Anomalous scaling for a passive scalar near the Batchelor limit

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A class of phenomenological Hopf equations describing mixing of a passive scalar by random flow close to the Batchelor limit (i.e., advection by random strain and vorticity) is analyzed. In the Batchelor limit multipoint correlators of the scalar are constructed explicitly by exploiting the SL $(N,\mathbb{R})$ symmetry of the Hopf operator. Hopf equations close to this ''integrable'' limit are solved via singular perturbation theory based on matched asymptotic expansions. The solution for the three-point correlator exhibits anomalous scaling indicating persistence of the small scale anisotropy for the scalar. In addition to the exponent, the full configuration dependence of the correlator is obtained. [S1063-651X(98)08703-0]

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I. INTRODUCTION

The advection of a passive scalar Θ by a turbulent velocity field [1] is of interest to experimentalists and theorists alike both in the context of the problem of turbulent mixing, and because of its similarities to the more challenging problem of turbulence itself. The governing equation is simply

$$\partial_t \Theta + \vec{v} \cdot \vec{\nabla} \Theta = \kappa \nabla^2 \Theta, \tag{1.1}$$

where κ is the diffusivity, which is analogous to the viscosity in the Navier-Stokes equation. Obukhov [2] and Corrsin [3] observed that when the velocity is in the Kolmogorov 1941 scaling [4] ($K41$) regime, the scalar should also display the same wave number spectrum and the scalar variance should cascade from large to small scales at a rate ϵ_θ determined by the large scale boundary conditions. The analogy with the statistics of the velocity fluctuations persists also in the manifestation of the violations of the $K41$ scaling as the fourth and higher order correlations of the scalar became increasingly non-Gaussian—the phenomenon known as intermittency [5].

One of the puzzling departures from Kolmogorov predictions [4], particular to the scalar, is that the derivative skewness $s_d = \langle(\partial_x\Theta)^3\rangle/\langle(\partial_x\Theta)^2\rangle^{3/2}$; observed in shear flows with an imposed large scale scalar (e.g., temperature) gradient, turns out to be of order one and Reynolds independent [6,1]. This quantity measures violation of parity symmetry on small scales. Kolmogorov phenomenology does not merely assume that the small scales are as universal as symmetry and dimensional considerations allow, but supplies a prediction as to how parity (and isotropy) breaking by a large scale gradient $\vec{g}$ influences the small scales, viz., $s_d \sim g/\langle(\partial_x\Theta)^2\rangle^{1/2} \sim R^{-(1/2)}$ or in the inertial range, $\delta\Theta_r = \Theta(r) - \Theta(0)$, $S_r = \langle \delta\Theta_r^3\rangle \sim r^{5/3}$ (versus r^1 in experiments). The force of this contradiction caused the early workers to carefully search for systemmatic errors in their probes, but the effect remained.

To model this effect in Eq. (1.1) it is convenient to assume that the large scale gradient is uniform and to shift $\Theta \to \theta - gr$. This puts a ''force'' gv on the right hand side of Eq. (1.1) which is a sensible idealization of how an experiment maintains a statistically steady state.

It is natural to ask whether the intermittency seen in the scalar field is merely a passive translation of that already in the velocity or whether it is intrinsic to Eq. (1.1), i.e., present for a Gaussian velocity field as well. Kraichnan long ago [7,8], argued that advection by a Gaussian δ-correlated random strain, gave nontrivial intermittency, and more recently Holzer and Siggia [9] showed the same numerically for a velocity field with Kolmogorov like power law correlator, and non-δ-correlated temporal correlations. In particular, for simulations with a mean gradient, the skewness was very similar to that in the shear flow experiments.

Recently a number of groups [10–12] realized that nontrivial exponents for scalar correlations of order 3 and greater are associated with the zero modes of the so-called Hopf operator that controls the temporal evolution of the equal time multipoint correlators. Following Kraichnan [7,8], this operator can be derived exactly [13] for a model with velocity that is white in time and Gaussian: the Kraichnan's δ-correlated model. Further approximations, either a closure for the dissipation term [14], an expansion for large dimension [10], or about the ''weak coupling'' molecular diffusion limit [11], or the ''strong coupling'' random straining limit [15]; are necessary to obtain explicit answers.

The work detailed in this article is devoted to another model [12,16] that remains more faithful to the temporal correlations of the velocity dictated by the Navier-Stokes equations at the expense of the exact derivation of the Hopf operator from Eq. (1). The models we consider are phenomenological and are best thought of by drawing an analogy between the Hopf operator and a Hamiltonian (for a quantum mechanical many-body system): the latter defines the evolution operator for a wave function, the former—the evolution operator for the multipoint (equal time) correlator. The study of the appropriate *effective* Hamiltonians is often fruitful even in the absence of their full microscopic derivation. Similarly we construct and investigate a class of phenomenological or effective Hopf operators the stationary modes of which approximate the correlators in question. The effective Hopf operator inevitably contains free parameters which are to be fixed through comparison with experiment; yet as long as the number of such parameters is small, the phenomenological model retains predictive power.

An important aspect of the physical velocity field is that the change in the relative distance of a pair of material points in the flow over the correlation time (the eddy turnover time) is of the order of the separation itself. The absence of any dimensional parameters in the inertial range implies that on any scale r, $\delta v(r)\tau(r)\sim r$, where τ is the Lagrangian correlation time. Thus the Lagrangian displacement over one correlation time is approximately described by an order one volume preserving mapping. The effective Hopf operator is then written as a sum of what we call the ''Batchelor-Kraichnan'' piece, which accounts for the large scale, coherent strain and vorticity [17,7]; and a second, dissipative, term, analogous to an eddy diffusion. The later expresses the effect of the small scales of the velocity field which fluctuate independently at the distinct points of the correlator. The derivation [12] of the resulting Hopf equation is only heuristic and will not be repeated here: instead we shall dwell on its analysis.

The Batchelor-Kraichnan operator is highly symmetric and in Sec. II we show that it is integrable by Lie algebraic methods [12]. (See also Refs. [18] and [19] for the analysis of the Batchelor limit.) Section III introduces the simplest model of dissipation, which we call the ''Laplacian model'' (or L model), which preserves some symmetry, and allows us to solve for the anomolous skewness and flatness exponents, both numerically via an ordinary differential equation shooting method, and via a matched asymptotic expansion. This dissipation model is not, however, consistent with Kolmogorov scaling when two points in the correlator coalesce. Hence in the remainder of Sec. III we introduce an improved model, we we call the ''pseudo-Kolmogorov model'' (or K model). The singular perturbation expansion is generalized to treat this more interesting model yielding the anomalous scaling exponent and the full configuration dependence of the three-point function. The results of the calculations described here have been previously reported in Refs. [12] and [15]. The singular perturbation theory described here has also been applied to the calculation of the anomalous skewness exponent in the δ-correlated velocity model near the Batchelor limit reported in Ref. [16].

II. PROPERTIES OF THE BATCHELOR-KRAICHNAN OPERATOR

A. Definitions and skewness in two dimensions

The Batchelor-Kraichnan operator describes the evolution of passive scalar correlators under the action of random large scale strain and vorticity [17,7,13],

$$\mathcal{L}_0 \equiv \sum_{ij} D_{ab}(r_i - r_j)\partial^a_{r_i}\partial^b_{r_j} \tag{2.1a}$$

with

$$D_{ab}(r) = (d+1)\delta^{ab}r^2 - 2r^a r^b, \tag{2.1b}$$

where the coefficients are constrained by incompressibility which demands $\partial_a D_{ab}=0$. Since the correlation functions are translationally invariant, they do not depend on the center of mass coordinate $1/N\Sigma_i \vec{r}_i$. It is convenient to define a set of reduced coordinates, e.g., for $N=3$:

$$\begin{bmatrix}\vec{\rho}_0\\ \vec{\rho}_1\\ \vec{\rho}_2\end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix}\vec{r}_1\\ \vec{r}_2\\ \vec{r}_3\end{bmatrix} \tag{2.2}$$

or $\vec{\rho}_i = M_{ij}\vec{r}_j$ in a more compact notation. Matrix M is orthonormal, $M_{ik}M_{jk}=\delta_{ij}$, which makes $\vec{\rho}_i$ independent so that $\vec{\partial}_i \equiv \partial/\partial\vec{\rho}_i = M_{ij}\vec{\partial}_{r_j}$, has the property $\partial^a_i \rho^b_k = \delta_{ab}\delta_{ik}$. The $\vec{\rho}_{1,2}$ are the relevant reduced vectors, independent of the center of mass variable $\vec{\rho}_0$. This definition is readily generalized to any N. It is easy to check that the characteristic inter-point distance, which we'll call the radius of gyration, is $R^2 = \Sigma_{jk}\vec{r}^{\,2}_{jk} = \Sigma_{i=1}^{N-1}\vec{\rho}^{\,2}_i$ and the Laplacian $\Sigma_{j=1}^N \partial^2_{r_j} = \Sigma_{i=1}^{N-1}\partial^2_{\rho_i}$. In the reduced variables

$$\mathcal{L}_0 = \sum_{a,b=1}^{d}\sum_{i,j=1}^{N-1}[(d+1)\vec{\rho}_i\cdot\vec{\rho}_j\delta^{ab} - 2(\rho^a_i\rho^b_j + \rho^b_i\rho^a_j)]\partial^a_i\partial^b_j. \tag{2.3}$$

The $\mathcal{L}_0$ operator turns out to be invariant under the group of general linear transformations $\vec{\rho}_i \to g_{ij}\vec{\rho}_j$ which mix different reduced coordinate vectors. (Below we shall on occasion refer to the reduced coordinate labels i as pseudospace.) This group factorizes into dilations $\vec{\rho}_i \to \beta\vec{\rho}_i$ and the volume preserving transformations, with $\det g = 1$, which constitute the SL $(N-1,\mathbb{R})$ group. The origin of this invariance traces to the fact that under large scale strain and vorticity Lagrangian coordinates evolve according to $\rho^a_i \to m_{ab}\rho^b_i$ (where $m_{ab} = \partial_a v_b$ is the strain-vorticity matrix): clearly this dynamics is invariant under $\vec{\rho}_i \to g_{ij}\vec{\rho}_j$. The infinitesimal SL transformations are generated by

$$G_{ij} \equiv \vec{\rho}_i\cdot\vec{\partial}_j - \frac{1}{n}\,\delta_{ij}\Lambda, \tag{2.4a}$$

where $n \equiv N-1$ for convenience and Λ is the dilation operator,

$$\Lambda \equiv \rho^a_i\partial^a_i. \tag{2.4b}$$

Remembering that in addition to dilation and SL, $\mathcal{L}_0$ is rotationally invariant, one can list all the invariant quadratic operators. They are, aside from Λ,

$$G^2 \equiv \tfrac{1}{2}\sum_{ij} G_{ij}G_{ji}, \tag{2.5a}$$

the Casimir operator of SL (n) (i.e., $[G_{ij}, G^2]=0$) and the total angular momentum square

$$L^2=\tfrac{1}{2}\sum_i \sum_{ab} (\rho_i^a\partial_i^b-\rho_i^b\partial_i^a)^2. \tag{2.5b}$$

Symmetry dictates that $\mathcal{L}_0$ can only be a linear superposition of these, which turns out to be:

$$\mathcal{L}_0=-(d+1)L^2+2dG^2+d(d-n)\left(\frac{\Lambda^2}{nd}+\Lambda\right). \tag{2.6}$$

In order to completely diagonalize $\mathcal{L}_0$ acting on $n\times d$ dimensional space we must find a solution with the same number of quantum numbers. Let us consider first the simplest case $n=2,d=2$. In addition to L^2, G^2, and Λ one can simultaneously diagonalize $G_y\equiv i(G_{12}-G_{21})$ which generates rotation in two-dimensional pseudospace. It is possible to directly construct a function $\Psi^{\lambda}_{\nu ql}(\rho)$, which is an eigenstate of all these operators:

$$T_\nu^{q,l}(\hat{\rho})=\int_0^{2\pi}d\psi\int_0^{2\pi}d\theta e^{il\theta+iq\psi}h_{\nu,q}\left(\frac{e_a(\theta)\rho_i^a n_i(\psi)}{\det\rho}\right), \tag{2.7a}$$

$$\Psi^{\lambda}_{\nu ql}(\rho)=[\det\rho]^{\lambda/2}T_\nu^{q,l}(\hat{\rho}), \tag{2.7b}$$

where $\hat{n}(\psi),\hat{e}(\theta)$ are 2D unit vectors parameterized by angles ψ, θ, respectively, and we introduced a homogeneous function $h_{\nu,q}(x)\equiv\mathrm{sign}(x)^q|x|^{2\nu}$. The $T_\nu^{q,l}(\hat{\rho})$ is just the transformation matrix for the ν representation [20] corresponding to SL(2) group elements $\hat{\rho}$ (normalized so as to make det $\hat{\rho}=1$). The set of functions $\Psi^{\lambda}_{\nu ql}$ forms a representation of SL(2)×SO(2)×Λ, i.e., transforms linearly under the action of the group elements.

It is sufficient to consider only $\nu\geqslant-1/2$ for which the integrals are well defined. The integration on θ and ψ projects onto l and q angular and pseudoangular momentum sectors. E.g., pseudospace rotation of ρ_i^a by angle ϕ, $\rho\to\mathbf{R}(\phi)\rho$, can be absorbed into redefinition $\psi\to\psi+\phi$ which leads to $T_\nu^{ql}(\mathbf{R}(\phi)\rho)=e^{iq\phi}T_\nu^{ql}$. The scaling dimension is λ: $\Lambda\Psi^{\lambda}_{\nu ql}=\lambda\Psi^{\lambda}_{\nu ql}$. We have used $\det\rho\equiv\epsilon_{ij}\epsilon_{ab}\rho_i^a\rho_j^b$—the area of triangle r_1, r_2, r_3 as the normalizing factor because it is invariant not only under spatial rotations but also under SL(2): it is easy to verify that G_{ij} det ρ=0. Finally, the eigenvalue of G^2 can be computed directly by differentiating (2.7a) and exploiting the homogeneity of h,

$$G^2\Psi^{\lambda}_{\nu ql}=\nu(\nu+1)\Psi^{\lambda}_{\nu ql}. \tag{2.8}$$

Thus from Eqs. (2.6) and (2.8) we have for the zero modes ($d=2$, $n=N-1=2$),

$$\mathcal{L}_0\Psi^{\lambda}_{\nu,q,l}=[-3l^2+4\nu(\nu+1)]\Psi^{\lambda}_{\nu,q,l}=0. \tag{2.9}$$

Curiously, for any $n=d$ in Eq. (2.6), λ does not appear directly. It does, however, enter indirectly via the boundary condition at $\det\rho=\vec{\rho}_1\wedge\vec{\rho}_2=0$. As shown in Appendix B (for $\nu\geqslant-\frac{1}{2}$) in the limit det $\rho\to0$,

$$\Psi^{\lambda}_{\nu ql}(\rho)\to|\det\rho|^{(\lambda/2)-\nu}. \tag{2.10}$$

On the physical grounds one must demand Ψ to remain finite, hence $\nu\leqslant\lambda/2$, or more strongly, differentiable: $\lambda/2=\nu+k$, integer $k\geqslant0$. Thus the lowest λ mode: $\lambda/2=\nu\geqslant-\frac{1}{2}$. (Furthermore, modes with $\lambda/2>\nu$ vanish at collinearity and therefore do not contribute to the structure function.) Equation (2.9) becomes $\lambda^2+2\lambda-3l^2=0$ and the $d=n=2$ spectrum,

$$\lambda(l,k)=2k-1+\sqrt{1+3l^2}. \tag{2.11}$$

The evolution equation (1.1) makes θ odd under reflections, so the $N=3$ correlator is odd under reflections and proportional to $\langle\vec{\nabla}\Theta\rangle$. Therefore l is odd, the lowest relevant mode is $l=1$, and the leading anomalous scaling exponent,

$$\lambda(1,0)=1. \tag{2.12}$$

Note, that the pseudoangular momentum quantum number, q, did not enter the eigenvalue equation (2.9), so that the spectral exponents are infinitely degenerate. This degeneracy is a consequence of the SL(2,$\mathbb{R}$) symmetry and is lifted by the dissipation term $\mathcal{L}_D$, which we will treat perturbatively in the next section.

B. Properties of the Batchelor-Kraichnan operator for $N>3$ or $d>2$

Let us now generalize the analysis of $\mathcal{L}_0$ to arbitrary N and d. First we find the spectrum of $\mathcal{L}_0$. The eigenvalue of angular momentum L^2 in d dimensions is $L^2=l(l+d-2)$. The spectrum of G^2 can be found by noting the following duality relation. Defining $J^{ab}\equiv\rho_i^a\partial_i^b-1/d\,\delta^{ab}\Lambda$, which generates SL($d,\mathbb{R}$) transformations acting on real space (rather than pseudospace) one can prove

$$G^2=J^2-\frac{d-n}{2}\left(\frac{\lambda^2}{nd}+\lambda\right). \tag{2.13}$$

Now, the spectrum of G^2 is determined by the structure SL($n,\mathbb{R}$) group which does not depend on d. Hence the spectrum of G^2 can be found from Eq. (2.13) evaluated for $d=1$ for which $J^2=0$, yielding

$$G^2=\frac{n-1}{2n}\lambda(\lambda+n) \tag{2.14a}$$

and by the same token,

$$J^2=\frac{d-1}{2d}\lambda(\lambda+d). \tag{2.14b}$$

Strictly speaking, λ, which enters here, is the homogeneity degree of the representation functions constructed from $d=1$ and may differ from the full Λ by an integer if invariants exist, e.g., det ρ for the $d=n$ case as we have seen already for $d=n=2$. Using Eq. (2.13) one can rewrite Eq. (2.6) more compactly:

$$\mathcal{L}_0=(d+1)L^2+2dJ^2. \tag{2.15}$$

Equation (2.14) will suffice for the calculation of the smallest λ, which from Eqs. (2.6) and (2.14b) obeys $(d+1)l(l+d-2)-(d-1)\lambda(\lambda+d)=0$, so that

$$\lambda(l)=\frac{d}{2}\left[\sqrt{1+\left(\frac{2}{d}\right)^2\frac{d+1}{d-1}\,l(l+d-2)}-1\right]. \quad (2.16)$$

Remarkably, the spectrum of leading zero mode exponent of $\mathcal{L}_0$ does not depend on N! This might have been expected because $\mathcal{L}_0$ represents only the advective part of the evolution so that points can be brought together, e.g., θ^2 behaves like θ. Hence the spectrum for any N is a subset of that for larger N.

Equally remarkably, we observe that for the s wave $\lambda(0)=0$, and for the p wave $\lambda(=1)=1$, independent of d! For $d\gg l$ we have $\lambda(l)\approx l+\mathcal{O}(1/d)$. The s-wave channel of course is relevant for the even order correlators (i.e., even N).

Certain of the eigenfunctions can be constructed via integral representations analogous to Eq. (2.7), e.g., for the flatness ($n=3$) in three dimensions we can write

$$\Psi^{\lambda}_{\nu;p,q;l,m}(\rho)=|\det\,\rho|^{(\lambda-2\nu)/3}$$
$$\times\int d\hat{\Omega}_1\int d\hat{\Omega}_2 Y^q_p(\hat{\Omega}_1)Y^l_m(\hat{\Omega}_2)h_\nu(\hat{\Omega}_{1a}\rho^a_i\hat{\Omega}_{2i}), \quad (2.17)$$

which has the spatial angular momentum (l,m), pseudospace angular momentum (q,p), and the $G^2=(n-1/n)\nu(2\nu+n)$ which agrees with Eq. (2.14) for $\lambda=2\nu$, which corresponds to the lowest λ state. Actually, Eq. (2.19) is not a complete set of states, because we only have six quantum numbers instead of nine. Additional quantum numbers can be introduced taking a somewhat more complex $h_\nu(x)$ and by replacing integrals over unit vectors $\int d\hat{\Omega}_1 Y^q_p(\hat{\Omega})\cdots$ by integrals over triads (i.e., rotation matrices): $\int d\mathbf{R}D^q_{pp'}(\mathbf{R})$ and $D^q_{pp'}(\mathbf{R})$ is an SO(3) representation matrix $-q\leqslant p,\ p'\leqslant q$. This will yield three real space angular quantum numbers plus three pseudospace, plus λ and ν for a total of eight. We believe this is the correct count and that the eigenvalue problem for G^2 in the nine-dimensional ρ space is nonintegrable. More explicitly, if we change from ρ to the Euler variables (see Appendix A) in G^2, impose the angular quantum numbers and λ, there remains a second order partial differential equation in two variables. Our assertion is that it cannot be solved by separation of variables and its solutions are labeled by ν alone.

The ''duality'' between the SL(n) acting on pseudo-space indices and the SL(d) acting in real space is particularly useful when constructing eigenfunctions for $n\neq d$ as it allows us to work with the smaller of the two. An interesting example of the $n\neq d$ case is the skewness in three dimensions, which can easily be adapted to describe the flatness in $d=2$. In contrast to the $n=d$ case, there is now a vector $\vec{\rho}_1\times\vec{\rho}_2$ which is invariant under SL(2); an arbitrary function of which can multiply the integral in Eq. (2.7). Thus,

$$\Psi^{\lambda}_{\nu;q;l,m,m'}(\rho)=|\vec{\rho}_1\times\vec{\rho}_2|^{(\lambda-2\nu)/2}Y^l_m\left(\frac{\vec{\rho}_1\times\vec{\rho}_2}{|\vec{\rho}_1\times\vec{\rho}_2|}\right)$$
$$\times\int_0^{2\pi}\frac{d\psi}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\,e^{iq\psi+im'\theta}$$
$$\times h_{\nu,q}(e_a(\theta)\rho^a_i n_i(\psi)), \quad (2.18)$$

where unit vector $\hat{e}(\theta)$ rotates about the $\vec{\rho}_1\times\vec{\rho}_2$ direction hence, $\hat{e}(\theta)\cdot\vec{\rho}_1\times\vec{\rho}_2=0$.

The $\mathcal{L}_0$ eigenfunction Eqs. (2.7) and (2.18) and the corresponding spectrum provide the point of departure for the perturbative calculation described next.

III. PERTURBATION ABOUT THE BATCHELOR LIMIT

A. Skewness for the Laplacian dissipation model (the *L* model)

Let us first develop the perturbation theory for the simplest case of the dissipation operator [21] $\mathcal{L}_D=\epsilon(\Sigma_i\vec{\partial}^2_i)(\Sigma_j\vec{\rho}^2_j)$ for the skewness $n=2$, in $d=2$.

It is convenient to work in the Euler coordinates defined in Appendix A and introduce a reduced variable $w\equiv\xi^{-1}=2\det\rho/R^2$ and write the Laplacian [see Eq. (A13)] acting on an eigenmode nondimensionalized by $\zeta\equiv\det\rho$, viz., $\Psi^{(\lambda)}/\zeta^{\lambda/2}$ as

$$\mathcal{L}_D=4\epsilon\Bigg[(1-w^2)\partial^2_w+[\lambda-(2+\lambda)w^2]w^{-1}\partial_w$$
$$+\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right)w^{-2}+\frac{1}{4}\,\frac{1}{1-w^2}\,(\partial^2_\chi+\partial^2_\phi+2w\partial_\chi\partial_\phi)\Bigg]. \quad (3.1a)$$

The Batchelor-Kraichnan operator in the same variables is

$$\mathcal{L}_0=4w^2\partial_w(1-w^2)\partial_w+\frac{w^2}{1-w^2}\,(\partial^2_\chi+\partial^2_\phi-2w^{-1}\partial_\chi\partial_\phi)$$
$$+3\partial^2_\phi. \quad (3.1b)$$

The ϕ and χ dependence is trivially diagonalized by going to the angular momentum representation: $\partial_\chi=iq,\ \partial_\phi=il$. The diagonalization reflects the fact that $\mathcal{L}_D$ is invariant with respect to rotations not only in space (i.e., $\phi\rightarrow\phi+\theta$) but also rotations in pseudospace (i.e., rotation acting on ''i'' index, $\chi\rightarrow\chi+\theta'$). The SL(2)×SO(2) is broken down to SO(2)×SO(2) but no further, and the q,l quantum numbers remain good. Thus in analogy with Eq. (2.7) we factorize the zero mode as $\Psi^\lambda_{q,l}=(\zeta/|w|)^{\lambda/2}\exp(il\phi+iq\chi)\varphi^\lambda_{ql}(w)$. The additional $w^{\lambda/2}$ factor means that φ is being nondimensionalized with R^2 rather than $\det\rho$.

We observe that while $\mathcal{L}_0$ scales as w to the zeroth power as $w\rightarrow0$, $\mathcal{L}_D$ scales as w^{-2} and hence for $\epsilon/w^2\gg1$ dominates over $\mathcal{L}_0$. Physically this region corresponds to nearly collinear configurations of points.

Let us first consider the leading term of the combined $\mathcal{L}=(\frac{1}{4})(\mathcal{L}_0+\mathcal{L}_D)$ in the $w\rightarrow0$ limit (which comes entirely from $\mathcal{L}_D$),

$$\left[\partial_w^2+\lambda w^{-1}\partial_w+\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right)w^{-2}\right]\varphi_{q,l}^{\lambda}/|w|^{\lambda/2}=0, \quad (3.2)$$

which implies $\varphi_{q,l}^{\lambda}(w)=A_{l,q}+B_{l,q}w$. As the area $\zeta\to 0$ (with $\mathbb{R}\sim$const) $w\sim\zeta$ and the overall eigenfunction

$$\Psi_{q,l}^{(\lambda)}\sim e^{il\phi+iq\chi}(A_{l,q}+B_{l,q}w)R^{\lambda}. \quad (3.3)$$

In order to work with fractional λ we have tacitly assumed $w>0$ or $\xi_1\xi_2=\vec{\rho}_1\wedge\vec{\rho}_2>0$. However, from the definition of the Euler coordinates (Appendix A) the interchange of $r_{1,2}$ corresponds to $\rho_1\to-\rho_1$, $\rho_2\to\rho_2$ or $\xi_1\to-\xi_1$, $\xi_2\to\xi_2$ (hence $w\to-w$) and $\chi\to-\chi$, $\varphi\to\varphi$. Any zero mode of $\mathcal{L}$ must be smooth [22] around $w=0$ since the Laplacian is dominant there. This can be insured by imposing the boundary conditions $A_{l,q}=A_{l,-q}$ and $B_{l,q}=-B_{l,-q}$. Because we are factoring the physical coordinates, there is a gauge like symmetry that must be imposed so that the factorization does not induce any spurious singularities [20].

To construct a global solution we must connect the $w^2\ll\epsilon$ region dominated by $\mathcal{L}_D$ to the $w^2\gg\epsilon$ region dominated by $\mathcal{L}_0$ where the effect of $\mathcal{L}_D$ is just a regular perturbation. Zero modes of $\mathcal{L}$ only occur for discrete values of λ. For given l, the $\mathcal{L}$ operator has an exact symmetry under $(w,q)\to-(w,q)$ just noted in connection with Eq. (3.3). Hence the eigenvalues λ corresponding to $\pm q$ must be identical. Numerically the eigenvalues can be determined by taking the dominant $O(1)$ solution around $w=0$, adding a constant b times the $O(w)$ solution and propagating the sum towards $w\sim 1$. There are two linearly independent solutions near $w=1$, only one of which is finite. Imposing finiteness at $w=1$ and insisting on $\lambda(q)=\lambda(-q)$ results in two conditions which determine λ and b.

To do the matching between $w=0,1$ perturbatively in ϵ, we go to a scaled variable $z=w/\sqrt{\epsilon}$ and expand the resulting equation in powers of $\epsilon^{1/2}$. The rescaling is chosen in such a way that the far field region of the ''boundary layer,'' $z\gg 1$, still resides (provided $z\ll\epsilon^{-1/2}$) within small w asymptotics of the outer solution $w\ll 1$, which is controlled by Batchelor-Kraichnan $\mathcal{L}_0$.

It is convenient to work in terms of φ defined above since it goes to a constant as $z\to 0$. [We have restricted to the $l=1$ angular momentum sector and have suppressed the λ,q quantum number labels on $\varphi(z)$.] It solves

$$\tilde{\mathcal{L}}\varphi(z)=0 \quad (3.4a)$$

with $\tilde{\mathcal{L}}=z^{\lambda/2}\mathcal{L}z^{-(\lambda/2)}$ or, explicitly,

$$\tilde{\mathcal{L}}=\tilde{\mathcal{L}}^{(0)}+\epsilon^{1/2}\tilde{\mathcal{L}}^{(1)}+\epsilon\tilde{\mathcal{L}}^{(2)}+\cdots . \quad (3.4b)$$

Since for $l=1$ we expect $\lambda\approx 1$ we define $\lambda=1+\epsilon\delta$ with δ to be determined. We have

$$\tilde{\mathcal{L}}^{(0)}=(1+z^2)\partial_z^2-z\partial_z, \quad (3.5a)$$

$$\tilde{\mathcal{L}}^{(1)}=\frac{q}{2}z, \quad (3.5b)$$

$$\tilde{\mathcal{L}}^{(2)}=\delta+\frac{1}{2}-\left(\frac{q}{2}\right)^2-\left(\frac{q}{2}\right)^2z^2-z^2(1+z^2)\partial_z^2-(2+\delta)z\partial_z - z^3\partial_z. \quad (3.6)$$

The two zero modes of $\tilde{\mathcal{L}}^{(0)}$ are easily found:

$$\varphi_1^{(0)}(z)=1, \quad (3.7a)$$

$$\varphi_2^{(0)}=[z\sqrt{1+z^2}+ln(\sqrt{1+z^2}+z)]. \quad (3.7b)$$

It is convenient to invert $\tilde{\mathcal{L}}^{(0)}$ and rewrite $\tilde{\mathcal{L}}\phi=0$ in the integral form

$$\varphi(z)=1+\int_0^z dz'\sqrt{1+z'^2}\times\left[C-\int_0^{z'}\frac{dz''}{(1+z''^2)^{3/2}}(\epsilon^{1/2}\tilde{\mathcal{L}}^{(1)}+\epsilon\tilde{\mathcal{L}}^{(2)})\varphi(z'')\right]. \quad (3.8)$$

The perturbative solution is obtained simply by iterating Eq. (3.8) starting with $\varphi=1$: $\varphi(z)=1+\epsilon^{1/2}\varphi^{(1)}(z)+\epsilon\varphi^{(2)}(z)+\cdots$,

$$\varphi^{(1)}(z)=\frac{q}{2}z, \quad (3.9a)$$

$$\varphi^{(2)}(z)=-\frac{z^2}{2}\left[\delta+\frac{5}{2}-\left(\frac{q}{2}\right)^2\right]+C_2\varphi_2^{(0)}(z). \quad (3.9b)$$

We have expanded the constant $C=C_0+\epsilon^{1/2}C_1+\epsilon C_2+\cdots$ and set $C_0=0$, $C_1=q/2$. The later conditions are required because $\varphi_2^{(0)}(z)\sim z^2$ for large z, which unless multiplied by constant of $O(\epsilon)$ would lead to the appearance of $\epsilon^{-1}w^2$ term—inconsistent with the asymptotics of the Batchelor regime. As is, we have in the matching region $1\ll z\ll\epsilon^{-1}$,

$$\varphi(w)=1+\frac{q}{2}w-\frac{1}{2}\left[\delta+\frac{1}{2}-\left(\frac{q}{2}\right)^2\right]w^2+C_2\left[w^2+\epsilon\ln\frac{2w}{\sqrt{\epsilon}}+\frac{\epsilon}{2}+O\left(\frac{\epsilon^2}{w^2}\right)\right], \quad (3.10)$$

The terms $O(\epsilon^2/w^2)$ are negligible because $w^2>\epsilon$ in the region of interest and we shall only be interested in terms of $O(\epsilon^0)$ anyway. To that order, the ''inner'' solution, Eq. (3.10) must be matched to the ''outer'' solution composed of the zero modes of $\mathcal{L}_0$, the $w\ll 1$ asymptotics of which is computed in Appendix C. These are the Legendre functions of $\nu=\frac{1}{2}$ and odd q. Scaling as for φ,

$$w^{\lambda/2}\mathcal{P}_{1/2}^{q,1}(w^{-1})\approx 1+\frac{q}{2}w-\left[\frac{q^2}{8}+\mathrm{sgn}(q)\left(\frac{q^2-1}{4}\right)\right]w^2+\mathcal{O}(\epsilon lnw). \quad (3.11)$$

Comparing Eqs. (3.10) and (3.11) to the leading order [22, 23] we identify

$$\delta=\frac{q^2-1}{2} \tag{3.12}$$

and

$$C_2=-\operatorname{sgn}(q)\left(\frac{q^2-1}{4}\right). \tag{3.13}$$

The identification of C_2 with the part odd under $q\to -q$ is forced by the fact that $C_2\varphi_2^{(0)}(z)\sim C_2 z$ for small z, and the analyticity across $z=0$ requires invariance under $q\to -q$, $z\to -z$, as mentioned earlier.

Thus we have calculated the correction to the scaling exponents of modes with different q:

$$\lambda(q)=1+\epsilon\,\frac{q^2-1}{2}+O(\epsilon^{3/2}). \tag{3.14}$$

Predictably the infinite degeneracy of the Batchelor limit is lifted. The threefold permutation symmetry of the correlator dictates $q=3n$. The requirement that l,q have the same parity makes q odd. The lowest λ thus corresponds to $q=\pm 3$: $\lambda_3\approx 1+4\epsilon$, which is in excellent agreement with the exponent found numerically [12], when the constant $2\epsilon\lambda$ is adjusted in the definitions of $\mathcal{L}_D$.

The Laplacian damping was convenient because it could be diagonalized for each q separately. However, this implies that the analytic behavior around $w\sim 0$ is obtained for all χ, in particular at $\chi=n\pi/3$. At these points, two of the $\vec{r}_i$ coalesce, e.g., $i=1,2$ and we expect the correlator to behave as $c_1+c_2 r_{12}^{2/3}$ (assuming $K41$). Thus the more physical model of dissipation must mix the q modes. Such a model will be analyzed in the next section. The results of the present (and the following) section is generalized to three dimensions in Appendix D.

B. Skewness for the pseudo-Kolmogorov model of dissipation (the *K* model)

The Laplacian dissipation model is unphysical as it forces analytic behavior of the correlation with a pair of points approaching coincidence. This can be remedied by replacing the $R^2=\vec{\rho}_1^{\,2}+\vec{\rho}_2^{\,2}$ factor in $\mathcal{L}_{\mathcal{D}}$ by a function $R^2F(\vec{\rho}_1/R,\vec{\rho}_2/R)$ which vanishes as the $\frac{4}{3}$ power—to reproduce Richardson diffusion—whenever two points coincide. A convenient form is

$$\mathcal{L}_D'=\epsilon R^2\left(\frac{r_{12}^2r_{23}^2r_{31}^2}{R^6}\right)^{2/3}\nabla^2, \tag{3.15}$$

which in Eulerian coordinates implies

$$\begin{aligned}F(w,\chi)=&(1-\sqrt{1-w^2}\,\cos 2\chi)^{2/3}\\&\times\left[1-\sqrt{1-w^2}\,\cos 2\left(\chi+\frac{2\pi}{3}\right)\right]^{2/3}\\&\times\left[1-\sqrt{1-w^2}\,\cos 2\left(\chi-\frac{2\pi}{3}\right)\right]^{2/3},\end{aligned} \tag{3.16}$$

so that $\mathcal{L}_D'$ has the same form as $\mathcal{L}_D$ in Eq. (3.1) but with ϵ replaced by $\epsilon F(w,\chi)$. As a consistency check, near $r_{1,2}\to 0$, $F\sim(w^2+4\chi^2)^{2/3}$ and the Laplacian behaves like $\partial_w^2+\frac{1}{4}\partial_\chi^2$.

All arguments concerning the singular nature of the perturbation near $w=0$ hold except the condition for $\mathcal{L}_{\mathcal{D}}'$ dominance, which becomes $w\ll\epsilon F(0,\chi)$. In analogy with Sec. III A we define $\Psi_l^\lambda(w,\chi)=(\zeta/w)^{\lambda/2}\Phi_l^\lambda(w,\chi)$ and expand $\tilde{\mathcal{L}}\Phi=0$. Repeating the steps that led to Eq. (3.5) we now find

$$\tilde{\mathcal{L}}^{(0)}=[f(\chi)+z^2]\partial_z^2-z\partial_z, \tag{3.17a}$$

where $f(\chi)\equiv F(0,\chi)$,

$$\tilde{\mathcal{L}}^{(1)}=-\frac{iz}{2}\,\partial_\chi, \tag{3.17b}$$

$$\begin{aligned}\tilde{\mathcal{L}}^{(2)}=&\,\delta+\tfrac{1}{2}f(\chi)+\tfrac{1}{4}[f(\chi)+z^2]\partial_\chi^2-[f(\chi)+z^2]z^2\partial_z^2\\&-[\delta+2\,f(\chi)+z^2]z\partial_z.\end{aligned} \tag{3.17c}$$

Because of the explicit χ dependence of the dissipation operator the crossover equation is no longer ''diagonal'' in q modes. Remarkably, however, because there are no derivatives with respect to χ in $\tilde{\mathcal{L}}^{(0)}$ it can still be inverted as before and the general solution can be constructed explicitly.

Inversion of $\tilde{\mathcal{L}}^{(0)}$ yields

$$\begin{aligned}\Phi(z,\chi)=&\,a(\chi)+\epsilon^{1/2}\int_0^z dz'\,\sqrt{f(\chi)+z'^2}\Bigg\{b(\chi)\\&-\int_0^{z'}\frac{dz''}{[f(\chi)+z''^2]^{3/2}}\,[\tilde{\mathcal{L}}^{(1)}+\epsilon^{1/2}\tilde{\mathcal{L}}^{(2)}]\Phi(z'',\chi)\Bigg\},\end{aligned} \tag{3.18}$$

where $a(\chi),b(\chi)$ introduce the zero modes of $\tilde{\mathcal{L}}^{(0)}$ and in analogy with the previous calculation we chose $b(\chi)=-i/2\,f(\chi)^{-1/2}\partial_\chi a(\chi)+i\epsilon^{1/2}\tilde{b}(\chi)$, which eliminates the term $\epsilon^{1/2}z^2$ for large z. We anticipate that $\tilde{b}(\chi)$ will not appear in the computation to the leading order. Iterating Eq. (3.18) results in

$$\begin{aligned}\Phi(w,\chi)=&\,a(\chi)-\frac{w^2}{2}\left[\frac{\delta}{f(\chi)}+\frac{1}{2}+\frac{1}{4}\,\partial_\chi^2\right]a(\chi)-\frac{i}{2}\,w\partial_\chi a\\&+i\tilde{b}(\chi)\int_0^w dw'\,\sqrt{\epsilon f(\chi)+w'^2}.\end{aligned} \tag{3.19}$$

Matching with the solution in the Batchelor regime will now require a superposition of many q modes,

$$\Phi(w,\chi)=w^{\lambda/2}\sum_q a_q e^{iq\chi}\mathcal{P}_{1/2}^{q,1}(w^{-1}), \tag{3.20}$$

which asymptotically for $w\ll 1$ becomes

$$\Phi(w,\chi)\approx a(\chi)+\frac{w^2}{8}\,\partial_\chi^2 a-\frac{i}{2}\,w\partial_\chi a+\frac{iw^2}{4}\,H[(1+\partial_\chi^2)a], \tag{3.21}$$

where $H[a(\chi)]$ denotes the Hilbert transform (which is defined simply the Fourier space by $\mathcal{F}_q[H(a)] = i\ \mathrm{sgn}(q)\mathcal{F}_q[a]$). Matching requires that $a(\chi)$ satisfy both Eq. (3.19) and Eq. (3.21) which is true only if

$$\frac{1}{2}\ \partial_\chi^2 a + \left[\frac{\delta}{f(\chi)} + \frac{1}{2}\right] a = 0. \tag{3.22}$$

Note that because the region of $\mathcal{L}_D'$ dominance $w \ll \epsilon f(\chi)$ is pinched whenever $f(\chi)=0$, which happens at $\chi = 1, \pm(\pi/3)$ corresponding to one of the $|r_{ij}|=0$. Equation (3.22) is singular at those points. Permutation of the $\vec{r}_i$ plus reflection, corresponds to $\chi \to \chi + (\pi/3)$. Thus Eq. (3.22) can be solved with antiperiodic boundary conditions on the interval $[0,\pi/3]$. Symmetry about $\chi=0$ [a consequence of $\rho_1 \to -\rho_1$ $\rho_2 \to \rho_2$ implies in addition that $a(\pi/6)=0$].

Local analysis near $\chi=0$ dictates

$$a(\chi) = a_0[1 + 6^{2/3}\delta\chi^{2/3} + \cdots] + a_1\left[\chi - \frac{6^{2/3}}{5}\ \delta\chi^{5/3} + \cdots\right]. \tag{3.23}$$

The matching condition (3.22) looks like a Sturm-Liouville problem with δ entering like an eigenvalue. Yet, for Eq. (3.22) to determine the eigenvalue δ we must specify the boundary condition at $\chi=0$. However, the matching procedure that led to this equation holds only for $\chi \gg \epsilon^{1/2}$ since for smaller χ the matching would have had to pass through the region $\chi \ll w \ll \epsilon^{1/2}$ whereas $\chi \gg w$ was assumed and necessary. In order to bridge the gap, let us consider the region $\chi \ll 1$ and $w \ll 1$, corresponding to near coincidence of the two observation points in the correlator, directly via the local expansion $\rho_1 \ll \rho_2$. In that limit the correlator has the form

$$\Psi^{(\lambda)}(\vec{\rho}_1,\vec{\rho}_2) = \vec{\rho}_2|\rho_2|^{\lambda-1}U\left(\frac{\vec{\rho}_1}{|\rho_2|}, \frac{\vec{\rho}_2}{|\rho_2|}\right), \tag{3.24}$$

or in the Euler coordinates,

$$\Psi^{(\lambda)}(w,\chi,\phi) = e^{i\phi}\zeta^{\lambda/2}w^{-\lambda/2}u(w,\chi). \tag{3.25}$$

The dissipation operator,

$$\mathcal{L}_D \approx \epsilon|\rho_2|^{2/3}|\rho_1|^{4/3}\partial_1^2 = 4\epsilon(w^2+4\chi^2)^{2/3} \times [\partial_w^2 + \tfrac{1}{4}\partial_\chi^2 + 2w^{-1}\partial_w\partial_\zeta + w^{-2}\zeta^2\partial_\zeta^2], \tag{3.26}$$

must be balanced against the leading terms of $\mathcal{L}_0$ [see Eq. (3.1b)] yielding

$$\left\{\epsilon(w^2+4\chi^2)^{2/3}(\partial_w^2 + \tfrac{1}{4}\partial_\chi^2) + \left[w^2\left(\partial_w^2 + \frac{1}{4}\ \partial_\chi^2\right) + \frac{i}{2}\ w\partial_\chi\right] + \left[\frac{\lambda}{2}\left(\frac{\lambda}{2}+1\right) - \frac{3}{4}\right]\right\} u(w,\chi) = 0. \tag{3.27}$$

The last term is due to the action of the $\rho_2\partial_2$ part of $\mathcal{L}_0$ on $\Psi^{(\lambda)}(\vec{\rho}_1,\vec{\rho}_2)$. Note that all of the terms can be balanced by rescaling $w = \epsilon^{3/2}\overline{w}, \chi = \epsilon^{3/2}\overline{\chi}$ although according to the $\lambda = 1 + \epsilon\delta$ assumption, the last term remains small $o(\epsilon)$.

Now, we observe that provided $w^2 \ll \epsilon[w^2 + (2\chi)^2]^{2/3}$ the second term in Eq. (3.27) can be neglected compared to the first and the third, so that the ''inner'' series solution balancing the latter two terms is valid in the narrow strip along $w = 0$ (see Fig. 1). We are interested in the solution that goes to a constant at $w=\chi=0$ and is locally s wave. Hence,

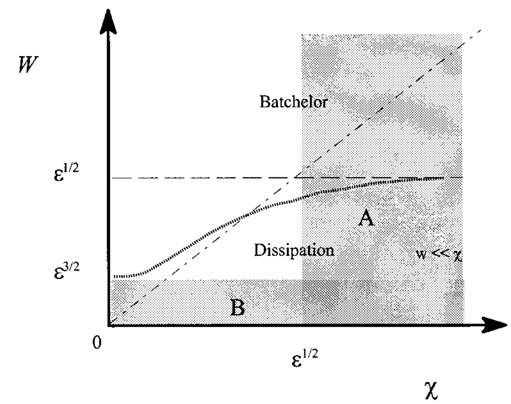

FIG. 1. A schematic drawing of different asymptotic domains. The dotted curve separates the region of $\mathcal{L}_0$ dominance (above) from the region of the $\epsilon\mathcal{L}_D$ dominance (below). The crossover occurs at $w \sim o(\epsilon^{1/2})$ for $\chi > \epsilon^{1/2}$ and at $w \sim o(\epsilon^{3/2})$ for $\chi < \epsilon^{1/2}$. The perturbative crossover solution holds in the shaded region A, which overlaps the domain of validity of the local expansion B about the w, $\chi=0$ singular point.

$$u(w,\chi) = 1 - \frac{9}{4}\ \frac{\lambda-1}{\epsilon}\ (\lambda+3)(w^2+4\chi^2)^{1/3} + \cdots\ , \tag{3.28}$$

which is valid in the domain extending to $\epsilon^{1/2} \ll \chi \ll 1$, $w \ll \chi$. Equation (3.28) must be compared with (3.23) but the two can only be reconciled by setting $\delta=0$.

Thus we conclude [24] that $\delta = 0 + o(\epsilon^{1/2})$, which is consistent with the numerical solution by Pumir [25]. Curiously, even though there is no correction to the exponent in the leading order in ϵ, Eq. (3.22) with $\delta=0$ leads to a nontrivial $a(\chi)$ and hence a nontrivial superposition of the degenerate q modes. In the limit $\epsilon \to 0$, singular perturbation *selects a particular superposition* of degenerate Batchelor modes.

The physical meaning of $a(\chi)$ is evident from Eqs. (3.20) and (3.21): it controls the behavior of the correlator with three points on one line and determines the superposition of the q modes away from collinearity. The solution of Eq. (3.22) for $\delta=0$ that satisfies the symmetry conditions is

$$a(\xi) = a_0\ \sin\left(\frac{\pi}{6} - |\xi|\right), \tag{3.29}$$

which has an apparent $|\xi|$ singularity at the origin. The configuration dependence of the correlator away from collinearity is found from Eq. (3.20) either as a sum over q modes given by Eq. (C3) or via integral representation (C1).

C. Flatness and higher order functions for the L model

It is not too difficult to apply the matched asymptotics perturbation theory developed in Sec. III A to the computation of the higher order multipoint functions for the Laplacian model. Let us consider even $n+1$ order correlators in $d=2$. The difference with the skewness calculation will be that we must start with the general n form of $\mathcal{L}_0+\mathcal{L}_D$ and study the $l=0$ angular momentum sector for which as we saw in Eq. (2.18) the unperturbed value of λ is equal to 0. As we shall see the matching will require $\lambda\sim\mathcal{O}(\sqrt{\epsilon})$.

As with the eigenfunctions of the Batchelor-Kraichnan operator in Eq. (2.9) we seek a solution in the form $\zeta^{\lambda/2}\Phi_\lambda(w,\hat{\chi})$, where the scaling dimension is carried by the determinant ζ and the arguments are scale invariant and depend only on the configuration of the $n+1$ polygon. The variables $w,\hat{\chi}$ are defined in Appendix A in terms of the Euler factorization of ρ_i^a. The operator $\mathcal{L}_D$ in these coordinates, acting on Ψ analogous to Eq. (3.1) has the form (see Appendix A, Eq. (A12), but note that here $n>d$)

$$\begin{aligned}\frac{1}{4\epsilon}\mathcal{L}_D=&(1-w^2)\partial_w^2+[\lambda-(2+\lambda)w^2]w^{-1}\partial_w\\&+\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right)w^{-2}-\frac{1}{4}\frac{1}{1-w^2}A_{12}^2\\&+(n-2)\left[(w^{-2}-1)w\partial_w-\frac{\lambda}{2}w^{-2}\right]\\&-\frac{1}{8}w^{-2}\sum_{\alpha=3}^{n}[A_{1,\alpha}^2+A_{2,\alpha}^2+\sqrt{1-w^2}(A_{1,\alpha}^2-A_{2,\alpha}^2)],\end{aligned}\tag{3.30}$$

where operators $A_{\alpha,\beta}$ rotating the pseudo-space basis $\hat{\chi}$ are defined in Appendix A, Eq. (A11). This differs from Eq. (3.1) (the $n=d=2$ case) in extra terms on the last two lines (provided we identify $A_{12}=i\partial_{\phi_{12}}$ as the derivative with respect to the angle of rotation in the $\hat{\chi}_{1,2}$ plane).

Let us consider only the leading part of $\mathcal{L}_D$ in the $w\to0$ limit, which scales as w^{-2}. According to the method presented in Sec. III A, this singular part of the perturbation operator will combine with the leading, $\mathcal{O}(1)$, part of the $\mathcal{L}_0$ to define the crossover equation [i.e., the analogue of Eq. (3.4)] in the scaled variable $z=w/\sqrt{\epsilon}$. Only a few of the terms in Eq. (3.30) survive:

$$\begin{aligned}\mathcal{L}_{\rm co}=&w^2\partial_w^2+4\epsilon\Bigg[\partial_w^2+(\lambda+n-2)w^{-1}\partial_w\\&+\frac{\lambda}{2}\left(\frac{\lambda}{2}+n-3\right)w^{-2}-\frac{1}{4}w^{-2}\sum_{\alpha=3}^{n}A_{1,\alpha}^2\Bigg].\end{aligned}\tag{3.31}$$

The $w\to0$ asymptotics of the solution to Eq. (3.31) is $w^{-\lambda/2\pm Q}U_Q(\hat{\chi})$. Let us seek the solution in the form

$$\Phi_\lambda(w,\hat{\chi})=w^{-\lambda/2}\sum_Q b_QU_Q(\hat{\chi})\varphi_Q(z),\tag{3.32}$$

where $U_Q(\hat{\chi})$ is the eigenfunction of $\Sigma_{\alpha=3}^{n}A_{1,\alpha}^2$ operator with the eigenvalue $-Q^2$. The $w^{-\lambda/2}$ divergence in Eq. (3.32) will be compensated by the determinant factor $\zeta^{\lambda/2}$ and the full solution will be well behaved at collinearity as long as $\varphi_Q(0)$ is bounded. We can rewrite Eq. (3.31) as

$$\begin{aligned}&\left[(1+z^2)\partial_z^2+(n-2)z^{-1}\partial_z-\frac{Q^2}{4}z^{-2}\right]\varphi_Q(z)\\&=-\lambda\left(\frac{2+\lambda}{4}-z\partial_z\right)\varphi_Q(z).\end{aligned}\tag{3.33}$$

Next we set seemingly arbitrarily at this point $\lambda=\sqrt{\epsilon}\delta$ with δ yet to be determined and keep only the leading term in ϵ. The operator on the left hand side of Eq. (3.33) is related to the hypergeometric equation and can be inverted. However, we shall only need an explicit solution for the $Q=0$ mode, which has the form

$$\begin{aligned}\varphi_0(z)=&1+c_2\int_0^z dz z^{2-n}(1+z^2)^{(n-2)/2}\\&-\sqrt{\epsilon}\frac{\delta}{2}\int_0^z dz z^{2-n}(1+z^2)^{(n-2)/2}\\&\times\int_0^z dz' z'^{n-2}(1+z'^2)^{n/2}+\mathcal{O}(\epsilon).\end{aligned}\tag{3.34}$$

The c_2 must be set to 0 to prevent divergence at $z=0$. In the matching region, $1\ll z\ll\epsilon^{-1/2}$ expression (3.34) reduces to

$$\varphi_0(z)=1-C_0z=1-\sqrt{\epsilon}\frac{\delta}{2}z\int_0^\infty dz'\frac{z'^{n-2}}{(1+z'^2)^{n/2}},\tag{3.35}$$

which must be compared to the asymptotic behavior of the Batchelor-Kraichnan eigenfunctions in the $w\ll1$ limit. The latter are Legendre-Jacobi functions with $\nu=0+\mathcal{O}(\sqrt{\epsilon})$ [see Eq. (C1)], which behave like

$$\mathcal{P}_0^{q,0}(w^{-1})\approx1-\frac{|q|}{2}w+\cdots,\tag{3.36}$$

where q is the eigenvalue of A_{12} and is not to be confused with Q^2 the eigenvalue $\Sigma_3^nA_{1\alpha}^2$; the two operators do not commute. The actual solution is a superposition of the q modes and the matching requires

$$\begin{aligned}&\sum_q a_q(\hat{\chi}_3,\ldots,\hat{\chi}_n)e^{iq\phi_{12}}\mathcal{P}_0^{q,0}(w^{-1})\\&=\sum_Q b_QU_Q(\hat{\chi}_\alpha)\varphi_Q(w/\sqrt{\epsilon}).\end{aligned}\tag{3.37}$$

It can be shown from Eq. (3.33) that $\varphi_Q(z)=1-C_Qz$ for large z; we have seen this explicitly for $Q=0$ in Eq. (3.35). [For $Q\neq0$ the C_Q constants can be computed from Eq. (3.33) with $\lambda=0$ via the hypergeometric function.] Hence we match separately the constant and the linear $z=w/\sqrt{\epsilon}$ terms,

$$\sum_q a_q(\hat{\chi}_3,\ldots,\hat{\chi}_n)e^{iq\phi_{12}}=\sum_Q b_QU_Q(\hat{\chi}_\alpha),\tag{3.38a}$$

$$\sum_q \frac{|q|}{2} a_q(\hat{\chi}_3,\ldots,\hat{\chi}_n)e^{iq\phi_{12}}=\frac{1}{2\sqrt{\epsilon}}\sum_{Q\neq 0} C_Q b_Q U_Q(\hat{\chi}_\alpha) + \delta \frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)} b_0 U_0(\hat{\chi}_\alpha), \tag{3.38b}$$

where on the right hand side we have separated out the $Q=0$ contribution and substituted the value of C_0 computed in Eq. (3.35). Unlike C_0, which is $\mathcal{O}(\sqrt{\epsilon})$, other C_Q turn out to be nonzero in the $\epsilon\rightarrow 0$ limit. Hence, to compensate for the $\sqrt{\epsilon}$ factor in the denominator on the right hand side of (3.38b) $b_Q\sim\mathcal{O}(\sqrt{\epsilon})$ and only $b_0=1$. This explains our choice of $\lambda\sim\mathcal{O}(\epsilon^{1/2})$: it was necessary since without it Eqs. (3.34) and (3.36) could not be matched.

Let us now determine δ from Eq. (3.38). Since all $Q\neq 0$ are higher order in $\epsilon^{1/2}$, the eigenvalue δ can be determined by projection onto the $Q=0$ mode. Borrowing Dirac's notation we rewrite Eq. (3.38) in the form

$$\sum_q a_q|p,q\rangle = b_0|p,Q=0\rangle+\mathcal{O}(\sqrt{\epsilon}), \tag{3.39a}$$

$$\langle p,Q=0|\left(\sum_q \frac{|q|}{2} a_q|p,q\rangle\right)=\delta\frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)} b_0. \tag{3.39b}$$

Because the perturbation operator $\mathcal{L}_D$ respects SO(n) rotation symmetry the total angular momentum, p [defined by $\frac{1}{2}\Sigma_{\alpha,\beta}A^2_{\alpha,\beta}=p(p+n-2)$] remains a good quantum number and can be used to label both q and Q^2 eigenstates. These eigenstates have the form of pth order harmonic polynomials in $\hat{\chi}_\alpha\cdot\hat{e}$ with an arbitrary pseudospace unit vector $\hat{e}$.

Since a_q can be found from Eq. (3.39a) by projection onto $|p,q\rangle$ states the computation of δ reduces to Clebsh-Gordon calculations. The calculation is particularly simple for the case of flatness, $n=3$, where q,Q are simply the eigenvalues of rotations A_{12} and A_{13} about the axis $\hat{\chi}_3$ and $\hat{\chi}_2$, respectively. The fourfold permutation symmetry of the physical correlator implies that the nontrivial mode of the lowest rank is $p=4$. We have $\langle p=4,q=4|p=4,Q=0\rangle=\sqrt{70}/16$ and $\langle p=4,q=2|p=4,Q=0\rangle=-\sqrt{5}/4\sqrt{2}$; the only nonzero overlaps, which contribute to Eq. (3.39b). Evaluating Eqs. (3.39) we conclude that the flatness exponent is

$$\lambda_4=\frac{45}{32}\epsilon^{1/2}+\mathcal{O}(\epsilon), \tag{3.40}$$

which agrees well with the direct numerical solution of $\mathcal{L}_0+\mathcal{L}_D=0$ found by the ''shooting'' method which does not involve perturbation theory in ϵ. The calculation of the scaling exponents for higher n is continued in Appendix E.

CONCLUSIONS

In the preceeding sections we have presented the singular perturbation theory tools for calculating low order multipoint correlators of the passive scalar near Batchelor limit. The upshot of the analysis was the calculation of the 3d order function for a phenomenological model (the K model), which appears sufficiently realistic to merit detailed comparison with the experiment. The experimental data for the scalar skewness [26] indicates that its exponent is very close to 1, which supports our argument that the passive scalar advected by the turbulent flow is described by a Hopf equation close to the Batchelor limit. Furthermore, the calculation of the full configuration dependence of the three-point correlator should allow a detailed test of the model even if the exponent is used to fix the unknown parameter of the model.

Many open questions remain. It would be interesting to calculate the four-point function for the K model in perturbation theory. Although the calculation for the L model indicates that the scaling exponent is $o(\epsilon^{1/2})$ the complete matching analysis of the K model is considerably more difficult (than the three-point case) because of the more complex structure of the singular manifold. There are also more fundamental issues. The n-point correlators are just the eigenvectors of the Hopf operator and are thus dependent on the details of the model. Are there more universal aspects of the problem? Perhaps the asymptotic large deviation behavior of the probability distribution function or the behavior near the center of the distribution? In addition to looking for less model-dependent objects one should seek to improve the effective Hopf models, perhaps by making them more properly hierarchical. Hopefully the contact with experiment will help direct further efforts in this subject.

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APPENDIX A

Euler parametrization

Here we describe the properties of the Euler parameterization of $\vec{\rho}_i$ which is defined by singular value decomposition. Consider the simplest case, $n=d=2$, first with

$$\rho_i^a\equiv\sum_\alpha \mathbf{R}_{\alpha,i}(\chi)\xi_\alpha\mathbf{R}_{\alpha,a}(\phi), \tag{A1}$$

$$\begin{bmatrix}\rho_1^x & \rho_2^x\\ \rho_1^y & \rho_2^y\end{bmatrix}=\begin{bmatrix}\xi_1C_\phi C_\chi+\xi_2S_\phi S_\chi & \xi_1C_\phi S_\chi-\xi_2S_\phi C_\chi\\ \xi_1S_\phi C_\chi-\xi_2C_\phi S_\chi & \xi_1S_\phi S_\chi+\xi_2C_\phi C_\chi\end{bmatrix} \tag{A2}$$

with the notation $C_\phi\equiv\cos\phi$ and $S_\phi\equiv\sin\phi$. For later convenience, signs are chosen in the rotation matrices so that $\mathbf{R}_{\alpha,a}(\phi)n_a(\theta)=n_\alpha(\theta-\phi)$. The triangle area $\zeta\equiv\vec{\rho}_1\wedge\vec{\rho}_2=\det\rho=\xi_1\xi_2$. The radius of gyration $R^2=\Sigma_{i=1}^2\vec{\rho}_i^2=\mathrm{Tr}\,\rho\rho^T=\xi_1^2+\xi_2^2$. The rotation matrix $\mathbf{R}_{\alpha,a}(\phi)$ describes the spatial orientation of the $(\vec{r}_1,\vec{r}_2,\vec{r}_3)$ triangle.

The pseudospace rotation $\mathbf{R}_{\alpha,i}(\chi)$ can be determined by diagonalizing a matrix of spatial invariants. Note that $\chi\to\chi+\pi$ (or $\phi\to\phi+\pi$) is equivalent to $\xi_1\to-\xi_1$ and $\xi_2\to-\xi_2$ while $\xi_1\leftrightarrow\xi_2$ interchange is equivalent to $\phi\to\phi+(\pi/2)$, $\chi\to\chi-(\pi/2)$. Therefore by an appropriate definition of χ,ϕ we can restrict to $0\leqslant|\xi_1|<\xi_2$. Also note that simultaneous $\chi\to\chi+\pi$ and $\phi\to\phi+\pi$ leaves ρ invariant, which means that the Euler representation is double valued. The coordinates $\xi_\alpha,\mathbf{R}(\chi)$ can be determined by diagonalizing

$$\rho\rho^T=\begin{bmatrix}\vec{\rho}_1^2 & \vec{\rho}_1\cdot\vec{\rho}_2\\ \vec{\rho}_1\cdot\vec{\rho}_2 & \vec{\rho}_2^2\end{bmatrix}=\mathbf{R}^T(\chi)\begin{bmatrix}\xi_1^2 & \\ & \xi_2^2\end{bmatrix}\mathbf{R}(\chi) \tag{A3}$$

from which

$$2\vec{\rho}_1\cdot\vec{\rho}_2=(\xi_1^2-\xi_2^2)\sin 2\chi, \tag{A4a}$$

$$\vec{\rho}_1^2-\vec{\rho}_2^2=(\xi_1^2-\xi_2^2)\cos 2\chi, \tag{A4b}$$

which is combined with

$$\vec{\rho}_1^2+\vec{\rho}_2^2=\xi_1^2+\xi_2^2, \tag{A4c}$$

$$\vec{\rho}_1\times\vec{\rho}_2=\xi_1\xi_2 \tag{A4d}$$

for complete determination of ξ_α,χ. Note that the $\xi_1=\xi_2$ point corresponding to $\vec{\rho}_1\cdot\vec{\rho}_2=0$, $\vec{\rho}_1^2=\vec{\rho}_2^2$ is special: in that case $\rho=\xi_1 R(\phi)R(\xi)$ and χ can be absorbed into redefinition of ϕ (alternatively $\phi\to\phi+\Delta$, $\chi\to\chi-\Delta$ is an additional ''gauge'' symmetry at that point). Explicitly, for $\xi_1\neq\xi_2$,

$$\chi=\frac{1}{2}\,tg^{-1}\,\frac{2\vec{\rho}_1\cdot\vec{\rho}_2}{\vec{\rho}_1^2-\vec{\rho}_2^2}. \tag{A5}$$

To relate these variables directly to the triangle configuration we express the r_{ij} distances:

$$r_{ij}^2=(\xi_1^2+\xi_2^2)\left[1+\frac{\xi_1^2-\xi_2^2}{\xi_1^2+\xi_2^2}\cos 2(\chi+\Delta_{ij})\right] \tag{A6}$$

with $\Delta_{12}=0$, $\Delta_{23}=-(2\pi/3)$, $\Delta_{31}=+(2\pi/3)$. The mapping of $\chi\epsilon[0,2\pi]$ into triangle configuration space is two to one.

For collinear configurations $\xi_1=0$ and we have

$$\frac{r_{12}}{r_{23}}=\frac{1-\cos 2\chi}{1-\cos 2\left(\chi-\dfrac{2\pi}{3}\right)}. \tag{A7}$$

The values $\chi=0$, $-(2\pi/3)$, $+(2\pi/3)$ (and those translated by π) correspond to $r_{12}=0$, or $r_{23}=0$, respectively. Quite generally, as is evident from Eq. (A6) the permutation of r_i points corresponds to the translation $\chi\to\chi+(2\pi/3)$. Interchange of r_1 and r_2 corresponds to $(\xi_1,\chi)\to-(\xi_1,\chi)$, leaving (ξ_2,ϕ) invariant.

More generally for $n\neq d$ (e.g., $n<d$) we can write

$$\rho_{ai}=\sum_{\alpha=1}^{n}\chi_i^\alpha\xi_\alpha\eta_a^\alpha, \tag{A8}$$

where $i=1,...n$ and $a=1,...,d$. The magnitude of the diagonal elements ξ_α^2 are defined as the eigenvalues of the $n\times n$ matrix $\rho^T\rho$, i.e., where we contracted on the real space index a. In Eq. (A8) χ is a square orthonormal matrix and $\vec{\eta}_\alpha$ are orthonormal vectors. Clearly Eqs. (A5), (A6), and (A7) apply for general d. We can describe the case $d<n$ by interchanging the pseudospace and real space labels.

Let us give explicit expressions for various differential operators in Euler variables:

$$\begin{aligned}\partial_{ai}=&\sum_{\alpha=1}^{n}\chi_i^\alpha\eta_a^\alpha\frac{\partial}{\partial\xi_\alpha}+\sum_{\alpha,\beta,k=1}^{n}(\xi_\alpha^2-\xi_\beta^2)^{-1}(\chi_i^\beta\eta_a^\alpha\xi_\alpha\\&+\chi_i^\alpha\eta_a^\beta\xi_\beta)\left(\chi_k^\beta\frac{\partial}{\partial\chi_k^\alpha}\right)+\sum_{b=1}^{d}\sum_{\alpha,\beta=1}^{n}(\xi_\alpha^2-\xi_\beta^2)^{-1}\\&\times(\chi_i^\alpha\eta_a^\alpha\xi_\alpha+\chi_i^\beta\eta_a^\alpha\xi_\beta)\left(\eta_b^\beta\frac{\partial}{\partial\eta_b^\alpha}\right)\\&+\sum_{\alpha=1}^{n}\xi_\alpha^{-1}\chi_i^\alpha\left(\delta_{ab}-\sum_{\beta=1}^{n}\eta_a^\beta\eta_b^\beta\right)\frac{\partial}{\partial\eta_b^\alpha}\end{aligned} \tag{A9}$$

where the last term contributes only when $d>n$. (The case of $n>d$ is obtained by interchanging χ and η matrices.)

The G^2 operator is

$$\begin{aligned}G^2=&-\frac{1}{2n}\left(\sum_{\alpha=1}^{n}\xi_\alpha\frac{\partial}{\partial\xi_\alpha}\right)^2+\frac{1}{2}\sum_{\alpha}^{n}\left(\xi_\alpha\frac{\partial}{\partial\xi_\alpha}\right)^2\\&+\frac{1}{2}\sum_{\alpha\neq\beta}\left(\xi_\alpha\frac{\partial}{\partial\xi_\alpha}-\xi_\beta\frac{\partial}{\partial\xi_\beta}\right)^2-\frac{1}{2}\sum_{\alpha\neq\beta}\frac{\xi_\alpha^2\xi_\beta^2}{(\xi_\alpha^2-\xi_\beta^2)^2}\\&\times\left[A_{\alpha\beta}^2+B_{\alpha\beta}^2-\frac{\xi_\alpha^2+\xi_\beta^2}{\xi_\alpha\xi_\beta}A_{\alpha\beta}B_{\alpha\beta}\right],\end{aligned} \tag{A10}$$

where

$$A_{\alpha\beta}\equiv i\sum_{i=1}^{n}\left(\chi_i^\alpha\frac{\partial}{\partial\chi_i^\beta}-\chi_i^\beta\frac{\partial}{\partial\chi_i^\alpha}\right) \tag{A11a}$$

and

$$B_{\alpha\beta}\equiv i\sum_{a=1}^{d}\left(\eta_a^\alpha\frac{\partial}{\partial\eta_a^\beta}-\eta_\alpha^\beta\frac{\partial}{\partial\eta_a^\alpha}\right) \tag{A11b}$$

are rotation operators.

It is straightforward but tedious to check that for $d=n=2$. Equation (A10) is just the Legendre operator in a reduced variable $\xi\equiv(\xi_1^2+\xi_2^2)/2\xi_1^2\xi_2^2$ and the angles χ and ϕ:

$$G^2=\left[\partial_\xi(\xi^2-1)\partial_\xi+\frac{\partial_\chi^2+\partial_\phi^2-2\xi\partial_\chi\partial_\phi}{4(\xi^2-1)}\right], \tag{A12}$$

the eigenfunctions of which are the Legendre-Jacobi functions

$$G^2[e^{iq\chi+il\phi}\mathcal{P}_\nu^{q,l}(\xi)]=\nu(\nu+1)[e^{iq\chi+il\phi}\mathcal{P}_\nu^{q,l}(\xi)]. \tag{A13}$$

Equation (2.7) provides an integral representation of these eigenfunctions as shown in Appendix B.

For the $n\times d$ Laplacian width $d\geqslant n$,

$$\sum_{\gamma=1}^{n}\partial_{a\gamma}^{2}=\sum_{\alpha=1}^{n}\frac{\partial^{2}}{\partial\xi_{\alpha}^{2}}+\frac{1}{2}\sum_{\alpha\neq\beta}^{n}\frac{\xi_{\alpha}^{2}+\xi_{\beta}^{2}}{(\xi_{\alpha}^{2}-\xi_{\beta}^{2})^{2}}\times\left[A_{\alpha\beta}^{2}+B_{\alpha\beta}^{2}+\frac{4\xi_{\alpha}\xi_{\beta}}{\xi_{\alpha}^{2}+\xi_{\beta}^{2}}A_{\alpha\beta}B_{\alpha\beta}\right]+\sum_{\alpha\neq\beta}^{n}\left[\frac{2\xi_{\alpha}}{\xi_{\alpha}^{2}-\xi_{\beta}^{2}}+(d-n)\xi_{\alpha}^{-1}\right]\frac{\partial}{\partial\xi_{\alpha}}+\sum_{\alpha=1}^{n}\xi_{\alpha}^{-2}\left(\delta_{ab}-\sum_{\beta=1}^{n}\eta_{a}^{\beta}\eta_{b}^{\beta}\right)\frac{\partial^{2}}{\partial\eta_{a}^{\alpha}\partial\eta_{b}^{\alpha}}-(d-n)\sum_{\alpha=1}^{n}\xi_{\alpha}^{-2}\eta_{a}^{\alpha}\frac{\partial}{\partial\eta_{a}^{\alpha}},\tag{A14}$$

where all repeated a,b indices are summed from 1 to d. For $n=d=2$ this can be rewritten in terms of ξ and $\zeta\equiv\det\rho$:

$$\mathcal{L}_{D}=4\epsilon\left[\partial_{\xi}(\xi^{2}-1)\partial_{\xi}+2(1-\xi^{-2})\xi\partial_{\xi}\partial_{\zeta}+\xi^{2}(\partial_{\zeta}^{2}-\partial_{\zeta})+\frac{1}{4}\frac{\xi^{2}}{\xi^{2}-1}(\partial_{\chi}^{2}+\partial_{\phi}^{2}+2\xi^{-1}\partial_{\chi}\partial_{\phi})\right].\tag{A15}$$

APPENDIX B

Integral representation of the eigenfunctions

Here we evaluate the eigenfunctions introduced in Eq. (2.7) explicitly writing ρ as in Eq. (A1), i.e., $\hat{n}\rho\hat{e}=(\mathbf{R}_{\chi}\hat{n}_{\psi})^{T}\Xi(\mathbf{R}_{\phi}\hat{n}_{\theta})$ where Ξ is a diagonal matrix with eigenvalues ξ_{α}. After substituting into the integral in Eq. (2.7) the $\mathbf{R}_{\phi}$ and $\mathbf{R}_{\chi}$ rotation matrices can be absorbed by $\psi\rightarrow\psi+\chi$ and $\theta\rightarrow\theta+\phi$ leading to

$$T_{\nu}^{ql}=(\xi_{1}\xi_{2})^{-\nu}e^{iq\chi+il\phi}\times\int_{0}^{2\pi}\frac{d\psi}{2\pi}\int_{0}^{2\pi}\frac{d\theta}{2\pi}e^{il\theta+iq\psi}h_{\nu,q}\times[\xi_{1}\cos\theta\cos\psi+\xi_{2}\sin\theta\sin\psi]$$
$$=C_{q}^{(\nu)}e^{iq\chi+il\phi}\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\pi}e^{il\theta+iqtg^{-1}[(\xi_{2}/\xi_{1})tg\theta]}\left[\frac{\xi_{1}}{\xi_{2}}\cos^{2}\theta+\frac{\xi_{2}}{\xi_{1}}\sin^{2}\theta\right]^{\nu},\tag{B1}$$

where the second line is obtained by shifting and integrating over ψ and using $h_{\nu,q}(x)=[\mathrm{sgn}(x)]^{q}|x|^{2\nu}$. Note that $\theta\rightarrow\theta+\pi$ or $\psi\rightarrow\psi+\pi$ change the sign of the argument of h. Hence, q,l must have the same parity for the integral to be nonzero. This parity is used in line two of Eq. (2.10) to half the domain of integration. The multiplicative constant is

$$C_{q}^{(\nu)}\equiv\int_{-\pi/2}^{\pi/2}\frac{d\psi}{\pi}\cos q\psi\cos^{2\nu}\psi=\frac{\sin\left[\pi\left(\frac{|q|}{2}-\nu\right)\right]\Gamma(1+2\nu)\Gamma\left(\frac{|q|}{2}-\nu\right)}{2^{2\nu}\Gamma\left(\frac{|q|}{2}+1+\nu\right)}.\tag{B2}$$

The integral appearing on the second line of Eq. (B1) defines the Legendre-Jacobi function $\mathcal{P}_{\nu}^{q,l}(\xi)$ with $\xi\equiv(\xi_{1}^{2}+\xi_{2}^{2})/2\xi_{1}\xi_{2}$. Note that its constant prefactor $C_{q}^{(\nu)}$ actually vanishes for integer and half-integer values of ν. To define the $\mathcal{P}$ function via the double integral representation appearing on the first line of Eq. (B1) in that case requires dividing by $C_{q}^{(\nu)}$ and taking a careful limit that effectively introduces a logarithm into the h function in the integrand.

Note that for near collinear configurations $\xi_{1}\xi_{2}=\det\rho\rightarrow0$ (e.g., $\xi_{1}\rightarrow0$ while $\xi_{2}=\mathrm{const}$). From Eq. (B1) it follows that in that limit $T_{\nu}^{q,l}\sim|\det\rho|^{-\nu}$.

The eigenfunction of $\mathcal{L}_{0}$ given in Eq. (2.7) in Euler variables becomes

$$\Psi_{\nu,q,l}^{\lambda}(\rho)=\zeta^{\lambda/2}e^{iq\chi+il\phi}\mathcal{P}_{\nu}^{q,l}(\xi).\tag{B3}$$

Similar manipulations allow us to reexpress the $N=3,d=3$ eigenfunction given in Eq. (2.18) in the Euler coordinates: $\rho_{i}^{a}=\Sigma_{\alpha=1}^{2}R_{\alpha,i}(\chi)\xi_{\alpha}\eta_{\alpha}^{a}$. Orthonormal vectors $\hat{\eta}_{1,2}$ span the plane of $\vec{\rho}_{1,2}$ while the third vector of Cartesian triad $\hat{\eta}_{3}$ is parallel to $\vec{\rho}_{1}\times\vec{\rho}_{2}$. Matrices $R_{\alpha,i}$ and η_{α}^{a} can be rotated away by shifting ψ and θ in the integral (see Eq. 2.18) yielding

$$\Psi_{\nu;q;l,m,m'}^{\lambda}(\rho)=|\vec{\rho}_{1}\times\vec{\rho}_{2}|^{\lambda/2}e^{iq\chi}D_{mm'}^{l}(\hat{\eta})\mathcal{P}_{\nu}^{qm'}(\xi),\tag{B4}$$

where m' is not summed.

APPENDIX C

Asymptotic behavior of Legendre-Jacobi functions

To match the zero mode between the regimes where the dissipation dominates, $w\sim0$ and $w\leqslant1$ (Sec. III) required the large $\xi=w^{-1}$ limit of $\mathcal{P}_{1/2}^{q,1}(\xi)$ (q odd) for the skewness and $\mathcal{P}_{0}^{q,0}(\xi)$ (q even) for the flatness. Using

$$\mathcal{P}_{\nu}^{q,l}=\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\pi}e^{il\theta+iqtg^{-1}[(\xi_{2}/\xi_{1})tg\theta]}\times\left[\frac{\xi_{1}}{\xi_{2}}\cos^{2}\theta+\frac{\xi_{2}}{\xi_{1}}\sin^{2}\theta\right]^{\nu}\tag{C1}$$

one finds the required asymptotic expressions:

$$\mathcal{P}_{0}^{0,2n}(\xi)=\left(\frac{\xi-1}{\xi+1}\right)^{|n|/2}\approx\left(1-|n|\xi^{-1}+\frac{n^{2}}{2}\xi^{-2}\right),\tag{C2}$$

$$\mathcal{P}_{1/2}^{2n+1,1}(\xi)=\sqrt{\xi+1}\left(\frac{\xi-1}{\xi+1}\right)^{n/2}\left(1+\frac{2n}{\xi+1}\right)$$
$$\approx \xi^{1/2}+(n+\tfrac{1}{2})\xi^{-1/2}$$
$$-\tfrac{1}{2}(3n^2+3n+\tfrac{1}{4})\xi^{-3/2}+\cdots, \quad \text{(C3)}$$

$$\mathcal{P}_{1/2}^{-2n-1,1}(\xi)=\sqrt{\xi-1}\left(\frac{\xi+1}{\xi+1}\right)^{n/2}$$
$$\approx \xi^{1/2}-(n+\tfrac{1}{2})\xi^{-1/2}$$
$$+\tfrac{1}{2}(n^2+n-\tfrac{1}{4})\xi^{-3/2}+\cdots. \quad \text{(C4)}$$

APPENDIX D

Skewness for the L model in $d=3$

For the skewness in $d=3$ the general expressions for G^2, Eq. (A10) and ∇^2, Eq. (A9) reduces to the following:

$$e^{-iq\chi}(\zeta)^{-\lambda/2}G^2\Psi=w^2\partial_w(1-w^2)\partial_w\gamma+\frac{1}{4}\frac{2wq\sigma_x-(q^2+\sigma_x^2)w^2}{(1-w^2)}\gamma, \quad \text{(D1)}$$

where $w\equiv\xi^-1$ and where $-B_{1,2}=\sigma_x$, $\gamma=e^{-iq\chi}\zeta^{\lambda/2}\Psi$. Also

$$e^{-iq\chi}(\zeta)^{-\lambda/2}\frac{R_g^2}{4}\nabla_\rho^2\Psi$$
$$=\Big[\partial_w(1-w^2)\partial_w+(1+\lambda)(1-w^2)w^{-1}\partial_w+\tfrac{1}{4}\lambda^2/w^2$$
$$+\tfrac{1}{4}(2qw\sigma_x-q^2-\sigma_x^2)/(1-w^2)$$
$$+\tfrac{1}{2}w^{-1}\left(\frac{\xi_2}{\xi_1}(\hat{\eta}_3\cdot\vec{\partial}_1)^2+\frac{\xi_1}{\xi_2}(\hat{\eta}_3\cdot\vec{\partial}_2)^2\right)\Big]\gamma, \quad \text{(D2)}$$

where $\hat{\eta}_3$ is understood as the cross product of $\hat{\eta}_1$ and $\hat{\eta}_2$ and differentiated accordingly. Our convention $|\xi_1|\leqslant\xi_2$ enters in the choice of root taken when we reexpress ξ_1/ξ_2 in terms of w.

For $l=1$ we superimpose functions of the form (2.18) that are linear in $\vec{\eta}_i$. However, anything linear in η_3 is ruled out because the skewness is even under reflections in a plane containing the external gradient. Thus the Hopf operator $\mathcal{L}_0+\mathcal{L}_\mathcal{D}$ reduces to a pair of second-order differential equations in w and it is convenient computationally to write $\gamma=\gamma_1(w)\eta_1+i\gamma_2(w)\eta_2$ in which case σ_x becomes the conventionally defined Pauli matrix acting on (γ_1,γ_2). With these conventions, γ behaves in the Batchelor limit as $w^{\lambda/2}\gamma_q^0=[\tfrac{1}{2}|q|w+\tfrac{1}{4}(1-q^2)w^2,(1-\tfrac{1}{8}q^2w^2)\mathrm{sgn}(q)]/(q^2-1)$.

The crossover equation becomes for $\varphi=\gamma/w^{\lambda/2}$:

$$(w^2\partial_w^2-w\partial_w+\tfrac{1}{4}w^2\partial_\chi^2-\tfrac{1}{2}iw\sigma_x\partial_\chi)\varphi$$
$$+\tfrac{2}{3}\alpha f(\chi)\left(\partial_w^2+\frac{1}{w}\partial_w+\frac{1}{4}\partial_\chi^2+\frac{5}{4}-\frac{1}{4}(1-\sigma_z)-\tfrac{1}{2}(1+\sigma_z)/w^2\right)\varphi+\tfrac{5}{3}(\lambda-1)\varphi=0, \quad \text{(D3)}$$

where $f(\chi)=1$ for the L model equals $F(0,\chi)$ [cf. Eq. (3.16) and below] for the K model.

The eigenvalue problem is derived by matching the component of φ that dominates near $w=0$, which is the second. If its amplitude is denoted by $a(\chi)$ then

$$\alpha f(\chi)(\partial_\chi^2+1)a+\tfrac{10}{3}(\lambda-1)a=0, \quad \text{(D4)}$$

when $f=1$, $a=e^{iq\chi}$, which determines the eigenvalue for the L model. The eigenvalue was also checked by solving $\mathcal{L}_0+\mathcal{L}_\mathcal{D}=0$ numerically, for α in the range 0.01–0.5, and fitting $(\lambda-1)/\alpha$ to a polynomial in $\alpha^{1/2}$. For the pseudo-Kolmogorov dissipation, the K model, the $w=0,\chi=0$ boundary condition dictates $(\lambda-1)/\alpha=0+o(\alpha^{1/2})$, by the same argument as in $d=2$, see Sec. III B.

APPENDIX E

Higher order functions for the L model

In the case of general n we observe that A_{12} and $Q^2=\Sigma_{\alpha=3}^n A_{1\alpha}^2$ are invariant under rotations in $n-2$-dimensional space orthogonal to the 12 plane. Below we will explicitly compute δ for the modes that are invariant under such rotations. In this case $Q=0$ condition implies that the eigenfunctions are also singlet under SO$(n-1)$ rotations about $\hat{\chi}_2$, the total angular momentum for which is $Q^2+\Sigma_{\alpha,\beta\geqslant 3}^n A_{\alpha\beta}^2$. This implies that the eigenfunctions are polynomials in $\hat{\chi}_2\cdot\hat{e}$ and $\Sigma_\alpha(\hat{\chi}_\alpha\cdot\hat{e})^2$. To simplify the notation let $\chi_\alpha\equiv\hat{\chi}_\alpha\cdot\hat{e}$ for the remainder of this section. We can choose

$$U_0^{(p)}(\chi)=H[\chi_2^p], \quad \text{(E1)}$$

where $H[\cdots]$ is the Harmonic projection operator which turns the pth order polynomial into an eigenstate of total angular momentum p; its explicit form will not be needed here but can be found in Ref. [20]. One can write

$$(\chi_2)^p=2^{-p}[(\chi_2+i\chi_1)+(\chi_2-i\chi_1)]^p$$
$$=2^{-p}\sum_{k=1}^{p}C_k^p(\chi_2+i\chi_1)^{p-k}(\chi_2-i\chi_1)^k \quad \text{(E2)}$$

and the kth term in the sum corresponds to $q=p-2k$. (Also note that because the projection operator H is linear and rotationally invariant, it commutes with A_{12}.) It follows that

$$a_{p-2k}=2^{-p}C_k^p b_0 \quad \text{(E3)}$$

is correct to $O(\epsilon^{1/2})$. We can now evaluate the left hand side of Eq. (3.39b)

$$2^{-p-1}\sum_{k=0}^{p}|p-2k|C_k^pH[(\chi_2+i\chi_1)^{p-k}(\chi_2-i\chi_1)^k]$$
$$=2^{-p-1}\sum_{k=0}^{p}|p-2k|C_k^p\sum_{m_1=0}^{p-k}C_{m_1}^{p-k}\sum_{m_2=0}^{k}C_{m_2}^k i^{m_1-m_2}$$
$$\times H[(\chi_2)^{p-m_1-m_2}(\chi_1)^{m_1+m_2}] \quad \text{(E4)}$$

and project it onto $Q=0$ SO$(n-1)$ singlet by averaging with respect to all rotations about χ_2. This average is nonzero only for terms with even $m_1+m_2=2l$ and replaces

$\chi_1^{2l} \to \beta_{2l}(\chi_\alpha^2 - \chi_2^2)^l$, where $\beta^{2l} = \Gamma(l+1/2)\Gamma((n-1)/2)/\Gamma(l+(n-1)/2)\Gamma(1/2)$. Now, the terms involving powers of (rotationally invariant) χ_α^2 do not survive the harmonic projection since they have total angular momentum less than p. Hence the required projection onto the $Q=0$ sector is found by reading of the coefficient of the χ_2^p term. This yields the following expression for δ and hence λ for arbitrary even $N=n+1$:

$$\lambda_N^{(p)} = \epsilon^{1/2} \frac{2\Gamma\left(\frac{N-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{N-2}{2}\right)} 2^{-p} \sum_{k=1}^{[p/2]} (p-2k) C_k^p \times \sum_{m_1=0}^{p-k} C_{m_1}^{p-k} \sum_{m_2=0}^{k} C_{m_2}^{k} (-1)^{m_1} \beta_{m_1+m_2}. \quad \text{(E5)}$$

As before in the case of flatness, the permutation symmetry of the $N=n+1$ points implies that the lowest nontrivial mode has $p=N$. Evaluating Eq. (E5) for $N=4$ we recover the result for the flatness (3.40). For higher N we find

$$\lambda_6 = 2.31\epsilon^{1/2} + \mathcal{O}(\epsilon), \quad \text{(E6a)}$$

$$\lambda_8 = 3.31\epsilon^{1/2} + \mathcal{O}(\epsilon). \quad \text{(E6b)}$$

Numerical evaluation of Eq. (E5) for large N yields an approximate expression: $\lambda_N \approx (-0.39 + 0.45N)\epsilon^{1/2}$. However, this perturbative result is only expected to hold for $\epsilon^{1/2}N \ll 1$. Finally, we note that an analogous calculation can be carried out for the odd moments.

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[21] The dissipation operator could be made Hermitian by adding the term $2\epsilon\Lambda$. Although this would appear in the leading order calculation, it is nonsingular and represents but a trivial modification of the calculation. Since there is no physical requirement of Hermiticity for the evolution operator we chose to work with the marginally simpler non-Hermitian form.
[22] It is natural to require the solution to be smooth along $w=0$ as long as no two points in the configuration do not coincide. The smoothness for noncoincident configurations can be satisfied even in the absence of molecular diffusion. Indeed, since the molecular diffusion operator is $\tau(R)\kappa\Sigma_i\vec{\partial}_i^2$ it is small compared to L_D as long as $\kappa < \epsilon R^{4/3}$.
[23] The matching of the $\mathcal{O}(\epsilon l n w)$ is not required to the leading order, but can be done by letting $\nu = \frac{1}{2} + \mathcal{O}(\epsilon)$.
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