Model for anisotropic directed percolation

V. Lien Nguyen
Institute of Physics, P.O. Box 429, Bo Ho, Hanoi 1000, Vietnam

Enrique Canessa*
The Abdus Salam International Centre for Theoretical Physics, P.O. Box 586, 34100 Trieste, Italy

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We propose a simulation model to study the properties of directed percolation in two-dimensional anisotropic random media. The degree of anisotropy in the model is given by the ratio μ between the axes of a semiellipse enclosing the bonds that promote percolation in one direction. At percolation, this simple model shows that the average number of bonds per site in two dimensions is an invariant equal to 2.8 independently of μ. This result suggests that Sinai's theorem proposed originally for isotropic percolation is also valid for anisotropic directed percolation problems. The invariant also yields a constant fractal dimension $D_f \sim 1.71$ for all μ, which is the same value found in isotropic directed percolation (i.e., $\mu = 1$). [S1063-651X(98)00702-8]

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I. INTRODUCTION

Critical phenomena [1] in anisotropic systems without equivalent nearest neighbors constitute an interesting research topic [2]. A universal formula for percolation thresholds, that involves the dimension of the anisotropic lattice and an arithmetic average of the coordination number for different anisotropic lattices, was recently postulated in Ref. [3]. The extension of these studies to more complex problems, such as directed percolation (DP), and more complex systems, such as anisotropic random systems, is yet to be addressed. In this context, random systems are good candidates to model anisotropy since they do not have equivalent nearest neighbors nor equivalent sites at all lengths.

In this work we propose a simple simulation model to study the properties of DP in two-dimensional (2D) anisotropic random media. The degree of anisotropy is computed by means of the ratio $\mu = y_c / x_c$ between the axes of a semiellipse enclosing the bonds that promote percolation in one direction, such that $y \leq x$ (see Fig. 1). As a function of the order parameter μ and at the percolation threshold, we measure the correlation length exponent ν and the fractal dimension D_f of the largest percolating clusters (in systems of up to 51 200 random sites). In the present model, the well-known scaling exponents of isotropic DP follow by simply setting $\mu = 1$.

At percolation threshold, our model shows that the average number of bonds per site for DP in anisotropic 2D random systems is an invariant ($B_c = 2.8$) independently of μ. This result suggests that the Sinai theorem, proposed originally for isotropic percolation (IP), is also valid for anisotropic DP problems. The invariant also yields a constant $D_f \sim 1.71$ for all μ, which corresponds to the value of isotropic DP.

The paper is organized as follows. In Sec. II, we outline our model. In Sec. III, we present the results of our simulations and discuss the effects of μ on the scaling exponents.

II. MODEL

In order to simulate DP in 2D anisotropic random media, we develop a simulation algorithm similar to the one used in Ref. [4]. The coordinates of N sites are generated at random in a square box of size $L = N^{1/2}$. The simulation length unit is chosen such that the density of sites, namely, n, in the box is always unity regardless of the total number of sites N. The percolation is then checked over sites from the left edge towards the right edge of the simulation box (i.e., along the x axis in Fig. 1). A periodical boundary condition is applied in the vertical y direction.

In Fig. 1 we show a ''particle'' that moves from i to j.

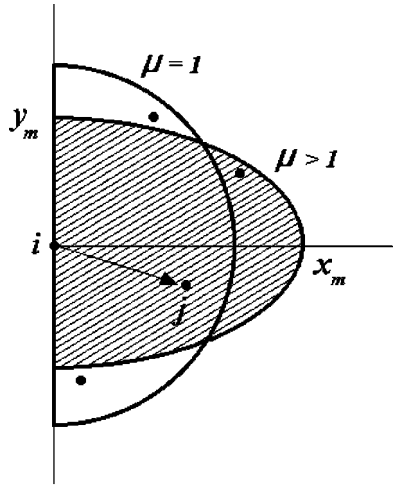

FIG. 1. The anisotropically directed percolation model. Percolation from site i to j is allowed whenever site j is contained within the shaded elliptical area. The degree of anisotropy is given by the ratio $\mu = x_m / y_m$, where x_m is the longer and y_m the shorter axis of the semiellipse. The case $\mu = 1$ yields the standard isotropic DP. At percolation, the average number of bonds per site (dots in this figure) is 2.8.

*Electronic address: canessae@ictp.trieste.it

The moving is allowed whenever the site j is contained within the shaded elliptical area. In our simulations, the degree of anisotropy is given by the parameter $\mu = x_m / y_m$, where x_m is the longer and y_m is the shorter axis of a semiellipse, i.e., μ is the ratio of the maximum ''hopping distances'' along the x and y axes.

In the standard 2D isotropic DP, there are three possible equivalent directions to move: up, down, and forward. This situation in our model is attained by setting $\mu = 1$. In the limit $\mu \to \infty$, the model tends to the one-dimensional percolation problem. Thus, simulation results using the present 2D percolation model will reveal features of the crossover from the standard (say, isotropic) DP to the 1D percolation problem. For intermediate values of $\mu > 1$ our model features anisotropic DP.

For a given value of the anisotropy parameter μ and for a given realization of random site coordinates $\{x_i, y_i\}$, in a sample of size $N = L \times L$, we study percolation from the left- to the right-simulation box edge. At the percolation threshold, we obtain the critical value of the semiellipse axis y_m, $y_c(N) = y_m(N)$, and the mass M of the critical cluster, $M(N)$ which is the total number of sites belonging to the largest cluster at percolation. These quantities are then averaged over a great number K of random realizations of site coordinates for the same sample size N which result on the average quantities $Y_c(N) = \langle y_c(N) \rangle$ and $\mathcal{M}(N) = \langle M(N) \rangle$, respectively.

In general, the dependence of the averages $Y_c(N)$ and $\mathcal{M}(N)$ on the samples size N is a consequence of the finite-size effects of the percolation problem. In order to quantify these effects, the present simulations were performed at different $N = 400$, 800, 1600, 3200, 6400, 12 800, 25 600, and 51 200. Accordingly, the number K decreases from 10^4 to 10^2 such that the product of the numbers $K \times N$ is approximately the same for all sample sizes N in our study.

Along with these average quantities, we also calculate the moments

$$\delta y_c(N) = \langle [y_c(N) - Y_c(N)]^2 \rangle^{1/2}, \tag{2.1}$$

$$\delta M(N) = \langle [M(N) - \mathcal{M}(N)]^2 \rangle^{1/2}, \tag{2.2}$$

and also the next-order moments, which are used to estimate the statistical errors of our simulation results.

The present measurements are performed for various values of $\mu = 1$, 2, 3, 4, 5, 6, 7, and 8. As can be seen from the results discussed in Sec. III, the greater the value of μ, the stronger the finite-size effects are. We verify that for $\mu = 8$ simulations can only be carried out in samples of size $N \geqslant 3200$.

Following the well-known finite-size scaling procedure suggested in Ref. [5], the critical exponent ν of the percolation problem is defined from the scaling expression

$$\delta y_c(N) \propto L^{-1/\nu}, \tag{2.3}$$

where $\delta y_c(N)$ is given in Eq. (2.1). Note that in the present study percolation is checked by the longitudinal direction only (the x axes in Fig. 1), then the exponent ν in Eq. (2.3) should be identified with the parallel $\nu_{||}$ (see Ref. [1]).

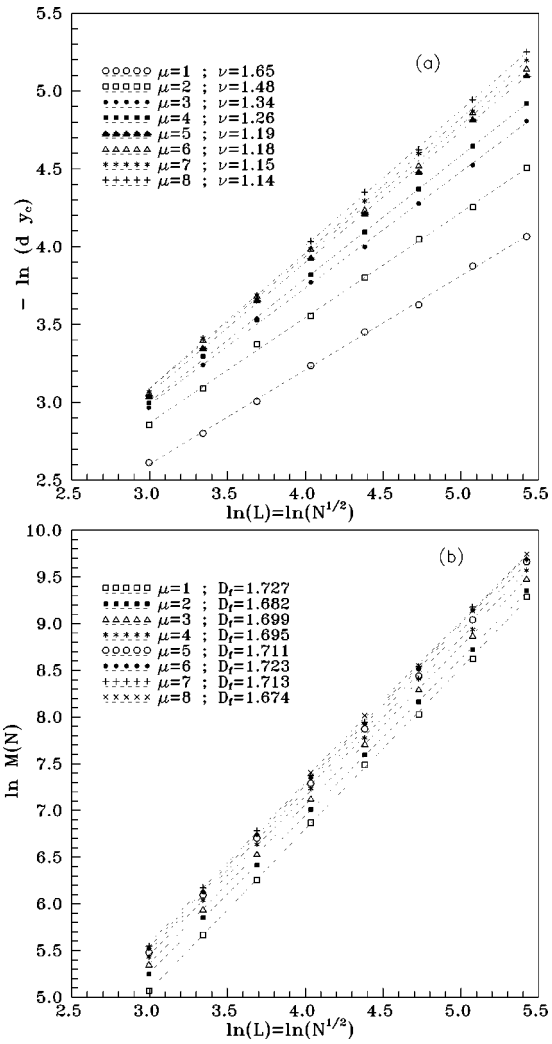

FIG. 2. Simulation data and linear fitting for (a) $-\ln \delta y_c(N)$ of Eq. (2.1) plotted against the sample size $\ln L$ for several values of the anisotropy parameter μ. (b) $\ln \mathcal{M}(N)$ of Eq. (3.3) plotted against the sample size $\ln L$ for several values of the anisotropy parameter μ. Within measurement errors, most fitted lines are parallel giving the average constant value $D_f \sim 1.71 \pm 0.02$. Note that fluctuations around this value are not systematic as μ increases.

III. RESULTS AND DISCUSSION

In Fig. 2(a) the quantities $-\ln \delta y_c(N)$ are plotted versus $\ln L \equiv \ln N^{1/2}$ for different values of the order parameter μ. The slopes of the fitting lines give the corresponding values for the exponent. Thus we found that for the largest $\mu = 8$, $\nu = 1.14$, and for $\mu = 2$ we measured $\nu = 1.48$. Other values are given in the figure. From these calculated moments and the linear fitting procedure, we estimate the statistical error to be less than 0.02 for all values of ν shown in this figure.

Results for the DP limiting case $\mu = 1$ were previously reported by one of us [4]. In this case, the value $\nu \approx 1.65 \pm 0.02$ is known as the universal value of $\nu_{||}$ for a whole class of isotropic DP models in two dimensions. As the amount of anisotropy increases, i.e., $\mu > 1$, the correlation length exponent ν decreases. Since this decrease is initially very fast to then become smoothly, it is not possible to obtain the whole crossover from 2D to 1D directed percolation for the behavior of ν. That is, the decrease from $\nu \approx 1.65$ in

2D DP to the limit $\nu=1$ in one dimension is limited by the size of our simulation box. The finite-size effects in correspondence to the different values of μ (and, therefore, to the different degrees of anisotropy in the 2D random systems) are in fact equivalent to those discussed in great detail in Refs. [4,6] for anisotropic percolation and isotropic DP.

By using the values of ν in Fig. 2(a), the critical *radius* $y_c = y_c(N\rightarrow\infty)$ *is determined from the scaling expression*

$$|y_c - y_c(N)| \propto L^{-1/\nu}. \tag{3.1}$$

In Figs. 3(a)–3(c) the quantities $y_c(N)$ are plotted versus $L^{-1/\nu}$. From these plots we obtain y_c by taking the asymptotic values $N\rightarrow\infty$ for all μ studied. The estimated values of y_c are also shown in this figure.

Very remarkably, our simulations show that for all μ considered the quantity $\mu y_c^2(\mu)$ is in fact a constant. Since $(\pi/2)\mu y_c^2 \equiv (\pi/2) x_c y_c$ is the area of the critical semiellipse at percolation, then our results suggest that Sinai's theorem [7], proposed originally for IP, is also valid for 2D anisotropic DP problems. In this respect, we emphasize again that our length unit should be taken as $n^{-1/2}$ for a system with site concentration n.

Thus, our simulations lead to the invariance

$$B_c^{(d)} = S_c n \equiv (\pi/2) n \mu y_c^2 = 2.82 \pm 0.02, \tag{3.2}$$

where n is the site concentration (e.g., the donor concentration in doped semiconductors), S_c is the area of the critical semiellipse, and $B_c^{(d)}$ is the mean number of connected bonds per site at percolation. The invariance of Eq. (3.2) may be somehow related to the fractal behavior of the critical clusters, as we shall discuss below.

Let us determine first the fractal dimension D_f of the critical percolation cluster using a standard procedure based on the scaling expression [8]

$$\mathcal{M}(N)_{L\rightarrow\infty} \propto L^{D_f}. \tag{3.3}$$

In Fig. 2(b) the quantities $\ln\mathcal{M}(N)$ are plotted against $\ln L$ for different values of the anisotropy parameter μ.

Very surprisingly, we found that the fractal dimensions D_f, as determined from the slopes of the fitting lines for various values of μ in Fig. 2(b), seem indeed to be constant and independent of μ within our simulation errors. We estimate $\mathcal{D}_f(\mu) \equiv D_f \approx 1.71 \pm 0.02$ for all μ, which corresponds to about the same value of the isotropic DP model with $\mu = 1$.

At first glance this result might raise some doubts, but we believe it can be understood in connection with the invariant given in Eq. (3.2). The invariance of $B_c^{(d)}$, with respect to changes in the anisotropy parameter μ, implies that the average number of connected bonds at percolation is independent of μ. If we assume the percolation process within an elementary semiellipse (as in Fig. 1) to be the ''originating percolation rule,'' then the invariance of Eq. (3.2) could mean that the law to generate percolation clusters remains unchanged as μ varies. If this conjecture is right, here we could suggest a more general statement for all types of percolation models which are related to each other by the Sinai theorem; in these cases, the fractal dimensions of the percolation clusters could all be the same.

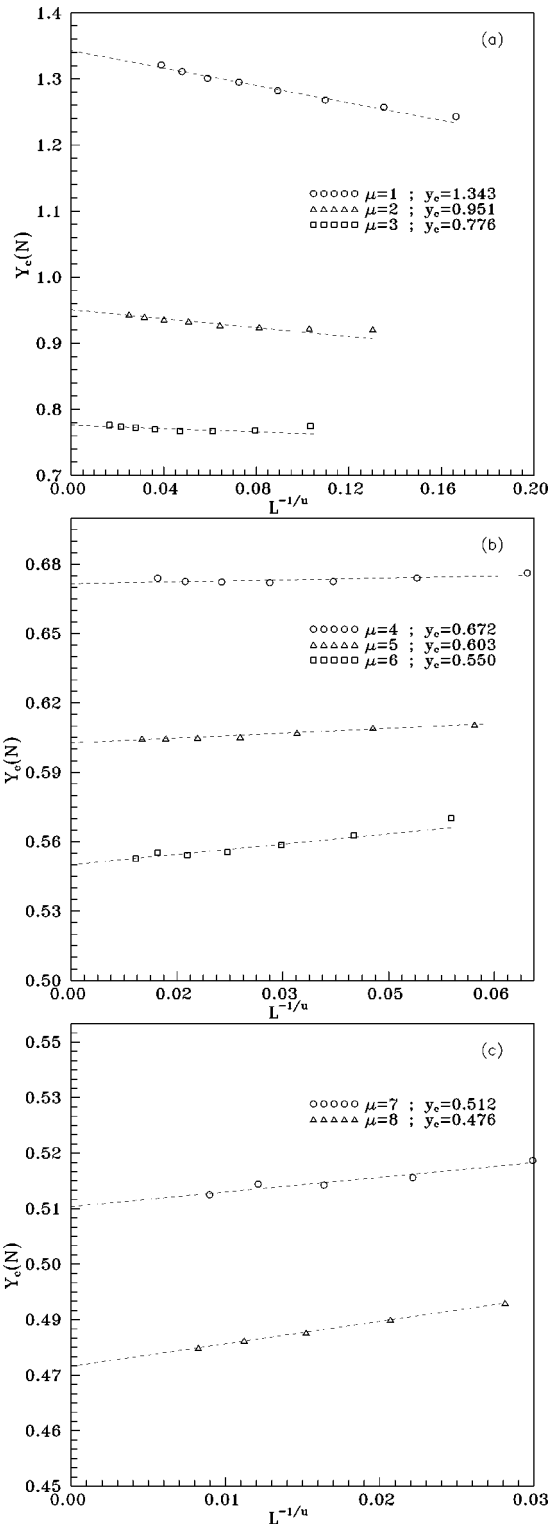

FIG. 3. Simulation data and linear fitting for $y_c(N)$ of Eq. (3.1) plotted against $L^{-1/\nu}$. (a) $\mu=1$, 2, and 3. (b) $\mu=4$, 5, and 6. (c) $\mu=7$ and 8. The threshold y_c for each μ studied is estimated from the asymptotic values $N\rightarrow\infty$.

It should be noted that our simulations are limited to $1 \leqslant \mu \leqslant 8$. It is in this range that we observed the invariance of B_c and the constant value for D_f. We believe that these features are maintained for a larger range of μ values. How-

ever, it is not feasible to increase μ and the sample size N simultaneously and get to the point where the present 2D simulation model crosses to the 1D case (i.e., $\mu \to \infty$).

To conclude, we suggested a model for anisotropic directed percolation (ADP) and have presented simulation results for the main critical exponents of the model in 2D random systems. Quite surprisingly, we found an invariance for the average number of connected bonds at percolation due to presence of a suitable external force (e.g., shear stress, magnetic field, etc.). Our simulations show that the product $\mu \times y_c^2$ is a constant for all μ's considered. This invariance should be in close relation to the value of D_f.

We strongly believe the present model of ADP could be important to describe some physical phenomena such as hopping conduction in anisotropic n-Ge and n-Si under strong electrical fields, where the impurity wave functions are anisotropic and the conduction band splits into one ellipsoid [9]. Our measurements could be useful, for instance, in the expressions for the hopping resistivity in 2D anisotropic random media. The invariance $B_c^{(d)} \sim 2.8$ could be used in these systems similarly to the invariance $B_c^{(i)} = 4.5$ for IP in a circle problem [9]. We hope the present model will stimulate further investigations on this direction.

ACKNOWLEDGMENT

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