Observer-based approach for controlling chaotic systems

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This paper presents a nonlinear state observer for a class of nonlinear systems which have an output dependent nonlinearity. By the observer design scheme proposed herein, an observer-based linear state feedback control approach is then derived to stabilize this class of systems. Analysis results indicate that both error dynamics and the subsequent closed-loop system can be made exponentially stable. The control strategy is also applied to two well-known chaotic systems: Rössler chaos and Lorenz chaos. Numerical simulations demonstrate the effectiveness of the proposed scheme. [S1063-651X(98)13802-3]

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I. INTRODUCTION

Several chaotic systems have been developed and thoroughly analyzed in recent decades. A chaotic system is a nonlinear deterministic system having a complex and unpredictable behavior. The sensitive dependence on initial condition and on the system parameter variation is a prominent feature of chaotic behavior. The controlling chaos problem of chaotic systems has received increasing attention [1–16]. In their pioneering work involving the controlling chaos, Ott, Grebogi, and Yorke [1] proposed a method (OGY method) which stabilizes unstable periodic orbits (UPO) embedded within a chaotic attractor by making small parameter perturbations. Adhering to the OGY method, several extensions have been successfully applied to many physical systems for various purposes [2,3]. Pyragas [4] proposed an alternative means of feedback stabilizing UPO by using a delayed self-controlling feedback, in which a continuous feedback term contains a delay variable and the delay corresponds to the period of UPO. Moreover, the delayed self-controlling method [4] and time delay coordinates strategy [5–7] can not only be applied without knowing *a priori* the dynamical equations but also be used for some rapid systems.

In addition, several methods known from standard control engineering have also been successfully applied to chaotic systems, e.g., entrainment and migration control [8,9], conventional engineering control [10–12], advanced nonlinear linearization technique [13], Lyapunov-based method [14], variable structure control method [15], and adaptive control theory [16]. The most common feature of these different control strategies is that the internal state variables are assumed to be available to construct the control forces; in addition, the controller structure is extremely complicated. However, under many circumstances, limited state information may be available and only the process output can be measured. Under such circumstances, a parallel state reconstruction, e.g., by means of a Kalman filter or a Luenberger-like type of observer, must be used to implement the control laws.

Recently, synchronization of chaotic systems has been linked to the concept of an observer in a control theoretical perspective [17,18]. The synchronization problem consists of forcing the transmitter system and receiver system to oscillate in a synchronous manner. The receiver system is usually a duplicate of the transmitter system, thereby accounting for why the receiver can be considered as an observer able to synchronously detect the state of the transmitter one. Hence, for an observer design similar to the synchronization design, knowledge of explicit dynamics of the controlled complex nonlinear systems is obviously a prerequisite.

The observer design of a general nonlinear system is a difficult problem in control and estimation theory. A variety of methods have been developed in recent years for some nonlinear systems. Four approaches are generally available for constructing nonlinear observers [19,20]. However, the observer error linearization and coordinate transformation are deemed necessary to construct the state observer.

In this paper, we address the problem of designing state observers for a class of nonlinear systems. The class of systems determined is allowed to have output dependent nonlinearity. By using the Bellman-Gronwall inequality lemma [21], under some structural assumptions in the nonlinearity, the exponential stability of the open-loop estimate error dynamics can be inferred. By such an observer scheme, the linear feedback control law based on such estimates is derived to stabilize this class of systems. By again using the Bellman-Gronwall inequality, in the case of the observer-based control law, the exponential stability of the subsequent closed-loop system can be inferred. Moreover, the proposed control scheme is applied to control two well-known chaotic systems: Rössler chaos and Lorenz chaos. Numerical simulations demonstrate the effectiveness of the proposed control strategy.

II. OBSERVER-BASED LINEAR CONTROLLER DESIGN FOR NONLINEAR SYSTEMS

A. Problem definition

Consider a class of single-input single-output nonlinear systems described by the following form:

$$\dot{x}=Ax+f(x,y)+B(u+d),$$
$$y=C^{T}x, \tag{1}$$

where $u,\ y\in\mathbb{R}$ denote the control input and system output, respectively, $x\in\mathbb{R}^{n}$ represents the state vector, $d\in\mathbb{R}$ is the

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dc bias of the controlled system, and A, B, and C denote constant matrices with appropriate dimensions, and $(\ )^T$ denotes the vector transpose. Furthermore, (A,B) represents a stabilizable pair and (C^T,A) is a detectable pair. f denotes a real analytic vector field on $\mathbb{R}^n$ with $f(0,y)=0$. Moreover, $f(x,y)$ satisfies the Lipschitz condition in x, i.e., there exists $\gamma>0$ such that $\|f(x_1,y)-f(x_2,y)\|\leqslant\gamma\|x_1-x_2\|$ for $x_1,\ x_2\in\mathbb{R}^n$, and for all $y\in\mathbb{R}$, where $\|(\cdot)\|$ denotes the appropriate norm of vector $(\cdot)$ and γ is the Lipschitz constant.

The class of nonlinear systems includes a wide variety of chaotic systems such as Rössler chaos and Lorenz chaos.

For stabilization purposes, a state feedback control law is designed herein to asymptotically stabilize the system (1) to the origin. In practice, this control law is of the form

$$u=K^Tx-d, \tag{2}$$

where $K\in\mathbb{R}^n$ is chosen such that $A+BK^T$ is an exponentially stable matrix, which is possible since the pair (A,B) can be stabilized. Therefore the control law in Eq. (2) asymptotically stabilizes the linear part of Eq. (1). If the state variables are unavailable, the conventional practice is to construct a state observer. Throughout this paper, an observer-based linear state feedback control scheme is derived such that the subsequent closed-loop system is exponentially stable. Some basic definitions and results used for developing the state feedback control scheme are summarized in the Appendix.

B. Nonlinear state observer

For estimating the state x of Eq. (1), we use a nonlinear state observer of the form

$$\dot{\hat{x}}=A\hat{x}+f(\hat{x},y)+B(u+d)+L(y-\hat{y}),$$
$$\hat{y}=C^T\hat{x}, \tag{3}$$

with $\hat{x}$ denoting the dynamic estimate of the state x and $f(\hat{x},y)$ representing the estimated vector of $f(x,y)$ based on the estimated state $\hat{x}$. The constant vector $L\in\mathbb{R}^n$ is chosen such that $A-LC^T$ is an exponentially stable matrix, which is also possible since the pair (C^T,A) is detectable. By allowing the state error $\hat{e}=x-\hat{x}$, the subsequent error dynamics can be written as follows:

$$\begin{aligned}\dot{\hat{e}}&=\dot{x}-\dot{\hat{x}}\\ &=Ax+f(x,y)+B(u+d)-A\hat{x}-f(\hat{x},y)-B(u+d)\\ &\quad -L(C^Tx-C^T\hat{x})=(A-LC^T)\hat{e}+f(x,y)-f(\hat{x},y).\end{aligned} \tag{4}$$

Given an initial condition $\hat{e}(0)$, the solution of the error dynamics (4) is as follows:

$$\hat{e}(t)=\exp[(A-LC^T)t]\hat{e}(0)+\int_0^t\exp[(A-LC^T)(t-\tau)]\times[f(x(\tau),y(\tau))-f(\hat{x}(\tau),y(\tau))]d\tau. \tag{5}$$

Because $A-LC^T$ is an exponentially stable matrix, positive constants m_1 and α_1 exist such that $\|\exp[(A-LC^T)t]\|\leqslant m_1\exp(-\alpha_1 t)$ for all $t\geqslant 0$. Therefore the inequality

$$\begin{aligned}\|\hat{e}(t)\|&\leqslant m_1\|\hat{e}(0)\|\exp(-\alpha_1 t)+m_1\int_0^t\exp[-\alpha_1(t-\tau)]\\ &\quad\times\|[f(x(\tau),y(\tau))-f(\hat{x}(\tau),y(\tau))]\|d\tau\\ &\leqslant m_1\|\hat{e}(0)\|\exp(-\alpha_1 t)+m_1\gamma\exp(-\alpha_1 t)\\ &\quad\times\int_0^t\exp(\alpha_1\tau)\|\hat{e}(\tau)\|d\tau\end{aligned} \tag{6}$$

is satisfied for all $t\geqslant 0$. Multiplying both sides of Eq. (6) by $\exp(\alpha_1 t)$ and defining $\overline{m_1}=\max\{m_1\|\hat{e}(0)\|,m_1\gamma\}$, as well as applying the Bellman-Gronwall lemma yield

$$\|\hat{e}(t)\|\leqslant\overline{m_1}\exp[-(\alpha_1-\overline{m_1})t], \tag{7}$$

which implies that the error dynamics exponentially converge to zero provided that $\alpha_1>\overline{m_1}$. Consequently, a sufficient condition is provided for the exponential convergence of the state error dynamics in the case of the Lipschitz condition in nonlinearity and a proper choice of the observer gain L.

C. Observer-based control law

In this subsection, we consider the case in which the control law (2) is implemented by observer state estimates:

$$u=K^T\hat{x}-d, \tag{8}$$

where measured state variables x are replaced by the corresponding estimates $\hat{x}$, as supplied by the proposed observer given in Eq. (3). The extended system describing the closed-loop $2n$-dimensional system constituted by Eqs. (1), (3), and (8) can be represented as follows:

$$\dot{x}=(A+BK^T)x-BK^T\hat{e}+f(x,y),$$
$$\dot{\hat{e}}=(A-LC^T)\hat{e}+f(x,y)-f(x-\hat{e},y). \tag{9}$$

The separation theorem [22] for the linear systems reveals that the eigenvalues of the linear part of system (9) are the union of the eigenvalues of $A+BK^T$ and $A-LC^T$.

By defining the augmented matrix A_c as

$$A_c=\begin{bmatrix}A+BK^T & -BK^T\\ 0 & A-LC^T\end{bmatrix}$$

we can obtain the solution of the closed-loop system (9) with the given initial conditions $x(0)$ and $\hat{e}(0)$ as follows:

$$\begin{bmatrix}x(t)\\ \hat{e}(t)\end{bmatrix}=\exp(A_ct)\begin{bmatrix}x(0)\\ \hat{e}(0)\end{bmatrix}+\int_0^t\exp[A_c(t-\tau)]\times\begin{bmatrix}f(x(\tau),y(\tau))\\ f(x(\tau),y(\tau))-f(x(\tau)-\hat{e}(\tau),y(\tau))\end{bmatrix}d\tau. \tag{10}$$

Because $A+BK^T$ and $A-LC^T$ are exponentially stable matrices, positive constants m and α exist such that $\|\exp(A_c t)\| \leqslant m\exp(-\alpha t)$ for all $t\geqslant 0$. Therefore the inequality

$$\left\|\begin{matrix} x(t) \\ \hat{e}(t)\end{matrix}\right\| \leqslant m\left\|\begin{matrix} x(0) \\ \hat{e}(0)\end{matrix}\right\| \exp(-\alpha t)+m\int_0^t \exp[-\alpha(t-\tau)] \times\left\|\begin{matrix} f(x(\tau),y(\tau)) \\ f(x(\tau),y(\tau))-f(x(\tau)-\hat{e}(\tau),y(\tau))\end{matrix}\right\| d\tau$$
$$\leqslant m\left\|\begin{matrix} x(0) \\ \hat{e}(0)\end{matrix}\right\| \exp(-\alpha t)+m\int_0^t \exp[-\alpha(t-\tau)] \times\left\|\begin{matrix} f(x(\tau),y(\tau))-f(0,y(\tau)) \\ f(x(\tau),y(\tau))-f(x(\tau)-\hat{e}(\tau),y(\tau))\end{matrix}\right\| d\tau$$
$$\leqslant m\left\|\begin{matrix} x(0) \\ \hat{e}(0)\end{matrix}\right\| \exp(-\alpha t)+m\gamma\exp(-\alpha t)\int_0^t \exp(\alpha\tau) \times\left\|\begin{matrix} x(\tau) \\ \hat{e}(\tau)\end{matrix}\right\| d\tau \qquad (11)$$

is satisfied for all $t\geqslant 0$. Multiplying both sides by $\exp(\alpha t)$ and defining $\overline{m}=\max\{m\|(\hat{e}(0),x(0))^T\|,m\gamma\}$, as well as applying the Bellman-Gronwall lemma, yield

$$\left\|\begin{matrix} x(t) \\ \hat{e}(t)\end{matrix}\right\| \leqslant \overline{m}\exp[-(\alpha-\overline{m})t]. \qquad (12)$$

The above equation implies that if $\alpha>\overline{m}$, then $(x(t),\hat{e}(t))^T$ exponentially converges to origin with the exponential rate $\alpha-\overline{m}$ and the closed-loop system is exponentially stable as well. Consequently, a sufficient condition is provided for the exponential convergence of the subsequent closed-loop system in the case of the Lipschitz condition in nonlinearity and a proper choice of both the observer gain L and the feedback gain K.

It can be easily verified that a class of chaotic systems, including the driven Rössler chaos and driven Lorenz chaos, belong to the class nonlinear systems mentioned above. In the following section, the observer-based feedback control approach proposed herein is applied to control this class of chaotic systems.

III. APPLICATION TO CHAOTIC SYSTEMS

A. Rössler system with control

This system is described by the following differential equations:

$$\dot{x}_1=-x_2-x_3, \quad \dot{x}_2=x_1+ax_2, \quad \dot{x}_3=c+x_3(x_1-b)+u, \qquad (13)$$

where a, b, and c denote positive parameters. By assuming that $u\equiv 0$ in the above equation, the Rössler system is obtained. This system has two equilibrium points:

$$x_e^{\pm}=(x_{e1}^{\pm},x_{e2}^{\pm},x_{e3}^{\pm})^T = \left(a\,\frac{b\pm\sqrt{b^2-4ac}}{2a},-\frac{b\pm\sqrt{b^2-4ac}}{2a},\frac{b\pm\sqrt{b^2-4ac}}{2a}\right)^T, \qquad (14)$$

where $b^2-4ac>0$. These parameters are selected in this study as $a=0.2$, $b=5.7$, $c=0.2$. By defining the state vector $x^T=[x_1\ \ x_2\ \ x_3]$ and the system output $y=x_1$, the system (13) can be represented by using Eq. (1) as follows:

$$\dot{x}=\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & -5.7\end{bmatrix}x+\begin{bmatrix} 0 \\ 0 \\ x_1x_3\end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 1\end{bmatrix}(u+0.2)$$
$$\equiv Ax+f(x,y)+B(u+d),$$
$$y=[1\quad 0\quad 0]x\equiv C^Tx, \qquad (15)$$

with $f(0,y)=0$, and the system (15) has a bounded, globally attracting set. Therefore state trajectories $x(t)$, $y(t)$ are always bounded and continuously differentiable. Consequently, $f(x_3,y)$ satisfies the Lipschitz condition for a bounded output y. The Lipschitz constant can be selected as $\gamma=\sup_{t\geqslant 0}y(t)$. Also, the linear part of the system described in Eq. (15) is verified in the Appendix as being controllable and observable and, moreover, being stabilizable and detectable.

While considering the state observer of Eq. (3), a nonlinear observer for the system (15) is given as follows:

$$\dot{\hat{x}}=\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & -5.7\end{bmatrix}\hat{x}+\begin{bmatrix} 0 \\ 0 \\ \hat{x}_3y\end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 1\end{bmatrix}(u+0.2)+L(y-\hat{y})$$
$$\equiv A\hat{x}+f(\hat{x},y)+B(u+d)+L(y-\hat{y}),$$
$$\hat{y}=[1\quad 0\quad 0]\hat{x} \qquad (16)$$

and the gain vector L is chosen as $L=[l_1\quad l_2\quad l_3]^T =[0.8\ \ -0.16\ \ -0.012]^T$ such that $A-LC^T$ is an asymptotically stable matrix. By regulating the state trajectory of system (15) to the origin, the observer-based state feedback control law (8) can be expressed as

$$u=K^T\hat{x}-d=K^T\hat{x}-0.2 \qquad (17)$$

and the state feedback gain is $K=[k_1\quad k_2\quad k_3]^T =[3.0\ \ -0.2\ \ 0.6]^T$. Figure 1 displays the numerical simulation results of the closed-loop system. The control was switched on at $t=0$ sec with the initial states ($x_1(0)=0.7$, $x_2(0)=0.4$, $x_3(0)=-0.8$) and the initial estimated states ($\hat{x}_1(0)=0.4$, $\hat{x}_2(0)=-0.1$, $\hat{x}_3(0)=1.2$).

B. Lorenz system with control

This system is described by the differential equations

$$\dot{x}_1=-\sigma x_1+\sigma x_2, \quad \dot{x}_2=rx_1-x_2-x_1x_3+u,$$
$$\dot{x}_3=x_1x_2-\beta x_3, \qquad (18)$$

where σ, r, and β denote positive parameters. By assuming that $u\equiv 0$ in the above equation, we obtain the Lorenz system. For $r>1$, this system has three equilibrium points:

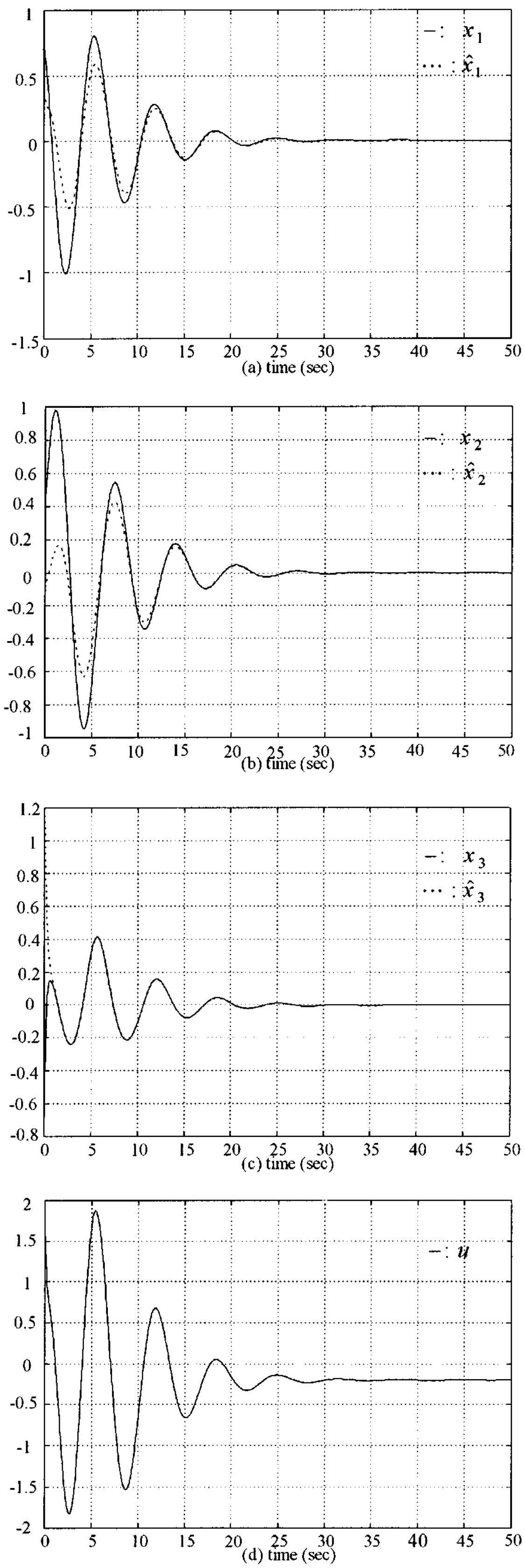

FIG. 1. System responses of the controlled Rössler chaos: (a) the actual state x_1 and the estimated state $\hat{x}_1$; (b) the actual state x_2 and the estimated state $\hat{x}_2$; (c) the actual state x_3 and the estimated state $\hat{x}_3$; (d) the control input u.

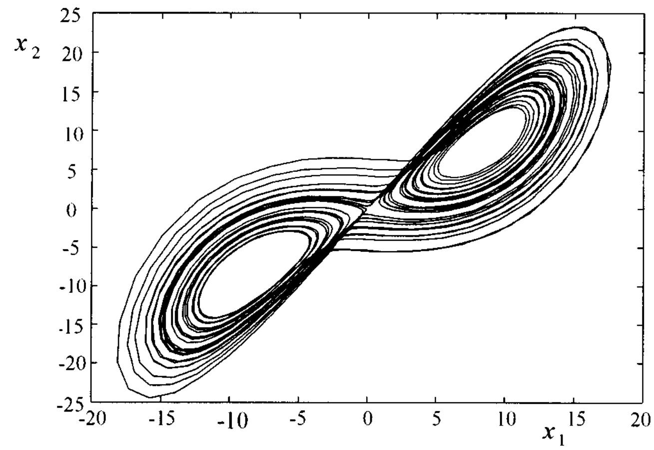

FIG. 2. Two-dimensional image of Lorenz chaos trajectory.

$$x_e^{\pm}=(\pm\sqrt{\beta(r-1)},\pm\sqrt{\beta(r-1)},r-1),\quad x_e^0=(0,0,0). \tag{19}$$

These parameters are selected herein as $\sigma=10$, $r=28$, $\beta=\frac{8}{3}$. Figure 2 depicts the chaos trajectory of the system (18) with the parameters given as above and $u\equiv 0$. By defining the state vector $x^T=[x_1 \quad x_2 \quad x_3]$ and the system output $y=x_1$, the system (18) can be represented by using Eq. (1) as follows:

$$\dot{x}=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & 0\\ 0 & 0 & -\frac{8}{3}\end{bmatrix}x+\begin{bmatrix}0\\ -x_1x_3\\ x_1x_2\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}u$$

$$\equiv Ax+f(x,y)+Bu,$$

$$y=[1 \quad 0 \quad 0]x, \tag{20}$$

with $f(0,y)=0$, and the system (20) has a bounded, globally attracting set. Therefore state trajectories $x(t)$, $y(t)$ are always bounded and continuously differentiable. Consequently, $f(x_3,y)$ satisfies the Lipschitz condition for a bounded output y. The Lipschitz constant can be selected as $\gamma=\sup_{t\geqslant 0}y(t)$. Also pointed out in the Appendix, the linear part of the system (20) is easily found to be both stabilizable and detectable.

By considering the state observer of Eq. (2), we obtain

$$\dot{\hat{x}}=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & 0\\ 0 & 0 & -\frac{8}{3}\end{bmatrix}\hat{x}+\begin{bmatrix}0\\ -y\hat{x}_3\\ y\hat{x}_2\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}u+L(y-\hat{y})$$

$$\equiv A\hat{x}+f(\hat{x},y)+B(u+d)+L(y-\hat{y}),$$

$$\hat{y}=[1 \quad 0 \quad 0]\hat{x}, \tag{21}$$

by selecting the gain vector L as $L=[l_1 \quad l_2 \quad l_3]^T=[-10 \quad 30 \quad 0]^T$ such that $A-LC^T$ is an exponentially stable matrix. By stabilizing the state trajectory of the system (20) to the equilibrium point x_e^0, the observer-based state feedback control (8) is given by

$$u = K^T(\hat{x} - x_e^0), \tag{22}$$

where the state feedback gain is $K = [k_1 \quad k_2 \quad k_3]^T = [-15.5 \quad -13.5 \quad 0]^T$. Figure 3 summarizes the numerical simulation results of the closed-loop system with the initial states $(x_1(0) = 0.3, x_2(0) = 0.4, x_3(0) = -0.8)$ and the initial estimated states $(\hat{x}_1(0) = 0.1, \hat{x}_2(0) = -0.1, \hat{x}_3(0) = -0.2)$.

Simulation results of these chaotic systems demonstrate that (i) the estimated state converges exponentially to the actual state and (ii) an observer-based linear state feedback control scheme can adequately control the chaos problem.

IV. CONCLUSIONS

This work presents a nonlinear state observer for a class of nonlinear systems with some structural assumptions. An observer-based linear state feedback control approach is also derived to stabilize this class of nonlinear systems. The control strategy is relatively simple and clearer than other either linear methods or nonlinear state feedback methods that require full state information. Analysis results confirm the exponential stability of the closed-loop system. The control scheme is also successfully applied to the controlling chaos problem. Moreover, numerical simulation results demonstrate the effectiveness of the proposed control scheme.

APPENDIX

For developing the observer-based controller of a class of nonlinear systems with its application to the problem of controlling chaos, some basic definitions and results of linear time-invariant control systems are briefly reviewed. The material is adopted from control system theory [21–23]. A reader who is unfamiliar with the results might find this information helpful.

Consider the following linear time-invariant system:

$$\dot{x} = Ax + Bu, \tag{A1a}$$

$$y = C^T x, \tag{A1b}$$

where $x \in \mathbb{R}^n$ represents the n-dimensional state vector, and $u, y \in \mathbb{R}$ denote the control input and system output, respectively. In addition, A, B, and C denote constant matrices with appropriate dimensions.

With a certain control input $u \equiv \overline{u}$, there exists a unique solution (or trajectory) of Eq. (A1a),

$$x(t) = x(t; 0, x(0), \overline{u}), \tag{A2}$$

which satisfies Eq. (A1a) under the initial conditions

$$x(0; 0, x(0), \overline{u}) = x(0). \tag{A3}$$

In many engineering applications, a need arises not only to drive the state trajectory to the equilibrium point $x = 0$ asymptotically but also to estimate how rapidly the trajectory approaches 0. This concept can be viewed as exponential stability of dynamical systems.

Definition 1 (exponential stability). The equilibrium point $x = 0$ of Eq. (A1a) is exponentially stable with a convergence rate α if there exist constants $\overline{m}, \alpha > 0$ such that

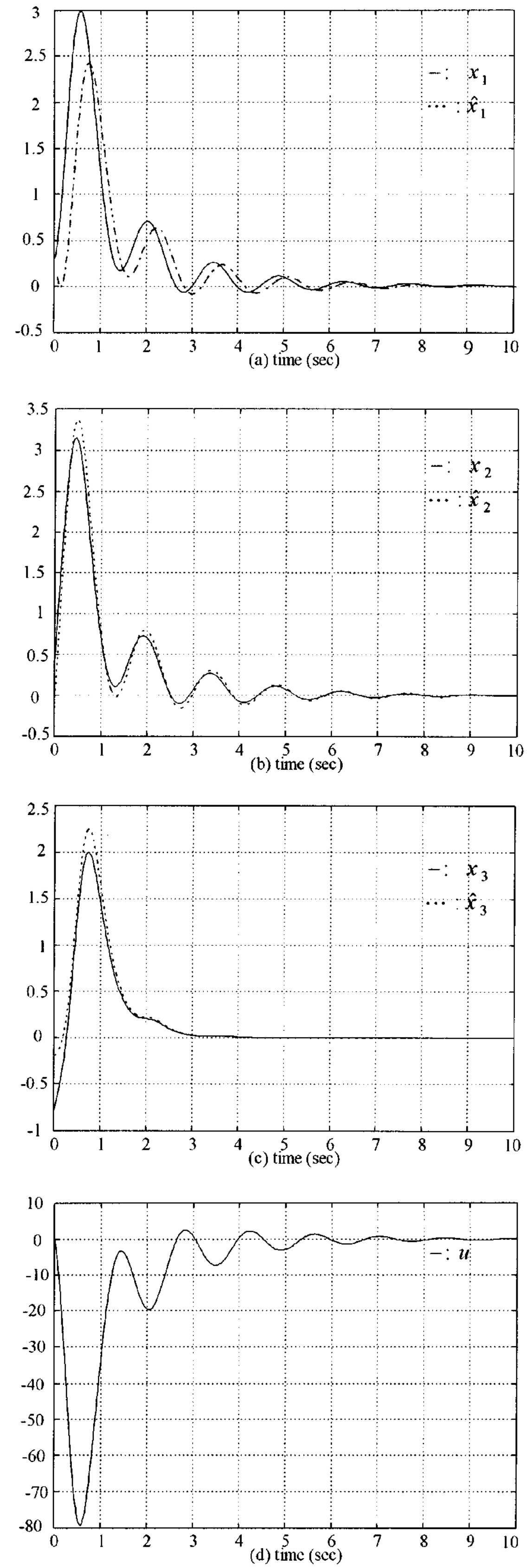

FIG. 3. System responses of the controlled Lorenz chaos: (a) the actual state x_1 and the estimated state $\hat{x}_1$; (b) the actual state x_2 and the estimated state $\hat{x}_2$; (c) the actual state x_3 and the estimated state $\hat{x}_3$; (d) the control input u.

$$\|x(t)\| \leqslant \overline{m}\|x(0)\| \exp(-\alpha t), \quad \forall t \geqslant 0. \tag{A4}$$

The problem of controllability is related to the possibility of driving the state trajectory $x(t)$ of the dynamical system given by Eq. (A1a) by means of the control input $u(t)$ in a finite interval of time.

Definition 2 (controllability). The system described by Eq. (A1) [or, in short, the pair (A,B)] is said to be *completely controllable*, if there exists a control input $u(\cdot)$ that can drive the system from any initial state $x(0)$ to any desired final state $x(t_f)$, $t_f < \infty$.

As indicated previously [22,23], the pair (A,B) is controllable if and only if the rank condition

$$\mathrm{rank}[B \quad AB \quad \cdots \quad A^{n-1}B] = n$$

is satisfied.

Relating u to the current state of the system (A1) in a feedback form, $u = K^T x$, $K \in \mathbb{R}^n$ is a column vector, Eq. (A1a) becomes

$$\dot{x} = (A + BK^T)x. \tag{A5}$$

The task for control theory involves designing a vector K such that the fixed point $x=0$ is exponentially stable with a convergence rate α. Related investigations have also confirmed [22,23] that, if the pair (A,B) is controllable, then for any given set of numbers $\mathrm{Re}(\mu_i) \leqslant -\alpha$, $i = 1,\ldots,n$ one can always find a vector K so that the matrix $A + BK^T$ has this set of numbers as its eigenvalues, i.e., all eigenvalues of $A + BK^T$ can be arbitrarily assigned to the open left-half complex plane or the matrix $A + BK^T$ is called an exponentially stable matrix.

The concept of observability closely resembles that of controllability. More specifically, observability refers to the possibility of determining the initial state $x(0)$ by measuring the input $u(t)$ and the output $y(t)$ over a finite interval of time.

Definition 3 (observability). The system given by Eq. (A1) [or, in short, the pair (C^T,A)] is said to be *completely observable*, if, for any initial state $x(0)$, there exists a finite time τ such that $x(0)$ can be determined (uniquely) from $u(t)$ and $y(t)$ for $0 \leqslant t \leqslant \tau$.

Previous investigations [22,23] have also confirmed that the pair (C^T,A) is observable if and only if the rank condition

$$\mathrm{rank}\begin{bmatrix} C^T \\ C^T A \\ \vdots \\ C^T A^{n-1} \end{bmatrix} = n$$

is satisfied.

The following stabilizability condition is weaker than controllability.

Definition 4 (stabilizability). The system described by Eq. (A1) [or, in short, the pair (A,B)] is said to be *stabilizable* if there exists a state feedback gain $K \in \mathbb{R}^n$ such that the closed-loop state equation $\dot{x} = (A + BK^T)x$ is exponentially stable.

The following detectability condition is weaker than the observability condition.

Definition 5 (detectable). The system given by Eq. (A1) [or, in short, the pair (C^T,A)] is said to be *detectable* if there exists an output injection gain $L \in R^n$ such that the closed-loop state equation $\dot{x} = (A - LC^T)x$ is exponentially stable.

Remark. A completely controllable pair can always be stabilized. Nevertheless, the opposite is not true. Intuitively, stabilizability can be viewed as stability of uncontrollable states. An equivalent interpretation of stabilizability is that all uncontrollable modes are exponentially stable. Similarly, an observable pair is also always detectable, and the converse is not true. Intuitively, detectability can be viewed as stability of unobservable states. An equivalent interpretation of detectability is that all unobservable modes are exponentially stable ones.

Example 1. Consider a linear system described by Eq. (A1) with the following triplicate matrices:

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & -5.7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C^T = [1 \quad 0 \quad 0].$$

Then it can easily be verified that $\mathrm{rank}[B \quad AB \quad A^2B] = 3$ and

$$\mathrm{rank}\begin{bmatrix} C^T \\ C^T A \\ C^T A^2 \end{bmatrix} = 3.$$

Therefore the pair (A,B) is both *controllable* and *stabilizable*, and the pair (C^T,A) is also both *observable* and *detectable*.

Example 2. Consider a linear system described by Eq. (A1) with the following triplicate matrices:

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C^T = [1 \;\; 0 \;\; 0].$$

Then it can easily be verified that $\mathrm{rank}[B \quad AB \quad A^2B] = 2$ and

$$\mathrm{rank}\begin{bmatrix} C^T \\ C^T A \\ C^T A^2 \end{bmatrix} = 2.$$

Moreover, all eigenvalues of the matrix A are $\{-22.8277, 11.8277, -2.6677\}$ and there exists only one uncontrollable and unobservable mode $\{-2.6677\}$, which is stable. Hence the pair (A,B) is not *controllable* but is *stabilizable*, and the pair (C^T,A) is not *observable* while it is *detectable*.

The following lemma plays a prominent role in deriving exponential stability of the closed loop in the observer-based control system.

Lemma 1. (Bellman-Gronwall inequality) [21]. Assume that $z(\cdot):\mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and $\alpha,\beta > 0$ are given constants. Under these conditions, if

$$z(t) \leqslant \beta + \int_0^t \alpha z(\tau) d\tau \quad \text{for all} \quad t \geqslant 0, \tag{A6}$$

then

$$z(t) \leqslant \beta \exp(\alpha t) \quad \text{for all} \quad t \geqslant 0. \tag{A7}$$

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