Bound states of dark solitons in the quintic Ginzburg-Landau equation

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(Received 27 January 1997)

We report results of systematic simulations of interactions between dark solitons in the complex quintic Ginzburg-Landau equation. Bound states of the solitons are found. The bound states (which are not possible in the cubic equation) exist in a wide range of parameters and are highly stable, providing an example of a stable bound state of solitary pulses in a generalized Ginzburg-Landau equation. [S1063-651X(97)08508-5]

PACS number(s): 42.65.Tg

I. INTRODUCTION

The particlelike nature of solitons most clearly demonstrates itself in interactions between them. For bright solitons, an outcome of the interaction depends on amplitudes, velocities, phases, and initial separation between the solitons [1,2]. For initially motionless ones, the interaction is attraction or repulsion, depending on their relative phase. For dark solitons, the picture is simpler, as two adjacent solitons may be in phase or π out of phase only. In addition, the amplitude and velocity of the dark soliton are related, hence one is dealing with fewer arbitrary parameters. Two initially motionless (i.e., black) dark solitons always repel each other [3].

The soliton interaction is a very sensitive effect that is greatly affected by perturbations [1,4–6]. It is known that for bright solitons, some perturbations tend to attenuate the interaction and may even lead to formation of bound states (BS's). For the complex Ginzburg-Landau (GL) equation [which may be regarded as a perturbed form of the nonlinear Schrödinger (NLS) equation], such a BS, with the phase difference 0 or π between the solitons was predicted in [7] (for more details see [8]). The existence of this BS has been confirmed numerically [9,10], although the subsequent studies have shown that apparently it is weakly unstable to perturbations of the relative phase between the two solitons [11,12]. The bound state has been observed also numerically in the dispersionless case [13]. Finally, it is relevant to notice that BS can exist in a different model, viz., the driven damped NLS equation, where the phases of the interacting solitons are independently locked to the driving force [14]. However, the separation between the bound solitons in this model is twice as large as in the GL equation.

The problem of existence of BS's of *dark* solitons was not addressed thus far. One of the reasons is that the perturbation theory for dark solitons, which allows one to analyze their dynamics under the action of perturbations in a consistent way (in particular, their stability and interactions), has been derived only recently [15]. The stable BS of the dark solitons is not only an interesting object for fundamental research, but may have practical importance too. Very recently, effective data transmission in nonlinear optical fibers has been demonstrated using dark solitons [16]. The relevance of the BS problem for the analysis of the operation of a soliton-based communication line is obvious. It is believed that dark solitons have some advantages over bright ones, in particular, a lower Gordon-Haus jitter [17].

In this paper we analyze interactions between the dark solitons in the presence of perturbations, namely, in the complex *quintic* GL equation (a perturbed NLS equation with a nonlinear saturable gain). Special attention is paid to the formation and stability of the two-soliton BS's. The eventual result is that the BS's do exist and appear to be fully stable in the quintic equation. As a matter of fact, this is an *example* of an absolutely stable BS of solitary pulses in an intrinsically driven generalized GL equation (the driven damped NLS equation, where BS's were found earlier [14], does not belong to this class of the models).

The rest of the paper is organized as follows. In Sec. II we introduce the GL equation and discuss the existence and stability of its dark-soliton solutions. Section III is devoted to an analysis of the interaction between the dark solitons. In Sec. IV we demonstrate the existence of stable BS's in the quintic GL equation. Section V concludes the paper.

II. THE GINZBURG-LANDAU EQUATION AND ITS DARK-SOLITON SOLITON SOLUTIONS

We take the perturbed NLS equation in the form

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-|u|^2u=iP[u], \tag{1}$$

where $u=u(z,t)$ is the complex field, t is the evolution variable ("time"), and x is a spatial variable ("space"). Notice that, in application to the nonlinear optical fibers, t is actually the propagation distance, while x is the so-called reduced time. The real term $P[u]$ represents a perturbation that combines gains and losses.

We are interested in dark-soliton solutions on a nonvanishing background, $|u|\to u_0$ for $|x|\to\infty$. We introduce the

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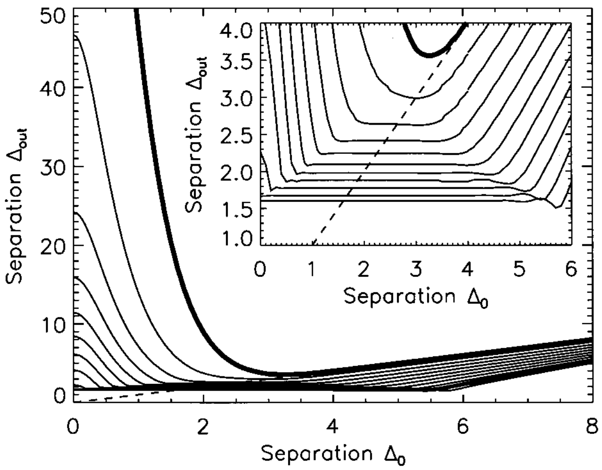

FIG. 1. Separation between the dark solitons at $\zeta=200$ vs the initial separation Δ_0. ϵ varies between 0 (top curve) and 1 (bottom curve). The inset shows the part of the figure related to the bound-state formation.

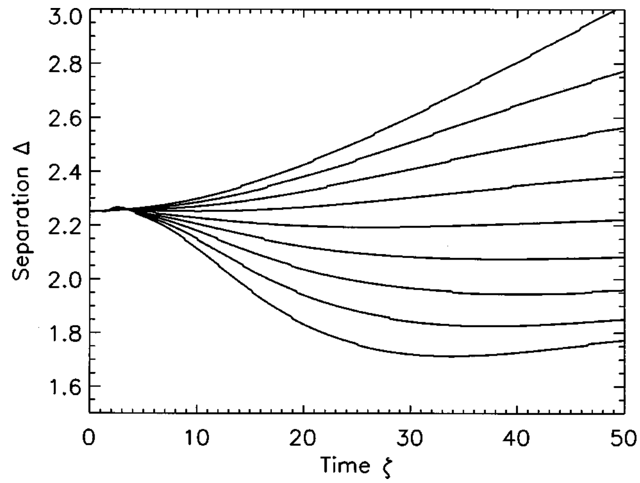

FIG. 2. Dynamics of the soliton interaction for different ϵ, varying from $\epsilon=0$ (top curve) to $\epsilon=0.8$ (bottom curve), at $\Delta_0=2.25$, $k=0.4$.

new variables $u\equiv u_0 e^{iu_0^2 t}v(t,x)$, $\zeta\equiv u_0^2 t$, and $\xi\equiv u_0 x$, in terms of which the equation takes the form

$$i\frac{\partial v}{\partial \zeta}+\frac{1}{2}\frac{\partial^2 v}{\partial \xi^2}-(|v|^2-1)v=iP[v]. \quad (2)$$

For $P=0$ this equation has the commonly known exact dark-soliton solution

$$v(\zeta,\xi)=\cos\phi\tanh Z-i\sin\phi, \quad (3)$$

where $Z\equiv(\xi-\zeta\sin\phi)\cos\phi$. The soliton amplitude and velocity are determined by the (constant) soliton phase angle ϕ, $|\phi|<\pi/2$. The phase shift across the soliton Φ is connected with ϕ by a simple relation $\Phi=\pi-2\phi$.

We now choose the perturbation in the form

$$P[v]=\delta v+\epsilon|v|^2 v+\mu|v|^4 v, \quad (4)$$

where δ, ϵ, and μ are, respectively, the coefficients of the linear, cubic, and quintic gain or dissipation (depending on their signs). We intentionally omit the diffusion term (spectral filtering, in terms of the fiber optics) because we suspect that it may slow down the interaction and also because an effect of this term on the dark soliton has not yet been studied by means of the perturbation theory. For convenience, we represent the coefficients in the form

$$\delta\equiv-k\epsilon, \quad \mu\equiv-(1-k)\epsilon. \quad (5)$$

In terms of this notation, both the cw background and dark solitons itself are known to be stable at $1/6<k<1/2$ [18].

If $\mu=0$, Eqs. (2) and (4) are known as cubic GL equations, which have an exact dark-soliton (''hole'') solution [19] ,

$$v(\xi)=\tanh(\alpha\xi)\exp[i\gamma\cosh\ln(\alpha\xi)], \quad (6)$$

where real constants α and γ may be found by direct substitution of Eq. (6) into Eq. (2). Note the nontrivial phase structure (''chirp,'' in terms of the fiber optics) of the dark soliton (6). This means that the dark soliton is in fact a sink that absorbs incident waves. If the system as a whole is in a stationary state (i.e., if the background does not decay), absorption of energy by the sink must be compensated for. There are other localized solutions (sources) that provide for this [20,21]. The source looks like a hump on the background; see, e.g., Figs. 2 and 4 of Ref. [22]. The existence of such humps prevents the dark solitons from an interaction, ''insulating'' them from each other.

It should be noted that the perturbation theory predicts that the dark solitons are unstable in the cubic GL equation [15], although the analysis did not take into account the above-mentioned diffusion (spectral filtering) term. Numerical simulations allowed one to identify a small region where the quiescent dark (black) soliton is stable [22] . However, one cannot use these stable solitons to study the interaction between them because everywhere in this region a direct interaction between dark solitons is prevented by a source that is formed between them.

We have found that the dark-soliton interaction and bound-state formation are possible in the full quintic GL equation. An exact solution in this case is known only in a limited range of parameters [21]. Nevertheless, the perturbation theory allows one to identify the stability range for dark solitons, which is much broader than that for the cubic GL equation [18]. In this case, the phase modulation is much weaker, sources are not formed, and the soliton interaction may be studied in detail.

III. SOLITON INTERACTION

First, we recall how the dark-soliton interaction occurs in the unperturbed NLS equation. Two initially motionless dark solitons always repel each other [3]. If one starts from an initial condition in the form

$$v_0(\xi)=\tanh(\xi-\Delta_0)\tanh(\xi+\Delta_0), \quad (7)$$

then two *gray* solitons are eventually formed. Their departure from the black one (i.e., the phase angle $|\phi|$) is roughly

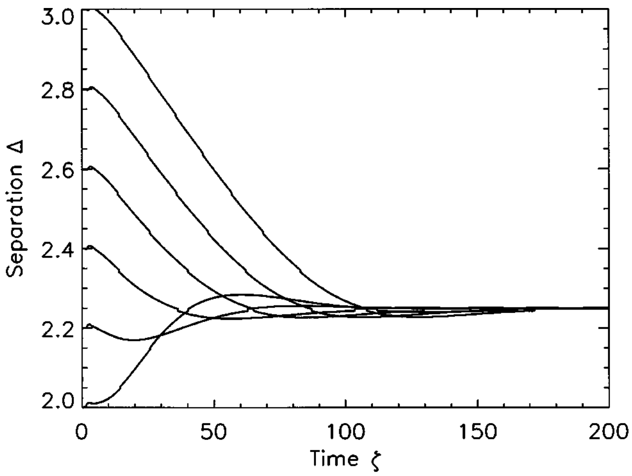

FIG. 3. Same as in Fig. 2 for $\Delta_0 = 2.2, \ldots, 3.0$ and fixed $\epsilon = 0.4$.

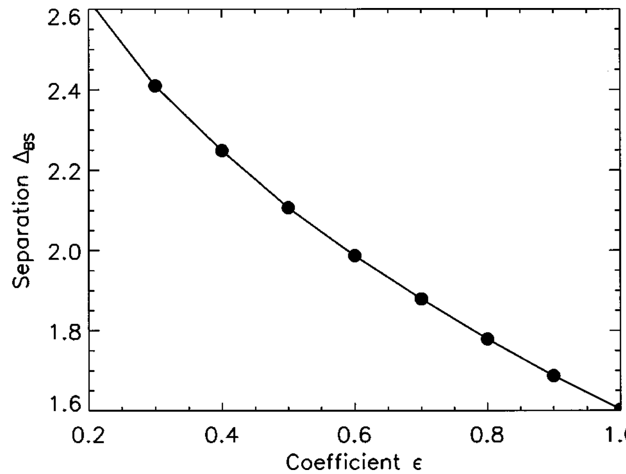

FIG. 5. Separation between the dark solitons in the bound state vs ϵ at $\zeta = 200$.

inversely proportional to Δ_0, i.e., for strongly overlapping solitons (small Δ_0) $|\phi|$ is large and vice versa. In the numerical experiments, the soliton interaction is usually studied in the form of a dependence of the output (final) separation Δ_{out} vs the input (initial) separation Δ_0. This dependence for the dark solitons in the unperturbed NLS equation is displayed in Fig. 1 (the bold curve). As one can see, the repulsion is strongest for small Δ_0 (although the difference from the case of the bright solitons is that Δ_{out} is finite at $\Delta_0 = 0$), then Δ_{out} attains a minimum, and finally it approaches Δ_0 at large Δ_0. The empirical solution for the soliton interaction has been given in Ref. [3] and it describes the soliton interaction with very good precision.

If we add the perturbation, we can expect two distinct scenarios in the simulations. In the first case, the perturbation affects each soliton but not the interaction. This means that each of two moving gray solitons that are initially formed from the initial condition (7) transforms into a black soliton, gradually dropping its velocity. Finally, two parallel propagating solitons are formed; however, this cannot be attributed to the bound-state formation (see, e.g., [18]). The interaction is affected only indirectly as the solitons acquire a different width. In the alternative case, the perturbation affects not only each soliton separately, but the interaction between them as well. In particular, a BS may be formed. The BS formation may be observed as independent of the output separation Δ_{out} of the input one Δ_0.

In the first series of simulations, we fix $\Delta_0 = 2.25$ and vary ϵ (see Fig. 2). As one can see, the perturbation (saturable gain) is braking the motion of the solitons. However, starting from $\epsilon = 0.4$, the motion of the solitons becomes nonuniform and this cannot be explained by the action of the perturbation on each soliton individually.

In the next series of simulations, we increase the propagation time from 50 to 200, fix $\epsilon = 0.4$, and study the interaction for several values of Δ_0 (Fig. 3). One can clearly see formation of the BS, with the separation between the solitons $\Delta_{BS} = 2.25$. Note the oscillatory type of the soliton trajectory for $\Delta_0 = 2$ and 2.2.

Now we can return to $\Delta_{out}(\Delta_0)$ dependence, which is most easily observed numerically (see the inset in Fig. 1). The BS formation may be observed starting from $\epsilon = 0.2$ as the nearly horizontal asymptotic part of the $\Delta_{out}(\Delta_0)$ curve. As ϵ increases, the size of this part also increases, and for $\epsilon > 0.9$ the bound state is formed for *any* initial separation in the [0,4] range.

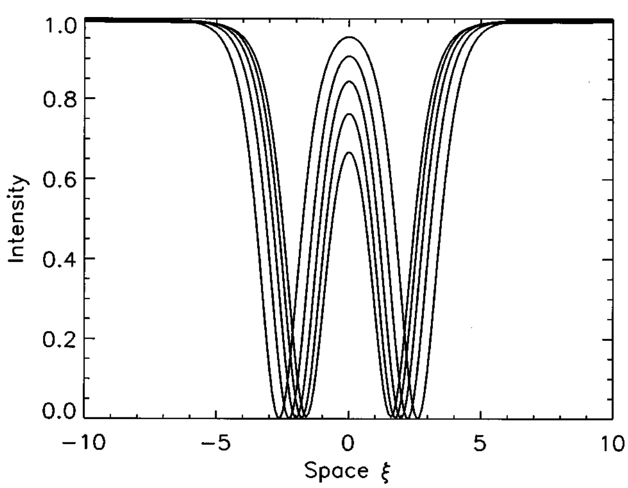

FIG. 4. Profiles of the bound states for ϵ taking values from 0.2 (top curve) to 1.0 (bottom curve).

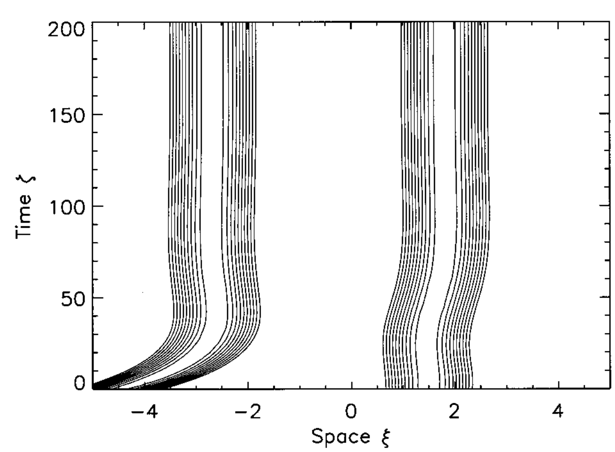

FIG. 6. Example of formation of the bound state from the solitons with $\phi_1 = 0$ and $\phi_2 = \pi/24$.

Figure 4 shows the profiles of the established BS's for several values of ϵ. As one can see, solitons are strongly overlapped in the bound states, even at small ϵ, and the overlapping increases with ϵ. Note that the separation between solitons in the BS rather weakly depends on ϵ (see Fig. 5).

To check the stability of the BS of two dark solitons, we used a perturbed asymmetric initial condition, viz., one soliton with $\phi_1=0$ and a second soliton with $\phi_2\neq 0$. The simulation shows that after some transition process, the bound state is formed and remains stable (Fig. 6).

Finally, we observed the multiple-soliton bound states, which consist of three, four, or more solitons. In fact, such structures have been predicted in Ref. [7] for bright solitons. However, the inner soliton in the multiple-pulse BS experiences different perturbation in comparison to outer solitons. So such BS have never been observed because of the instability to phase perturbations. In our case, the BS is highly stable and it may consist of an arbitrary number of solitons.

IV. CONCLUSION

We have simulated interactions of dark solitons in the quintic Ginzburg-Landau equation. We observe stable two-soliton bound states. This is an example of a stable bound state of solitary pulses governed by a generalized Ginzburg-Landau equation.

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