

$$\frac{\omega^2}{2\omega_p^2}=-\frac{\cos^2\theta}{\gamma_m v_m^2}\frac{n_\parallel^2-(1+\delta_1)}{[n_\parallel^2-(1+\delta_2)\cos^2\theta](n_\parallel^2-1/v_m^2)}. \tag{57}$$

The A mode depends on the values of $n_\parallel^2$ corresponding to the zeros (at $n_\parallel^2=1+\delta_1$) and poles [at $n_\parallel^2=(1+\delta_2)\cos^2\theta$ and $n_\parallel^2=1/v_m^2$] of the right-hand side of Eq. (57) [note that $(1+\delta_2)\cos^2\theta<1+\delta_1$].

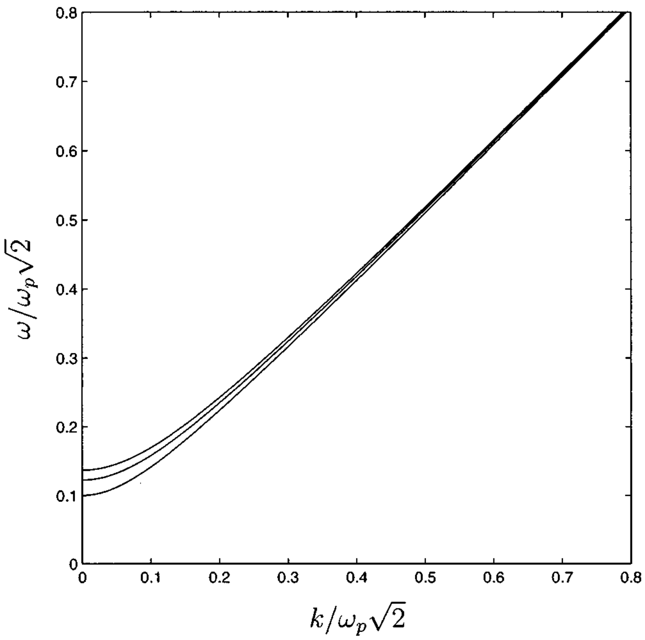

FIG. 3. Same as Fig. 2 but for ω and a function of k. The mode starts as the longitudinal Langmuir mode at $k=0$ and becomes the transverse O mode as ω approaches k.

a. Case 1. $(1+\delta_2)\cos^2\theta<1+\delta_1<1/v_m^2$. The L-O mode requires $0<n_\parallel^2<(1+\delta_2)\cos^2\theta$. The cutoff frequency is $\omega\to\omega_p\sqrt{2}/v_m$ and the mode is purely longitudinal (Langmuir mode) when $n_\parallel\to0$. For $\omega\to\infty$ the L-O mode becomes transverse (O mode) when $n_\parallel^2\to(1+\delta_2)\cos^2\theta<1$. The A mode has parallel refractive index in the range from $n_\parallel^2=1+\delta_1$ (where $\omega\to0$) to $n_\parallel^2\to1/v_m^2$ (where $\omega\to\infty$). This mode is transverse in the whole frequency range. Since $v_m<1$ this is the only possible case in the infinite magnetic field limit, where $\delta_1=\delta_2=0$.

b. Case 2. $(1+\delta_2)\cos^2\theta<1/v_m^2<1+\delta_1$. The only difference from case 1 is that the refractive index for the A mode decreases now from $(1+\delta_1)^{1/2}$ to $1/v_m$ with increasing frequency.

c. Case 3. $1/v_m^2<(1+\delta_2)\cos^2\theta<1+\delta_1$. This order is possible only for $(1+\delta_2)\cos^2\theta>1$, that is, in the quasiparallel regime. The L-O mode becomes electromagnetic (O mode) with $n_\parallel\to1/v_m$ for $\omega\to\infty$. The A mode has $(1+\delta_2)\cos^2\theta<n_\parallel^2<1+\delta_1$.

2. ''Hard bell'' distribution

The waterbag distribution has $d\tilde{F}_0/du_z=0$ everywhere except at the end points, where it is infinite. This precludes damping due to the Cherenkov resonance. In order to study damping it is necessary to consider a distribution with $d\tilde{F}_0/du_z\neq0$. Here we consider the hard bell distribution

$$\tilde{F}_0=\frac{3\gamma_m^2}{4u_m^3}\frac{v_m^2-v_z^2}{1-v_z^2}H(v_m^2-v_z^2)$$
$$=\frac{3}{4u_m^3}(u_m^2-u_z^2)H(u_m^2-u_z^2). \tag{58}$$

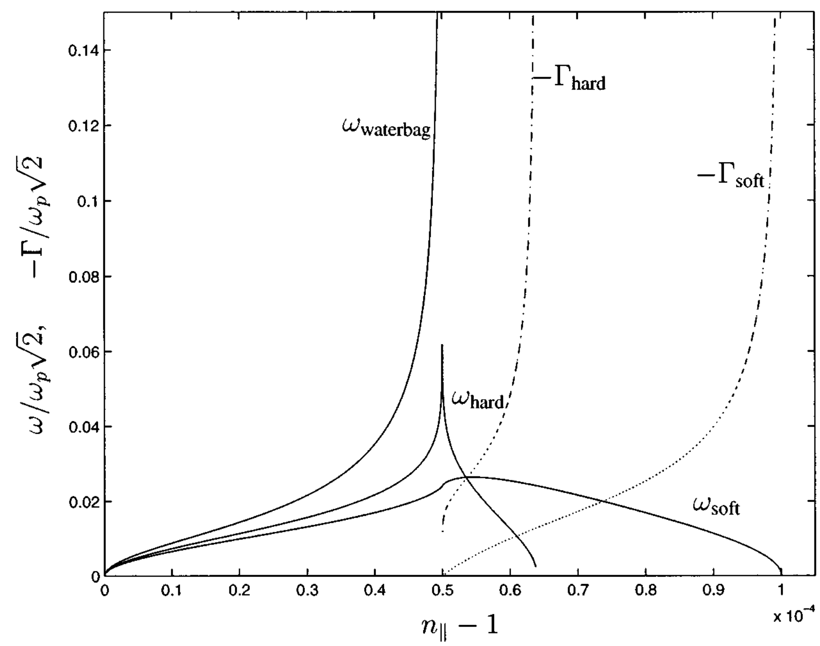

FIG. 4. Frequency (solid line) and damping rate (dash-dotted line) for the subluminous A mode in the range $n_\parallel > 1$. For given $1 < n_\parallel < 1/v_m$ the frequency is highest for the waterbag and lowest for the soft bell. There is a singularity ($\omega \to \infty$) for the waterbag and the hard bell cases at $n_\parallel = 1/v_m$. In the soft bell case there is no such singularity. In both hard bell and soft bell cases the wave damps for $n > 1/v_m$.

The dispersion function (44) becomes

$$W(n_\parallel) = \frac{3}{2u_m^3(n_\parallel^2-1)^2}\left[n_\parallel \ln\left|\frac{n_\parallel v_m+1}{n_\parallel v_m-1}\right| - u_m\gamma_m(n_\parallel^2-1) - \frac{n_\parallel^2+1}{2}\ln\left|\frac{1+v_m}{1-v_m}\right|\right] - i\,\frac{3\pi n_\parallel}{2u_m^2(n_\parallel^2-1)^2}\,H(n_\parallel v_m - 1). \tag{59}$$

This function has a logarithmic singularity, $W \to \infty$ at $n_\parallel = 1/v_m$, but does not change sign there, in contrast with the waterbag case. The sign change occurs at some $n_\parallel = 1/v_* > 1/v_m$ (the exact value of which is of no importance here), so that the plasma is nontransparent for waves with $n_\parallel > 1/v_*$. The imaginary part of $W(n_\parallel)$ is nonzero for all $n_\parallel > 1/v_m$. The mode behavior again depends on the relative positions of the zeros and the singularities, with $(1+\delta_2)\cos^2\theta < 1+\delta_1$, $1/v_m^2 < 1/v_*^2$. We consider two cases.

a. Case 1. $(1+\delta_2)\cos^2\theta < 1+\delta_1 < 1/v_m^2 < 1/v_*^2$. The most important new feature is the appearance of a new mode, whose parallel refractive index is in the range $1/v_m < n_\parallel < 1/v_*$ with $n_\parallel \to 1/v_m$ for $\omega \to \infty$ and $n_\parallel \to 1/v_*$ for $\omega \to 0$. We refer to this as the sub-∥ mode. The damping rate for this mode is

$$\frac{\Gamma}{\omega} = \frac{\text{Im } W}{2 \text{ Re } W} = -\frac{3\pi n_\parallel}{4u_m^2(n_\parallel^2-1)^2 \text{ Re } W}, \tag{60}$$

which is large.

b. Case 2. $(1+\delta_2)\cos^2\theta < 1/v_m^2 < 1+\delta_1 < 1/v_*^2$. In this case the additional sub-∥ mode exists for $1+\delta_1 < n_\parallel^2 < 1/v_*^2$, and tends to low frequencies ($\omega \to 0$) at both ends of the range. It is strongly damped everywhere, and hence is of little practical interest.

3. "Soft bell" distribution

The waterbag distribution is discontinuous at $v_z = v_m$, and while the hard bell distribution is continuous at v_z

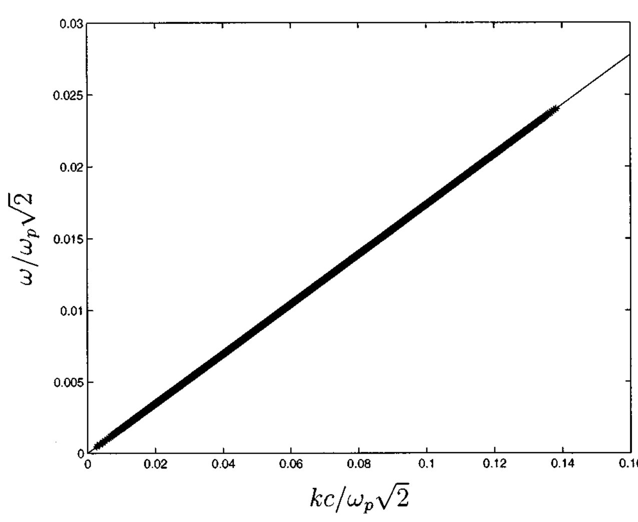

FIG. 5. Dispersion relation for subluminous A mode in the nondamping range $1 < n_\parallel < 1/v_m$. The existence of a maximum frequency for the soft bell case (heavy line) is apparent. The thin line extending beyond the soft bell cutoff corresponds to the waterbag and hard bell cases, which are nondistinguishable in the low-frequency range and do not have an upper frequency limit.

$=v_m$, it has a discontinuous derivative there. The new sub-$\parallel$ mode that we identify for a hard bell distribution exists only near $n_\parallel=1/v_m$, and may well be associated with the discontinuous derivative there. To explore this point, we consider a soft bell distribution, which is continuous with a continuous derivative at $v_z=v_m$:

$$\tilde{F}_0=\frac{15\gamma_m^4}{16u_m^5}\left[\frac{v_m^2-v_z^2}{1-v_z^2}\right]^2 H(v_m^2-v_z^2)$$
$$=\frac{15}{16u_m^5}(u_m^2-u_z^2)^2 H(u_m^2-u_z^2). \tag{61}$$

The dispersion function is

$$\begin{aligned}W(n_\parallel)=&\frac{15\gamma_m^2}{4u_m^5(n_\parallel^2-1)^3}\left\{\frac{1}{8}\ln\left|\frac{1+v_m}{1-v_m}\right|[(3+v_m^2)(3n_\parallel^2+1)\right.\\&-n_\parallel^2(3v_m^2+1)(n_\parallel^2+3)]\\&+\frac{1}{4}u_m\gamma_m(n_\parallel^2-1)(3v_m^2n_\parallel^2+v_m^2-n_\parallel^2-3)\\&\left.+n_\parallel(n_\parallel^2v_m^2-1)\ln\left|\frac{n_\parallel v_m+1}{n_\parallel v_m-1}\right|\right\}\\&-i\,\frac{15\pi\gamma_m^2 n_\parallel(n_\parallel^2v_m^2-1)}{4u_m^5(n_\parallel^2-1)^3}H(n_\parallel v_m-1).\end{aligned} \tag{62}$$

The main difference from the hard bell case is that now there is no singularity at $n_\parallel=1/v_m$. The sign change of $W(n_\parallel)$ occurs at some $n_\parallel=1/v_*$ and the explicit value of v_* is of no particular significance here. For $(1+\delta_2)\cos^2\theta<1+\delta_1<1/v_*^2$ the sub-$\parallel$ mode has $1+\delta_1<n_\parallel^2<1/v_*^2$, as for the ''hardbell'' case, but now with $\omega\to 0$ at both ends of the range. There is substantial damping for $n_\parallel>1/v_m$, and the damping rate increases rapidly with increasing $n_\parallel$. We conclude that the sub-$\parallel$ mode is only of possible interest for $n_\parallel<1/v_m$.

The frequency of the A mode is now limited from above, through the existence of a maximum frequency. The mode ceases to exist for $\omega>\omega_{\max}$, with $\omega_{\max}$ determined by the behavior of the dispersion function, implying $\omega\to\omega_{\max}$ for $n_\parallel=1/\nu_m$.

4. Illustrations

To make the above analysis more comprehensible we illustrate the mode features for these three distributions graphically in Figs. 1–5. For this purpose we choose $\gamma_m=100$ and $\theta=80°$. Since the parameters $\langle\gamma\rangle$ and $\langle\gamma^{-3}\rangle$ enter the dispersion relations, it is of interest to compare them for these three cases. Numerically we find $\langle\gamma\rangle_w=50$ and $\langle\gamma^{-3}\rangle_w=0.1$ for the waterbag, $\langle\gamma\rangle_h=37.5$ and $\langle\gamma^{-3}\rangle_h=0.16$ for the hard bell, and $\langle\gamma\rangle_s=31.2$ and $\langle\gamma^{-3}\rangle_s=0.2$ for the soft bell distributions. As could be expected, the softer the distribution, the lower is $\langle\gamma\rangle$ and the higher is $\langle\gamma^{-3}\rangle$. In the three cases $\langle\gamma^{-3}\rangle\langle\gamma\rangle\approx 0.5$–$0.6$.

The frequency-refractive index relation for the superluminous L-O mode ($\omega/\omega_p\sqrt{2}$ as a function of $n_\parallel$) for all three cases in the infinite magnetic field limit is shown in Fig. 2. This mode exists for $0<n_\parallel<\cos\theta$. The cutoff frequency $\omega_0=\omega_p\sqrt{\langle\gamma^{-3}\rangle}$ is nearly the same for all three distributions. Figure 3 shows that the L-O wave becomes almost electromagnetic and transverse ($\mathbf{E}\perp\mathbf{k}$) already at $\omega\approx 2\omega_0$.

Figure 4 shows the frequency-refractive index (solid lines) and damping rate–refractive index (dash-dotted lines) relations for the subluminous A mode in the range $n_\parallel>1$ for the same parameters as above. In the case of waterbag distribution the plasma is not transparent for this mode for $n_\parallel>1/v_m$. In the hard bell and soft bell cases the wave propagates, but the damping rate becomes comparable to or even larger than the wave frequency. Thus no weakly damped A wave exists for $n>1/v_m$ in any of the three cases. The wave frequencies are not limited from above for the waterbag and the hard bell distributions (although in the latter case the logarithmic singularity does not allow us to show this in the figure). The dispersion relation $\omega(k)$ for this mode in the transparency range $1<n_\parallel<1/v_m$ and the upper frequency limit for the soft bell distribution (heavy line) are seen in Fig. 5. The thin line corresponds to the two other cases, which are not distinguishable from the soft bell case in this low-frequency limit, but extend to $\omega\to\infty$.

To summarize, the A mode exists only in the very narrow range of refractive indices, $(n_\parallel-1)\lesssim 10^{-4}$, within which it is well approximated by the dispersion relation $\omega=k\cos\theta$. It is a low-frequency wave and ceases to exist when its frequency becomes of the order of the Langmuir wave cutoff frequency $\omega_p\sqrt{\langle\gamma^{-3}\rangle}$.

5. Relativistic thermal distribution

All the above distributions have a high energy cutoff and have discontinuous first or higher derivatives. An example of a distribution which extends to arbitrarily high particle energies and has all its derivatives continuous is the one-dimensional relativistic thermal (Jüttner-Synge) distribution

$$\tilde{F}_0=Ae^{-\rho\gamma}, \tag{63}$$

where $\rho=m/T$ is the inverse of the temperature in units of the rest mass, and where the normalization constant A is of no particular interest. If one interprets the exponential function in Eq. (63) as a smoothed form of cutoff of the distribution, then this is analogous to a soft bell distribution with $\rho=1/\gamma_m$ or $v_m^2=1-1/\rho$. The mean values $\langle\cdots\rangle$, defined by Eq. (45), may be evaluated in terms of known functions for the distribution (63), and compared with the results for other choices of distribution function. In the ultrarelativistic limit, relations between these averages are insensitive to the form of the cutoff of the distribution. One specific approximate relation required below is $\langle\gamma^{-3}\rangle\sim\langle\gamma\rangle^{-1}$, which applies to within a factor of order unity. The exact value of this factor depends on the details of the distribution.

The dispersion function (44) for the distribution (63) is transcendental, and may be written in terms of various relativistic plasma dispersion functions. For example, it may be written

$$W(1/z)=Az^2\frac{\partial}{\partial z}T(z,\rho), \tag{64}$$

with $z=1/n_\parallel$, where the relativistic plasma dispersion function is that introduced and discussed by [19]

$$T(z,\rho)=\int_{-1}^{1}dv\,\frac{e^{-\rho\gamma}}{v-z}. \tag{65}$$

[A result analogous to Eq. (64) was derived by [14] in terms of different functions.] In the ultrarelativistic limit, $\rho\ll 1$, one has

$$\frac{\partial T(z,\rho)}{\partial z}\approx 2(z^2-1)^{-1} \tag{66}$$

for $1-z\gg\rho^2$. An expansion given by [19] implies $\partial T(z,\rho)/\partial z\approx 4\rho^{-2}$ for $|1-z|\ll\rho^2$. The damping rate is determined by $\mathrm{Im}\,T(z,\rho)=\pi H(1-z)e^{-\rho\gamma_r}$, with $\gamma_r=1/(1-z^2)^{1/2}=n_\parallel/(n_\parallel^2-1)^{1/2}$. This implies strong damping for $n_\parallel^2\gtrsim 1-\rho^2$, analogous to $n_\parallel^2\gtrsim 1/v_m^2$ for the hard bell and soft bell distributions.

Comparison with the results for the other three distributions considered above, with $\rho\sim 1/\gamma_m\ll 1$, suggests that the dispersive function is not particularly sensitive to the choice of distribution function for $n_\parallel<1(z>1)$. For $n_\parallel>1$, the function $\partial T(z,\rho)/\partial z$ has a zero at $0<z-1\ll 1$, analogous to the zero of $W(n_\parallel)$ at $n_\parallel=1/v_*$ for the other distributions. However, unlike the other three cases, the function (64) has no unusual properties corresponding to $n_\parallel\sim 1/v_m(n_\parallel^2-1\sim\rho^2)$, suggesting that the properties of the new sub-$\parallel$ mode may be an artifact of the discontinuous derivatives of the hard bell and soft bell distributions. The strong damping for $n_\parallel^2\gtrsim 1+\rho^2$ is analogous to that for $n_\parallel^2\gtrsim 1/v_m^2$ for the hard bell and soft bell distributions.

F. Wave properties for more general distributions

With the foregoing examples as guides, we now draw some general conclusions concerning the properties of the wave modes for a wider class of distributions of highly relativistic particles. We consider distributions that are nonzero in a range $-u_{m-}<u_z<u_{m+}$, where $u_{m\pm}$ are large and positive, with $u_{m-}\neq u_{m+}$ in general. We also assume that $\tilde{F}_0$ decreases monotonically with the increasing γ, that is, there are no beams.

For the A mode to exist and be of interest, one requires that the frequency be below the maximum allowed frequency, and that the damping be weak. In the ultrarelativistic case, $\tilde{F}_0$ is approximately constant below a cutoff at $\gamma=\gamma_m$, and then normalization to unity implies $\tilde{F}_0\sim 1/\gamma_m$. Continuity of $d\tilde{F}_0/du_z$ at $u_z=u_m$ results in the existence of a maximum A wave frequency that can be estimated by setting $n_\parallel^2=1/v_m^2=1+1/\gamma_m^2$. Using Eq. (44) to estimate $W(n_\parallel)$ at this value, one finds $W(n_\parallel)\sim\zeta\langle\gamma\rangle$, with ζ a coefficient of order unity that depends on the details of the distribution function. It follows that the maximum allowed frequency for A waves is $\omega_{\max}\sim\omega_p(2/\zeta\langle\gamma\rangle)^{1/2}$. Damping results from continuity of $\tilde{F}_0$ at $u_z=u_m$ and the damping is strong for $1+\delta_1>1+1/u_m^2$. To within a factor of order unity, this condition implies that the damping is strong for $\Delta\gtrsim 1/\langle\gamma\rangle^3$. It follows that A waves exist in one-dimensional plasmas only when the magnetic field is sufficiently strong that this condition is satisfied.

We conclude that the A mode exists and has dispersion relation approximated by $\omega_A=k\cos\theta$ within limits set by the maximum frequency, $\omega_{\max}\sim\omega_p/\langle\gamma\rangle^{1/2}$, and by the onset of strong damping, with weakly damped waves confined to the range $\Delta\lesssim 1/\langle\gamma\rangle^3$. With the parameters used here, the maximum frequency for A waves is $\sim 10^9\ \mathrm{s}^{-1}$ near the pulsar surface, decreasing to $\lesssim 10^6\ \mathrm{s}^{-1}$ above $R\sim 10^2R_0$. The strong damping implies that A waves cannot exist at $R\gtrsim 10^3R_0$, where $\Delta\gtrsim 1/\langle\gamma\rangle^3$. Presence of an exponential tail for $\gamma\to\infty$, as in the distribution (63), does not change these semiquantitative conclusions.

For the L-O mode one has $n_\parallel<1$. For an ultrarelativistic plasma the dispersion function may be approximated by

$$W(n_\parallel)=\int_{-\infty}^{\infty}\frac{\tilde{F}_0du_z}{\gamma^3(1-n_\parallel v_z)^2}$$
$$\approx\langle\gamma^{-3}\rangle\left[\frac{1+a}{2(1-n_\parallel)^2}+\frac{1-a}{2(1+n_\parallel)^2}\right], \tag{67}$$

where a is the measure of the distribution asymmetry. In the high phase velocity limit $n_\parallel\ll 1$ the L-O mode describes oblique Langmuir waves with approximate dispersion relation

$$\omega_l^2=2\omega_p^2\langle\gamma^{-3}\rangle+k^2. \tag{68}$$

As noted above, one has $\langle\gamma^{-3}\rangle\sim\langle\gamma\rangle^{-1}$ in an ultrarelativistic plasma. With the parameters used here the cutoff frequency implied by Eq. (68) is $\omega_p\langle\gamma^{-3}\rangle^{1/2}\sim 10^9\ \mathrm{s}^{-1}$, near the pulsar surface, and $\sim 10^6\ \mathrm{s}^{-1}$ at $R\sim 10^2R_0$.

In the high-frequency limit, $n\to 1$, Eq. (67) in Eq. (55) describes O-mode waves with dispersion relation

$$\omega_O^2=k^2+\frac{2\omega_p^2\langle\gamma^{-3}\rangle}{\sin^2\theta}(1+2a\cos\theta+\cos^2\theta), \tag{69}$$

where the condition $k\gtrsim\omega_p\langle\gamma^{-3}\rangle^{1/2}/\sin\theta$ is assumed to be satisfied.

G. Parallel propagation

The above expressions are valid for oblique propagation, when $\theta\gg(\Delta\langle\gamma\rangle)^{1/2}$, $1/\langle\gamma\rangle$. In the opposite limit the waves should be considered as effectively parallel. For completeness we summarize the properties of the waves propagating parallel to the external magnetic field.

The t and A waves become circularly polarized and have the same dispersion relation $\omega^2=k^2v_A^2(1-\Delta\lambda)$. The dispersion relation for the Langmuir wave in the limit $k\ll\omega_0$ becomes

$$\omega^2=\omega_0^2+\frac{4\omega_p^2k\langle u_z\gamma^{-4}\rangle}{\omega_0}+\frac{6\omega_p^2k^2\langle u_z^2\gamma^{-5}\rangle}{\omega_0}. \tag{70}$$

The most significant change from the oblique case is that the parallel Langmuir mode crosses the line $n=1$ and becomes subluminous at $\omega^2\approx 4\omega_p^2\langle\gamma\rangle$.

VI. DISCUSSION AND CONCLUSIONS

Our study of the low-frequency waves in a one-dimensional, relativistic pair plasma is motivated by their possible application to pulsar radio emission. We describe

TABLE I. Plasma parameters for different locations in a pulsar magnetosphere. The choice of parameters is discussed in Sec. II.

Distance, R_0	ω_p (s^{-1})	Ω (s^{-1})	$\omega_p(\langle\gamma\rangle)^{1/2}$ (s^{-1})	$\omega_p/(\langle\gamma\rangle)^{1/2}$ (s^{-1})	$\Omega/\langle\gamma\rangle$ (s^{-1})	$\omega_p^2\langle\gamma\rangle/\Omega^2$
1	2×10^{10}	2×10^{19}	2×10^{11}	2×10^{9}	2×10^{17}	10^{-16}
10	6×10^{8}	2×10^{16}	6×10^{9}	6×10^{7}	2×10^{14}	10^{-13}
10^2	2×10^{7}	2×10^{13}	2×10^{8}	2×10^{6}	2×10^{11}	10^{-10}
10^3	6×10^{5}	2×10^{10}	6×10^{6}	6×10^{4}	2×10^{8}	10^{-7}
10^4	2×10^{4}	2×10^{7}	2×10^{5}	2×10^{3}	2×10^{5}	10^{-4}

these waves in the plasma rest frame. Observed frequencies are higher than those in the plasma rest frame by a factor $\sim\gamma_p$, due to the Lorentz transformation, with $\gamma_p=10^3$ assumed here. The basic parameters characterizing the low-frequency waves in the pulsar plasma for different conditions are given in Table I. Depending on the location in the pulsar magnetosphere, slightly different sets of obliquely propagating modes exist.

It can be seen from the table that above $R\gtrsim10^3R_0$ the radio range waves are no longer nonresonant, since $\omega\sim\Omega/\langle\gamma\rangle$. On the other hand, the finite magnetic field corrections are negligible up to $R\sim10^3R_0$, and the infinite magnetic field approximation must be applied. The maximum A wave frequency and minimum Langmuir wave frequency, which are of the same order, $\sim\omega_p/(\langle\gamma\rangle)^{1/2}$, are above the radio range near the pulsar surface, but below it at $R\gtrsim10^2R_0$. The subluminous A wave and superluminous L-O wave apparently complement each other to ensure the number of allowed oblique modes at any given frequency equals two. In the parallel propagation case there are always two complementary subluminous transverse waves (t and A waves degenerate), while the parallel Langmuir wave gradually enters the radio range with increasing distance. At $R\sim10^2R_0$ the whole undamped part of the Langmuir wave is in the radio range.

There is observational evidence in favor of the location of the emission zone well inside the magnetosphere, at $R\sim10^{-2}R_L$, which corresponds to $10R_0\lesssim R\lesssim10^2R_0$. Taking into account that this zone is only a small part of the magnetosphere, one finds that the emission zone should be located somewhere between $10R_0$ and 10^2R_0 for a typical 1 s pulsar, and its size is $\sim R_0$. In this case the only waves which may participate in the local spectrum formation (for example, due to nonlinear processes) are t and A waves, for which the approximate dispersion relations $\omega_t=k$ and $\omega_A=k\cos\theta$ are appropriate. The Langmuir-O mode has a dispersion relation approximated roughly by $\omega_l^2=\omega_p^2\langle\gamma^{-3}\rangle+k^2$.

To summarize, we propose a method for studying waves with frequencies much lower than the relativistic gyrofrequency in relativistic pair plasmas. We derive the concise general dispersion relations for these low-frequency waves without making any additional simplifying assumptions. We analyze the effects of gyrotropic terms on the waves in the radio frequency range and find them negligible except for the polarization in the parallel propagation case. Working in the plasma rest frame we derive the dispersion relations in various limits. Our representation differs from that of Ref. [13] in that that our choice of the plasma rest frame avoids having the dispersion relations depend on Doppler shift effects. This choice of frame allows us to concentrate on the effects related to the intrinsically relativistic distribution of electrons and positrons. Finally, using the pulsar plasma parameters typical for the polar cap cascade models, we perform mode-location mapping and establish which modes can participate in the processes of the formation of the radio emission spectrum for a typical pulsar.

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[1] M. Hoshino, J. Arons, Y. A. Gallant, and A. B. Langdon, Astrophys. J. **390**, 454 (1992); R. V. E. Lovelace, J. C. L. Wang, and M. E. Sulkanen, *ibid.* **315**, 504 (1987); M. C. Begelman, R. D. Blandford, and M. D. Rees, Rev. Mod. Phys. **56**, 255 (1984); S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972); R. Misra and F. Melia, Astrophys. J. **449**, 813 (1995); G. Z. Xie, B. F. Liu, and J. C. Wang, *ibid.* **454**, 50 (1995).

[2] R. N. Manchester and J. H. Taylor, *Pulsars* (Freeman, San Francisco, 1977).

[3] C. F. Kennel and F. V. Coroniti, Astrophys. J. **283**, 694 (1984); Y. A. Gallant and J. Arons, *ibid.* **435**, 85 (1994).

[4] D. B. Melrose, J. Astrophys. Astron. **16**, 137 (1995).

[5] P. A. Sturrock, Astrophys. J. **164**, 529 (1971); M. A. Ruderman and P. G. Sutherland, *ibid.* **196**, 51 (1975); J. Arons, Space Sci. Rev. **24**, 437 (1979); F. C. Michel, Rev. Mod. Phys. **54**, 1 (1982); J. K. Daugherty and A. K. Harding, Astrophys. J. **252**, 337 (1982); J. Arons, *ibid.* **266**, 175 (1983); A. E. Shabad and V. V. Usov, Nature **295**, 215 (1982); V. V. Usov and D. B. Melrose, Astrophys. J. **464**, 306 (1996).

[6] D. B. Melrose, in *Pulsars as Physics Laboratories*, edited by R. D. Blandford, A. Hewish, A. G. Lyne, and L. Mestel (Oxford University Press, London, 1993), p. 105.

[7] J. G. Lominadze, G. Z. Machabeli, and V. V. Usov, Astrophys. Space Sci. **90**, 19 (1983).

[8] M. E. Gedalin and G. Z. Machabeli, Sov. Astron. Lett. **8**, 153 (1982).

[9] M. E. Gedalin and G. Z. Machabeli, Sov. J. Plasma Phys. **9**, 592 (1983).

[10] D. B. Melrose, Plasma Phys. Controlled Fusion **39**, 93 (1996).

[11] L. S. Kuz'menkov and P. A. Polyakov, Astrophys. **19**, 463 (1983).
[12] M. E. Gedalin and G. Z. Machabeli, Sov. Astrophys. **19**, 91 (1983).
[13] J. Arons and J. J. Barnard, Astrophys. J. **302**, 120 (1986).
[14] P. A. Polyakov, Plasma Phys. Rep. **23**, 172 (1997).
[15] J. M. Cordes, in *The Magnetospheric Structure and Emission Mechanisms of Radio Pulsars*, edited by T. H. Hankins, J. M. Rankin, and J. A. Gill (Pedagogical University Press, Zielona Góra, Poland, 1992), p. 253.
[16] M. Gedalin, Phys. Plasmas **1**, 1159 (1994).
[17] M. Gedalin, Phys. Rev. E **47**, 4354 (1993).
[18] A. S. Volokitin, V. V. Krasnoselskikh, and G. Z. Machabeli, Sov. J. Plasma Phys. **11**, 531 (1985).
[19] B. B. Godfrey, B. S. Newberger, and K. A. Taggart, IEEE Trans. Plasma Sci. **PS-3**, 60 (1975).