Wave packets, rays, and the role of real group velocity in absorbing media

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In an absorbing medium, where the vector $\mathbf{W}=\partial\omega/\partial\mathbf{k}$ usually is complex for real values of the wave vector $\mathbf{k}$, the group velocity $\mathbf{W}$ may become real for some complex values of $\mathbf{k}$. The role of real group velocity in the propagation of one-dimensional wave packets in homogeneous absorbing media is examined. Applying the saddle point method to an analysis of the asymptotic behavior of the Gaussian wave packets shows that for absorbing media, at large times and distances, the real group velocity appears as a local characteristic of any small section of a wave packet. For each section we can find the complex values of the local wave number and the local frequency defining a real group velocity. Thus, the real group velocity concepts in absorbing media do not have to be based on the signals having real wave vectors or real frequencies. The analysis of the exact solution for a Gaussian wave packet in a medium with a complex law of dispersion describing whistler waves in a collisional plasma is performed. It is shown that at all times the initial carrier wave number exists as a real part of the local complex wave number at some point of the Gaussian envelope and this point moves with a constant real group velocity. For large times the local wave group with the initial carrier wave number can be found far away from the envelope center. [S1063-651X(98)05601-3]

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I. INTRODUCTION

The concepts of wave packet(s) (WP), group velocity, and ray tracing come up in many areas of physics—quantum mechanics, optics, plasma physics, fluid mechanics, solid state physics, geophysics, and astrophysics. The group velocity concept seems to have been first introduced by Rayleigh [1] for the transverse sound waves propagating in thin elastic rods. Since then this concept was applied to studying WP and signals based on various kinds of waves in dispersive, nonabsorbing media. The theory of WP in dispersive media without absorption of the wave energy has been amply discussed in the literature [2,3]. As is well known, any WP in a homogeneous medium is constructed by the continuous superposition (integration) of the elementary plane waves, sinusoidal in both space and time, with neighboring values of the wave vector $\mathbf{k}$ and the frequency ω. Usually such a superposition is presented in the form of the integral of the function $A(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]$ in the $\mathbf{k}$ space. The dispersion equation $D(\omega,\mathbf{k})=0$ characterizes the properties of the medium with respect to wave propagation. The spatial Fourier transform of the WP at $t=0$ is the function $A(\mathbf{k})$ concentrated in some vicinity of the carrier wave vector $\mathbf{k}_c$. The spatial maximum of the WP envelope propagates with the group velocity $\mathbf{W}=\partial\omega/\partial\mathbf{k}$ calculated at $\mathbf{k}=\mathbf{k}_c$.

The concept of rays in the theory of WP propagation in nonabsorbing media appears in studies of the asymptotic behavior of the WP for large values of t and $|\mathbf{x}|$. A powerful tool that leads to the asymptotic formulas for the WP solutions is the saddle point (SP) method for the Fourier integrals. Whitham [3] shows that for large times each small section of the WP can be characterized by the instantaneous values of the local wave vector $\tilde{\mathbf{k}}(\mathbf{x},t)$ and the local frequency $\tilde{\omega}(\mathbf{x},t)=\omega[\tilde{\mathbf{k}}(\mathbf{x},t)]$, where the dependence $\omega=\omega(\mathbf{k})$ is determined by the dispersion equation $D(\omega,\mathbf{k})=0$. These local characteristics of the WP remain constant along the straight-line trajectories $\mathbf{x}=\mathbf{X}(t)$ termed the rays. The value of the local wave vector $\tilde{\mathbf{k}}$ at the point $\mathbf{x}=\mathbf{x}_m(t)$ corresponding to the spatial maximum of the WP envelope equals $\mathbf{k}_c$. The description of the rays associated with the WP propagation admits the Hamiltonian formalism: the vectors $\mathbf{X}(t)$ and $\tilde{\mathbf{k}}$ determine the position of the dynamic system in the configuration (coordinate) space and in representation (momentum) space, respectively, while the frequency $\omega(\tilde{\mathbf{k}},\mathbf{X})$ plays the role of the Hamiltonian. For a homogeneous medium ω depends only on $\tilde{\mathbf{k}}$ and the Hamiltonian differential equations have the form

$$\frac{d\mathbf{X}}{dt}=\frac{\partial\omega}{\partial\tilde{\mathbf{k}}}\equiv\mathbf{W}(\tilde{\mathbf{k}}),\quad \frac{d\tilde{\mathbf{k}}}{dt}\equiv\left[\frac{\partial}{\partial t}+\mathbf{W}(\tilde{\mathbf{k}})\ \frac{\partial}{\partial\mathbf{x}}\right]\tilde{\mathbf{k}}=-\frac{\partial\omega}{\partial\mathbf{X}}=\mathbf{0}. \tag{1}$$

The solutions of this Hamiltonian system represent the straight lines in the phase space $\{\mathbf{X},\tilde{\mathbf{k}}\}$ given by

$$\mathbf{X}=\mathbf{X}_0+\mathbf{W}(\tilde{\mathbf{k}}_0)t,\quad \tilde{\mathbf{k}}=\tilde{\mathbf{k}}_0, \tag{2}$$

where $\mathbf{X}_0$ and $\tilde{\mathbf{k}}_0$ are arbitrary constants of integration along each ray. If an initial distribution of the wave vector in space is specified, then $\tilde{\mathbf{k}}_0$ can be considered as a known function of $\mathbf{X}_0$.

In dissipative (absorbing) or active (amplifying) media the elementary waves that are harmonic in space decay or grow in time, while the time-harmonic waves decay or grow in space. This leads to a formally defined complex group velocity vector [4–6] $\mathbf{W}=\partial\omega/\partial\mathbf{k}$. The complex Hamiltonian equations and their generalizations for the spatially inhomogeneous absorbing media where $\omega=\omega(\mathbf{k},\mathbf{x})$ lead to the complex rays $\mathbf{X}=\mathbf{X}(t)$. The complex ray tracing was used as a

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mathematical tool for solution of some problems of wave propagation in absorbing ionosphere [6,7] and in a hot tokamak plasma [8].

When the medium is absorbing, the WP acquires some new features in comparison with its behavior in nonabsorbing media. Thus, the velocity of the WP envelope maximum $\mathbf{V}_m$ changes with time even in the case when the medium is homogeneous [4,9,10]. In addition, the local wave number that can be determined at the point of the envelope maximum also changes with time [4]. Consequently the vectors $\mathbf{V}_m$ and $\mathbf{W}$ are not identical. In an absorbing medium vector $\mathbf{W}$ is complex at the point of the envelope maximum. Although the velocity $\mathbf{V}_m$ can be expressed in terms of both real and imaginary parts of $\mathbf{W}$, from the physical point of view the concept of the complex group velocity becomes obscure. On the other hand, several examples existing in the literature show that $\mathbf{W}$ may represent a real physical velocity in an absorbing [5,9,11,12] or amplifying [13] medium if under some circumstances the vector $\mathbf{W}$ becomes real. The case of an absorbing system described by the one-dimensional heat-conduction equation was analyzed in [11] where a purely exponential ''carrier wave'' and a slow envelope in the form of hyperbolic sine were considered. The carrier wave satisfied the condition $\mathrm{Im}W=0$, which was fulfilled for purely imaginary values of k and the real group velocity W represented the velocity of propagation of the zero temperature point. As was suggested in [12], the WP and the group velocity concepts in absorbing media do not have to be based on the signals having real wave vectors or real frequencies. To some extent, this suggestion is confirmed by the analysis of the WP given in the present paper.

For a homogeneous absorbing medium the condition $\mathrm{Im}\,\mathbf{W}=0$ leads to some relationship between $\mathrm{Re}\,\mathbf{k}$ and $\mathrm{Im}\,\mathbf{k}$ (and between $\mathrm{Re}\omega$ and $\mathrm{Im}\omega$ as well). Under this condition the solutions of the Hamiltonian system (1) presented by Eqs. (2) still determine the real rays $\mathbf{X}=\mathbf{X}(t)$ along which the SP wave vector $\tilde{\mathbf{k}}$ keeps constant complex values that are different for different rays. The purpose of the present paper is to examine the role of the real rays provided by the requirement $\mathrm{Im}\mathbf{W}=0$ in the propagation of the one-dimensional WP in the homogeneous absorbing media and to show that such rays determine all local characteristics the WP for large times and distances.

The paper is organized as follows: In Sec. II the SP method is employed for calculating the asymptotic form of the Fourier integrals containing the complex eikonal $S(k,x,t)=kx-\omega(k)t$ and describing the WP in an absorbing medium. Both spatial and temporal WP corresponding to the initial-value and boundary-value problems are considered, including the asymptotic form for the Gaussian packets and for the Green functions. It is demonstrated that for the imaginary part of the eikonal $\mathrm{Im}S$ considered as a function of k (or ω), the condition $\mathrm{Im}W=0$ is satisfied in the SP $k=\tilde{k}$ (or $\omega=\tilde{\omega}$) of the function $\mathrm{Im}S$. In Sec. III the main properties of the complex SP wave number $\tilde{k}(x,t)$ are analyzed. It is shown that for absorbing media the role of the SP wave number in the asymptotic behavior of the WP is similar to what was described by Whitham [3] for the WP in the purely dispersive media where the values of $\tilde{k}$ are always real. Analogously to the nonabsorbing case, the real group velocity W represents the velocity of propagation of the SP wave number with the distinction that the wave number becomes complex.

In Sec. IV we present and investigate the exact analytical solution for a Gaussian WP in a medium characterized by the quadratic complex law of dispersion $\omega=(\alpha-i\beta)k^2$ to illustrate the features of the WP that are associated with the local group velocity, which remains real in the presence of absorption. This solution can describe the propagation of the whistler WP in a collisional magnetosphere. In the particular cases $\beta=0$ or $\alpha=0$ the solution is converted to Gaussian packets for the systems described by the Schrödinger equation or the heat conduction equation, respectively. Some properties of a Gaussian whistler WP in the presence of absorption were briefly discussed by Muschietti and Dum in [4] with the emphasis on the propagation of the maximum of the WP envelope. The main advantage of the exact solution analyzed in Sec. IV is in the fact that it gives a complete description of the transition from the initial stage to the asymptotic stage of the WP evolution in the presence of the k-dependent wave damping.

The structure of the exact solution allows one to determine the local complex wave number and the local complex frequency, which differ from the SP values and approach them for large times.

It is shown that the trajectory, along which the real part of the local wave number is equal to the carrier wave number, represents the straight line and it is a genuine Hamiltonian ray $X(t)$ satisfying Eqs. (1) with $\mathrm{Im}W=0$. This property demonstrates the importance of the concept of the real group velocity for absorbing media. For the whistler WP the real value of W represents the velocity of propagation of the carrier wave number and it differs from the velocity of propagation of a Gaussian peak studied in [4].

The main results are summarized and discussed in Sec. V.

II. THE SADDLE POINT METHOD FOR WAVE PACKETS IN ABSORBING MEDIA

The description of linear waves propagating in a uniform, dispersive, and absorbing medium is commonly based on the elementary exponential solutions

$$\exp(iS)=\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]. \tag{3}$$

The relationship between the wave frequency ω and the wave vector $\mathbf{k}$ is given by the dispersion equation

$$D(\omega,\mathbf{k})=0. \tag{4}$$

For absorbing or active media the dispersion equation is complex: its solution $\omega=\omega(\mathbf{k})$ determines the complex frequency $\omega=\omega_r+i\omega_i$ for real values of the wave vector $\mathbf{k}$. This means that the initial perturbation having the sinusoidal form in space will decay in time if $\omega_i<0$ and grow in time if $\omega_i>0$. The first case corresponds to an absorbing medium, while the situation with $\omega_i>0$ is related to an active medium, in which the steady state is unstable with respect to small perturbations. We consider below only one-dimensional problems and assume that the vector $\mathbf{k}$ is directed along the x axis, so that the complex eikonal S in Eq. (3) has the form

$$S=kx-\omega(k)t, \tag{5}$$

where $\omega(k)$ is the solution of Eq. (4).

There are two main problems associated with the propagation of WP. The first is the initial-value (Cauchy) problem where we investigate the evolution of the initial distribution in space $u(x,0)$, of some wave field $u(x,t)$, satisfying a partial differential equation

$$\mathsf{L}(\partial/\partial t,\partial/\partial x)[u]=0. \tag{6}$$

Here the linear differential operator L is a symbolic polynomial in $\partial/\partial t$, $\partial/\partial x$ with constant coefficients. The dispersion equation (4) and the wave equation (6) are related by

$$D(\omega,k)=\mathsf{L}(-i\omega,ik). \tag{7}$$

If the order n of the polynomial L with respect to $\partial/\partial t$ is larger than 1, then the solution of the Cauchy problem can be determined when the function u and its derivatives $\partial^s u/\partial t^s$ for $s=1,2,...,n-1$ are specified for $t=0$. For $n=1$ the solution of the Cauchy problem can be presented in the form of the Fourier integral

$$u(x,t)=\int_{-\infty}^{\infty}A(k)\exp[iS(k,x,t)]dk, \tag{8}$$

where S is the eikonal given by Eq. (5) and $A(k)$ is the Fourier transform of the initial distribution

$$A(k)=(2\pi)^{-1}\int_{-\infty}^{\infty}f(x)\exp(-ikx)dx, \quad f(x)=u(x,0). \tag{9}$$

If $n>1$ and all roots $\omega_j(k)$ of the dispersion equation are different, then the solution of the Cauchy problem is given by a sum of n integrals of the type (8) with different $A_j(k)$ and $S_j(k,x,t)=kx-\omega_j(k)t$.

The spatial WP is the continuous superposition of the planar waves (1), whose wave numbers k are concentrated at $t=0$ in the vicinity of the carrier wave number k_c. An often used model of WP is the Gaussian packet, for which

$$A(k)=\frac{C}{\Delta\sqrt{\pi}}\exp\left[-\frac{(k-k_c)^2}{\Delta^2}\right]. \tag{10}$$

The corresponding initial distribution $u(x,0)=f(x)$ is the Gaussian WP in the x space

$$f(x)=C\exp(ik_cx-x^2/h^2) \quad (h=2/\Delta), \tag{11}$$

where C is the amplitude of the WP envelope and h is the half-width of the envelope at $t=0$. The half-width of the WP in the k space is $\Delta=2/h$. In the limit $h\to\infty$ Eq. (11) represents the harmonic wave in space and Eq. (9) results in $A(k)=C\delta(k-k_c)$. The transition to the opposite limit $h\to 0$ with $C=(h\sqrt{\pi})^{-1}$ results in the initial distribution $u(x,0)=\delta(x)$ and $A(k)=1/(2\pi)$. The solution $u(x,t)$ corresponding to this initial condition is the Green function for the Cauchy problem

$$G(x,t)=(2\pi)^{-1}\int_{-\infty}^{\infty}\exp[iS(k,x,t)]dk. \tag{12}$$

The second problem is the boundary-value problem when one needs to obtain the solution $u(x,t)$ in the half-space $x>0$, which is resulted from a given signal $g(t)$ applied to the boundary $x=0$,

$$u(0,t)=g(t)=\int_{-\infty}^{\infty}B(\omega)\exp(-i\omega t)d\omega, \tag{13}$$

where $B(\omega)$ is the Fourier transform of the boundary signal $g(t)$. The solution $U(x,t)$ of the boundary-value problem can be presented in the form of the Fourier integral

$$U(x,t)=\int_{-\infty}^{\infty}B(\omega)\exp[i\Sigma(\omega,x,t)]d\omega. \tag{14}$$

The eikonal Σ in Eq. (14) is given by

$$\Sigma=k(\omega)x-\omega t. \tag{15}$$

Unlike the solution $u(x,t)$ of the initial-value problem, which is determined by the complex function $\omega(k)$ for real values of k, the solution $U(x,t)$ of the boundary-value problem can be determined when the complex function $k(\omega)$ is specified for real values of ω. If for a given value of ω Eq. (4) admits more than one solution, then the solution $U(x,t)$ of the boundary-value problem may be presented as a sum of integrals of the type (14) with different functions $\Sigma_j=k_j(\omega)x-\omega t$.

The Green function of the boundary-value problem $G_b(x,t)$ is the solution corresponding to the boundary signal $g(t)=\delta(t)$. Similarly to Eq. (12), this solution can be determined by the Fourier integral of $\exp(i\Sigma)$ with the complex eikonal Σ:

$$G_b(x,t)=(2\pi)^{-1}\int_{-\infty}^{\infty}\exp[i\Sigma(\omega,x,t)]d\omega. \tag{16}$$

One of the fundamental problems associated with the propagation of WP in an absorbing medium is the asymptotic behavior of WP for large values of t and x when the WP traverses a distance that is much longer than its initial width. For most cases the asymptotic behavior of WP can be investigated by the saddle point (SP) method. The SP method has been extensively employed for the WP in lossless dispersive media [3]. Extensions of the SP method to the cases of absorbing media were made in [4,9,10]. Applications of the SP method to calculating the Fourier integrals of the type (8) usually includes three elements [2]: (i) determining the SP $\tilde{k}$ for the function Im S considered as a function of the complex variable $k=k_r+ik_i$; (ii) changing the integration contour (8) from the real k axis to some other path in the complex k plane, which passes through the SP $k=\tilde{k}$ and yielding the same value of the initial integration apart from contributions of singular points of the integrand if such exist; (iii) evaluating the asymptotic value of the integral over the new path making use of the fact that the integrand changes rapidly in the vicinity of the SP.

The SP of the function Im$S(k)$ is determined from the equation

$$\left[\frac{\partial S(k,x,t)}{\partial k}\right]_{k=\tilde{k}}=x-t\left[\frac{d\omega(k)}{dk}\right]_{k=\tilde{k}}=0. \tag{17}$$

Since $d\omega/dk$ is in general a complex quantity, for real x and t Eq. (17) is equivalent to two real equations:

$$\mathrm{Re}\left(\frac{d\omega}{dk}\right)_{k=\tilde{k}}=\frac{x}{t},\quad \mathrm{Im}\left(\frac{d\omega}{dk}\right)_{k=\tilde{k}}=0. \tag{18}$$

For a lossless medium, where $d\omega/dk$ is always real for real values of k, Eqs. (18) prescribe the real SP $k=\tilde{k}_r$. The quantity $d\omega/dk$ is recognized in the theory of WP in lossless media as the group velocity $W(k)$. Thus, for a purely dispersive medium the SP of the function ImS is characterized by the conditions

$$\tilde{k}=\tilde{k}_r,\quad \tilde{k}_i=0,\quad (d\omega/dk)_{k=\tilde{k}}=W(\tilde{k})=x/t. \tag{19}$$

In the general case where Eq. (4) is complex,

$$W(k)=\mathrm{Re}(d\omega/dk)+i\ \mathrm{Im}(d\omega/dk) \tag{20}$$

is complex too for real values of k. The SP $k=\tilde{k}$ of the eikonal S satisfying Eq. (18) is therefore complex. We shall assume that at the point $k=\tilde{k}$ the second derivative of the function $S(k)$ does not vanish:

$$(d^2S/dk^2)_{k=\tilde{k}}=-it(d^2\omega/dk^2)_{k=\tilde{k}}\neq 0. \tag{21}$$

In the complex k plane one can find the line of steepest descent γ passing through the point $k=\tilde{k}$, along which Im$S(\tilde{k})$ takes the minimal value and, therefore, $|\exp iS|=\exp(-\mathrm{Im}S)$ is maximized. The line of steepest descent can be found from the equation

$$\mathrm{Re}S(k,x,t)=\mathrm{Re}S(\tilde{k}(x,t),x,t). \tag{22}$$

For given x and t, which are considered as parameters in Eq. (22), this equation gives the relation between k_r and k_i and, therefore, it determines some curve in the complex k plane.

The main advantage of the SP method for the asymptotic calculations of the WP's as $t\to\infty$ stems from the fact that after replacing the path of integration the integral along γ can be easily evaluated due to the presence of a very sharp maximum of the integrand at $k=\tilde{k}$ [2]. The value of the integral (8) can be calculated by taking the integral along γ:

$$u(x,t)=u_\gamma(x,t)=\int_\gamma A(k)\exp[iS(k,x,t)]dk. \tag{23}$$

Recasting the eikonal in the form $S=t[kx/t-\omega(k)]$ and employing the well-known asymptotic formula [1] for the integrals of the type (23) results in the asymptotic expression for $u_\gamma(x,t)$ when $t\to\infty$ and $|x/t|$ does not exceed any given constant value:

$$u_\gamma(x,t)\approx\left(\frac{2\pi}{t(d^2\omega/dk^2)_{k=\tilde{k}}}\right)^{1/2}A(\tilde{k})\exp[iS(\tilde{k},x,t)-i\pi/4]. \tag{24}$$

Here $\tilde{k}=\tilde{k}(x/t)$ is the complex coordinate of the SP determined by Eq. (18). In the case when $A(k)$ is given by Eq. (10), the asymptotic expression for the spatial Gaussian packet is obtained from Eq. (24) as

$$u(x,t)\approx\frac{Ch}{\sqrt{2t(d^2\omega/dk^2)_{k=\tilde{k}}}}\times\exp\left[-\frac{h^2(\tilde{k}-k_c)^2}{4}+iS(\tilde{k},x,t)-i\frac{\pi}{4}\right]. \tag{25}$$

For $Ch=1/\sqrt{\pi}$ and $h\to 0$, Eq. (25) is reduced to the asymptotic formula for the Green function of the Cauchy problem

$$G(x,t)\approx\frac{1}{\sqrt{2\pi t(d^2\omega/dk^2)_{k=\tilde{k}}}}\exp[iS(\tilde{k},x,t)-i\pi/4]. \tag{26}$$

As is seen from the comparison of Eq. (24) with Eq. (26), the asymptotics of the general solution of the Cauchy problem is determined by the value of the spectral function $A(k)$ at the SP and by the asymptotics of the Green function $G(x,t)$:

$$u(x,t)\approx 2\pi A[\tilde{k}(x/t)]G(x,t). \tag{27}$$

The application of the SP method to the solution of the boundary-value problem presented by Eq. (14) contains the same steps as the asymptotic calculation of the Fourier integral (8). The complex SP ($\omega=\tilde{\omega}$) of the eikonal $\Sigma(\omega)$ is determined from two real equations

$$\mathrm{Re}\left(\frac{dk}{d\omega}\right)_{\omega=\tilde{\omega}}=\frac{t}{x},\quad \mathrm{Im}\left(\frac{dk}{d\omega}\right)_{\omega=\tilde{\omega}}=0. \tag{28}$$

These equations are similar to Eqs. (18) and they indicate that in the SP of the eikonal $\Sigma(\omega)$ the group velocity $W=d\omega/dk$ is real and its value is x/t, while the SP itself $\tilde{\omega}=\tilde{\omega}_r+i\tilde{\omega}_i$ is complex, and both quantities $\tilde{\omega}_r$ and $\tilde{\omega}_i$ are functions of x/t. The same value of W is obtained at the complex SP of the eikonal $S(k)$. The analog of the asymptotic formula (25) can be obtained also for the solution $U(x,t)$ of the boundary-value problem.

It must be emphasized that our whole discussion pivots on the choice of Eqs. (18) and (28) (which for us is physically plausible) that x,t are real. Mathematically it is equally consistent to assume complex rays, e.g., as done by Connor and Felsen [9].

III. LOCAL REAL GROUP VELOCITY AND THE SADDLE POINT COMPLEX WAVE NUMBER AS ASYMPTOTIC CONCEPTS

We concentrate here on the concept of the real group velocity $W=x/t$ as the velocity of propagation of the constant wave number $k=\tilde{k}(x/t)$ and the constant frequency ω

$=\tilde{\omega}(x/t)$, which corresponds to the SP for the eikonal $S=kx-\omega(k)t$ in the complex k plane. This is the generalization of the concept of group velocity given by Whitham for WP in dispersive lossless systems [3] to the case of dispersive and absorbing media. For a spatial WP in an absorbing medium the function $\exp[iS(\tilde{k},x,t)]$ determining the asymptotics of the Green function for large values of t and for a finite ratio x/t is not purely harmonic and $|\exp(iS)|$ is varying in time and space. The value of S at the SP is complex and given by

$$\tilde{S}(x,t)\equiv S[\tilde{k}(x/t),x,t]=\Theta(x,t)+i\Phi(x,t),$$

$$\Theta(x,t)=\tilde{k}_r\left(\frac{x}{t}\right)x-\tilde{\omega}_r\left(\frac{x}{t}\right)t,\quad \Phi(x,t)=\tilde{k}_i\left(\frac{x}{t}\right)x-\tilde{\omega}_i\left(\frac{x}{t}\right)t. \tag{29}$$

Here $\tilde{\omega}=\tilde{\omega}_r+i\tilde{\omega}_i=\omega[\tilde{k}(x/t)]$, the SP frequency, is the value of the frequency calculated from the complex dispersion equation $\omega=\omega(k)$ at the point $k=\tilde{k}$, the SP wave number. In general both $\tilde{k}$ and $\tilde{\omega}$ are complex. Equations (18) indicate that for $k=\tilde{k}$ and hence for $\omega=\tilde{\omega}$, the group velocity $W=d\omega/dk=x/t$ is real. This allows defining the real straight-line rays $x/t=\text{const}$ in the (x,t) plane as the trajectories, along which the wave number $\tilde{k}$ and the frequency $\tilde{\omega}$ remain constant.

Thus, the real group velocity $W=x/t$ appears in asymptotic analysis as the velocity of propagation of the constant SP wave number $k=\tilde{k}(x/t)$. It should be noted that only this concept of real W (and not the concept of group velocity as the velocity of propagation of maximum of the WP envelope) will persist in the presence of absorption. The local central wave number that can be determined in the vicinity of the envelope maximum does *not* remain constant in an absorbing medium. Thus, for whistler WP in a collisional plasma, the initial carrier wavelength of spatial oscillations for large values of t can be found in one of two wings of the WP but not in its center. Such behavior will be demonstrated in Sec. IV.

It can be easily checked that the SP values $k=\tilde{k}$ and $\omega(\tilde{k})=\tilde{\omega}$ can be expressed through the partial derivatives of the SP phase function $\tilde{S}$ defined in Eq. (29):

$$\tilde{k}=\partial\tilde{S}/\partial x,\quad \tilde{\omega}=-\partial\tilde{S}/\partial t. \tag{30}$$

Therefore, $\tilde{k}$ and $\tilde{\omega}$ satisfy the continuity equation

$$\frac{\partial\tilde{k}}{\partial t}+\frac{\partial\tilde{\omega}}{\partial x}=\frac{\partial\tilde{k}}{\partial t}+W(\tilde{k})\,\frac{\partial\tilde{k}}{\partial x}=0, \tag{31}$$

which clearly indicates that the real group velocity $W(\tilde{k})$ is the velocity of propagation of the complex wave number $\tilde{k}$. Equations (30) and (31) are generalizations of the equations obtained by Whitham for purely dispersive homogeneous media [3]. For an absorbing medium Eqs. (30) and (31) are complex.

It should be noted that for nonabsorbing media the analog of the continuity equation (31) appears also in the ray theory of quasiharmonic signals propagating in inhomogeneous media, including two- and three-dimensional problems [2]. Such signals usually are considered in the form $A(\mathbf{r},t)\exp[iS(\mathbf{r},t)]$ where A is the slowly varying amplitude and S is the rapidly varying eikonal function. The local wave vector is defined by $\mathbf{k}=\partial S/\partial\mathbf{r}$, which is equivalent to the Sommerfeld-Runge law of refraction $\nabla\times\mathbf{k}=\mathbf{0}$ [14]. The local frequency is defined by $\omega=-\partial S/\partial t$. These two relations ensure the uniqueness of the eikonal integral representation given by integration of $\mathbf{k}\cdot d\mathbf{r}-\omega\,dt$ and provide the equation $\partial\mathbf{k}/\partial t+\partial\omega/\partial\mathbf{r}=\mathbf{0}$ [compare with Eq. (31)]. The ray trajectories in an inhomogeneous medium are described by the analog of Eqs. (1) with the distinction that $\partial\omega/\partial\mathbf{X}$ does not vanish. This analogy stems from the fact that for large times and distances, in both situations the spatial and temporal variations of the WP envelope or the wave amplitude are slow in comparison with the eikonal variations.

Employing the Fourier integral (16) for studying the asymptotic behavior of the Green function for the boundary-value problem $G_b(x,t)$ for large values of x and finite ratio t/x facilitates the definition of the complex SP frequency $\tilde{\omega}=\tilde{\omega}(x/t)$ on the basis of Eqs. (28). The analog of Eq. (29) can be obtained for the value of the eikonal Σ at the SP,

$$\tilde{\Sigma}(x,t)\equiv\Sigma[\tilde{\omega}(x/t),x,t]=\tilde{k}x-\tilde{\omega}t, \tag{32}$$

where $\tilde{k}=k(\tilde{\omega})$. The functions $\tilde{k}$ and $\tilde{\omega}$ satisfy the analog of Eq. (30), in which $\tilde{S}$ should be replaced by $\tilde{\Sigma}$. Since both $\tilde{\omega}$ and $\tilde{k}$ are functions of $x/t=d\tilde{\omega}/d\tilde{k}=W(\tilde{\omega})$, the real group velocity again appears as the velocity of propagation of the SP complex frequency and the SP complex wave number.

IV. EXACT SOLUTION FOR A GAUSSIAN WAVE PACKET IN AN ABSORBING MEDIUM WITH A QUADRATIC COMPLEX DEPENDENCE $\omega(k)$

In this section we consider the evolution of the spatial Gaussian WP characterized by the dispersion equation

$$\omega=(\alpha-i\beta)k^2 \tag{33}$$

in a dispersive absorbing medium, where α and β are nonnegative parameters. In accordance with Eq. (7), the partial differential equation corresponding to Eq. (33) has the form

$$\mathsf{L}[u]\equiv\frac{\partial u}{\partial t}-(i\alpha+\beta)\,\frac{\partial^2 u}{\partial x^2}=0. \tag{34}$$

The two popular particular forms of Eq. (34) are the Schrödinger equation $(\beta=0)$ and the heat conduction equation $(\alpha=0)$. The complex law of dispersion given by Eq. (33) when both parameters α and β are nonzero describes the whistler waves propagating in the terrestrial magnetosphere along the geomagnetic field lines [15]. The parameters α and β for whistlers are determined by

$$\alpha=c^2\omega_{ce}/\omega_{pe}^2,\quad \beta=\alpha\nu_e/\omega_{ce}. \tag{35}$$

Here c is the vacuum light velocity, ω_{ce} is the electron gyrofrequency, ω_{pe} is the electron plasma frequency, and ν_e is the mean collision frequency of electrons. Similar waves with a complex quadratic law of dispersion termed helicons are known in semiconductors and metals in the presence of an external magnetic field [16]. For the helicons propagating along the magnetic field the parameters of the dispersion equation (33) can be calculated from the same formulas (35) as for whistlers.

When the initial distribution $u(x,0)=f(x)$ is a Gaussian WP given by the real part of Eq. (11), the solution of the Cauchy problem for the law of dispersion given by Eq. (33) has the form

$$\overline{u}=\frac{u(x,t)}{C}=\frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\cos(kx-\alpha k^2 t)\times\exp[-\beta k^2 t-\Delta^{-2}(k-k_c)^2]dk. \tag{36}$$

This integral allows an exact analytical representation [17,18] as a function of parameters x, t, α, β, k_c, and Δ. Introducing the dimensionless quantities

$$\xi=k_c x,\quad \tau=\beta k_c^2 t,\quad N=k_c/\Delta,\quad a=\alpha/\beta, \tag{37}$$

Eq. (38) can be recast as a function of two dimensionless variables ξ and τ containing two dimensionless parameters N and a:

$$\overline{u(\xi,\tau)}=E(\xi,\tau)\cos[\theta(\xi,\tau)],$$

$$E(\xi,\tau)=\frac{\exp[-\psi(\xi,\tau)]}{[(1+\tau/N^2)^2+a^2\tau^2/N^4]^{1/4}}, \tag{38}$$

where

$$\psi(\xi,\tau)=\frac{(1+\tau/N^2)[\xi-\xi_m(\tau)]^2}{4[(N+\tau/N)^2+(a\tau/N)^2]}+\frac{\tau}{1+\tau/N^2} \tag{39}$$

and

$$\xi_m(\tau)=2a\tau/(1+\tau/N^2) \tag{40}$$

and

$$\theta(\xi,\tau)=\vartheta(\xi,\tau)-\varphi(\tau),$$

$$\vartheta=\frac{\xi(1+\tau/N^2)-a\tau+a\xi^2\tau/(4N^4)}{(1+\tau/N^2)^2+a^2\tau^2/N^4},$$

$$\varphi=\frac{1}{2}\arctan\left(\frac{a\tau}{\tau+N^2}\right). \tag{41}$$

In the case $a\gg 1$ the function $\vartheta(\xi,\tau)$ represents the fast varying spatial and temporal oscillations, while $\varphi(\tau)$ is the slow varying phase of these oscillations. In the particular case $\beta=0$ the right-hand side of Eq. (38) is reduced to the solution for a Gaussian WP in a nonabsorbing medium [19]. The transition to the limit $\beta\to 0$ means that in Eqs. (38)–(41) $a\to\infty$, $\tau\to 0$ and $a\tau\to\alpha k_c^2 t$. The result of this limiting case is

$$\overline{u}=\frac{\exp[-\Delta^2(x-2\alpha k_c t)^2/4(1+\alpha^2\Delta^4 t^2)]\cos[(k_c x-\alpha k_c^2 t+\alpha\Delta^4 x^2 t/4)/(1+\alpha^2\Delta^4 t^2)-\frac{1}{2}\arctan(\alpha\Delta^2 t)]}{[1+\alpha^2\Delta^4 t^2]^{1/4}}. \tag{42}$$

In the opposite limiting case ($\alpha=0$) Eq. (38) becomes the solution of the Cauchy problem for the heat conduction equation with the initial condition in the form of a Gaussian WP

$$\overline{u}=(1+\beta\Delta^2 t)^{-1/2}\exp\left[-\frac{\Delta^2 x^2+4\beta k_c^2 t}{4(1+\beta\Delta^2 t)}\right]\cos\left(\frac{k_c x}{1+\beta\Delta^2 t}\right). \tag{43}$$

In the limit $\Delta\to\infty$, for $G(x,t)=C\tilde{u}$ with $C=\Delta/(2\sqrt{\pi})$, Eq. (43) results in a well-known formula for the Green function of the Cauchy problem for the heat conduction equation (see, for example, Ref. [20]).

The analysis of the exact solution given by Eqs. (38)–(41) when both parameters α and β are nonvanishing shows the following features of WP:

(i) As follows from Eqs. (38) and (39), the form of the WP envelope determined by the function $E(\xi,\tau)$ remains Gaussian for all times. The magnitude of the envelope's maximum (at the point $\xi=\xi_m$) decays exponentially for $t \ll 1/(\beta\Delta^2)$ and undergoes a power decay according to $1/\sqrt{t}$ at $t\to\infty$. The width of the envelope increases with time and behaves as a linear function of time for $t\ll 1/(\beta\Delta^2)$. For $t\to\infty$ the envelope width increases as $\sqrt{t}$. Thus, the asymptotic behavior of the envelope for $t\to\infty$ is determined by the wave absorption.

(ii) The spatial maximum of the WP envelope $x=x_m(t)=\xi_m/k_c$ propagates with the velocity

$$V_m(t)=\frac{dx_m}{dt}=\frac{2\alpha k_c}{(1+\beta\Delta^2 t)^2}=\frac{W_r(k_c)}{(1+\beta\Delta^2 t)^2}. \tag{44}$$

As is seen from Eq. (44), in the presence of absorption the velocity of the envelope maximum differs from the real part of the group velocity $W_r(k_c)$ calculated for the initial carrier wave number k_c. Only for $t=0$ do these two velocities coincide. The envelope center is decelerating and its velocity $V_m(t)$ tends to zero as $1/t^2$ when $t\to\infty$. The evolution of the WP envelope is shown in Fig. 1.

(iii) When $N^2\gg 1$, i.e., the characteristic number of spatial oscillations inside the WP is sufficiently large, the denominator of the ratio in right-hand side of Eq. (38) is a slowly varying function of t in comparison with the functions $\exp(-\psi)$ and $\cos\theta$. In this case it is reasonable to define the local complex wave number k^* and the local complex frequency ω^* as

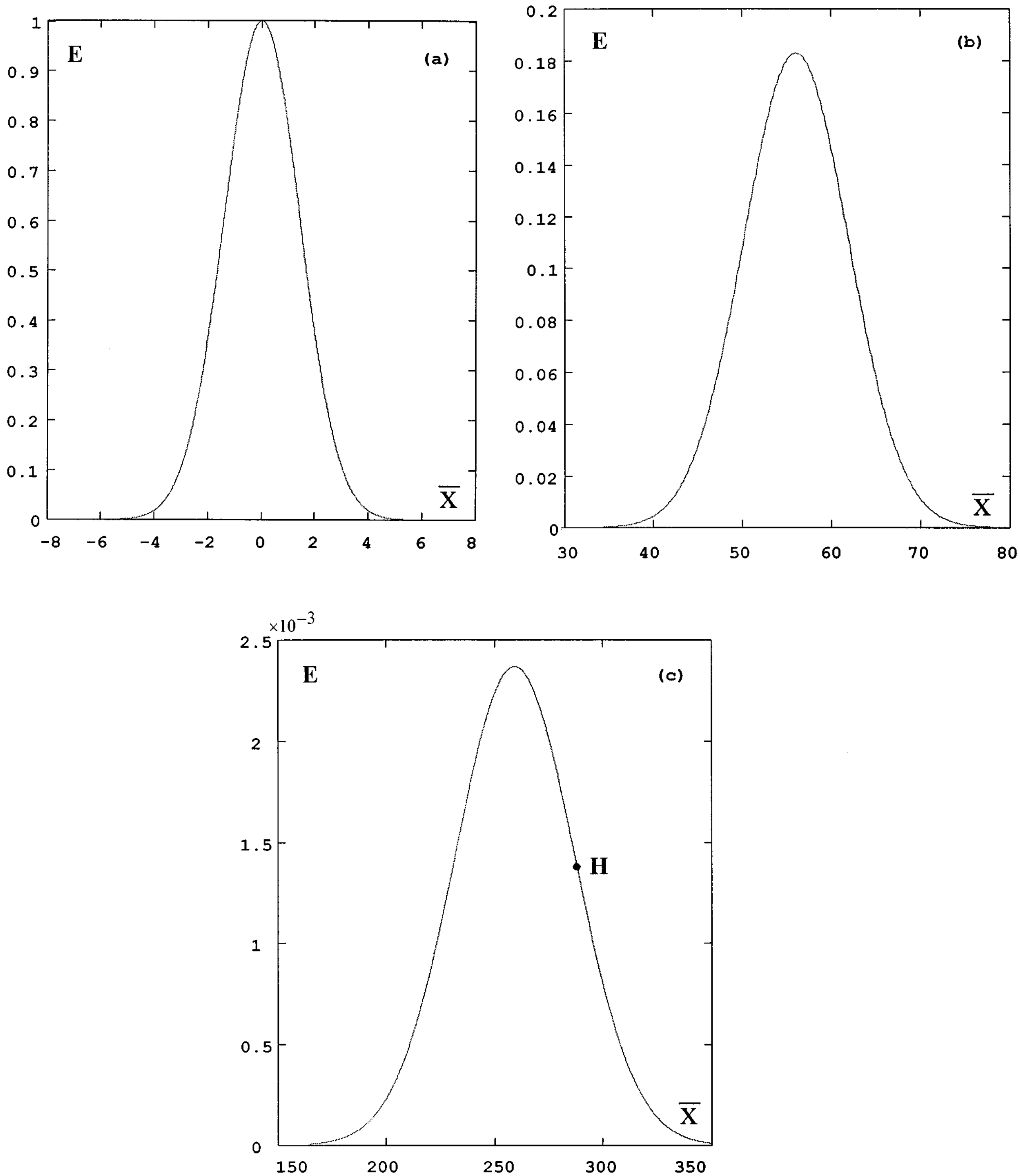

FIG. 1. The spatial envelope of the WP determined by Eqs. (38) and (39) with $a=200$, $N=7$ as a function of the dimensionless coordinate $\overline{x}=\Delta x$ for various values of the dimensionless time variable $\tau=\beta k_c^2 t$: (a) $\tau=0$, (b) $\tau=1$, (c) $\tau=5$. In (c) H is the point in which the real part of the local wave number is equal to the initial carrier wave number k_c and the local group velocity is real.

$$k_r^* = \frac{\partial \vartheta}{\partial x} = k_c \frac{\partial \vartheta}{\partial \xi}, \quad k_i^* = \frac{\partial \psi}{\partial x} = k_c \frac{\partial \psi}{\partial \xi},$$

$$\omega_r^* = -\frac{\partial \vartheta}{\partial t} = -\beta k_c^2 \frac{\partial \vartheta}{\partial \tau}, \quad \omega_i^* = -\frac{\partial \psi}{\partial t} = -\beta k_c^2 \frac{\partial \psi}{\partial \tau}. \tag{45}$$

The quantity k_r^* characterizes an inverse wavelength of the spatial oscillations in the vicinity of the point (x,t), while the quantity k_i^* represents an inverse length of the exponential decay of these oscillations in space in the vicinity of the same point. The quantities ω_r^* and $-\omega_i^*$ have analogous meaning with respect to the WP behavior in time.

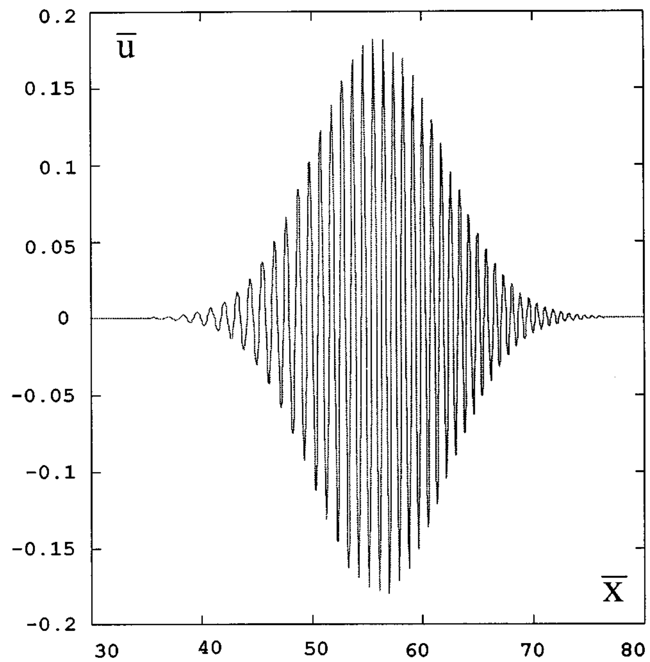

FIG. 2. The spatial oscillations within the Gaussian envelope calculated from the exact solution with $a=200$, $N=7$, and $\tau=1$. The dimensionless coordinate and time are defined as $\overline{x}=\Delta x$ and $\tau=\beta k_c^2 t$.

For large values of t, the local wave number k^* and the local frequency ω^* approach the SP values, $\widetilde{k}$ and $\widetilde{\omega}$, respectively:

$$k^*(x,t)\to\widetilde{k}(x/t), \quad \omega^*(x,t)\to\widetilde{\omega}(x/t)$$
$$(t\to\infty, \ |x/t|<\text{const}), \quad (46)$$

where $\widetilde{k}$ and $\widetilde{\omega}$ are given by

$$\widetilde{k}=\frac{(\alpha+i\beta)x}{2(\alpha^2+\beta^2)t}, \quad \widetilde{\omega}=(\alpha-i\beta)\widetilde{k}^2=\frac{(\alpha+i\beta)x^2}{4(\alpha^2+\beta^2)t^2}. \quad (47)$$

As is seen from Eq. (47), the complex quantities $\widetilde{k}$ and $\widetilde{\omega}$ are constant along the straight-line rays $x/t=C$. These rays are the asymptotic lines for the curvilinear trajectories, along which one of the quantities $k_r^*, k_i^*, \omega_r^*, \omega_i^*$ is constant.

(iv) The trajectories $k_r^*=\text{const}$ determined by the first of Eqs. (45) and by Eq. (41) are as follows:

$$x=X(t)=\frac{2k_c\{[(1+\beta\Delta^2 t)^2+\alpha^2\Delta^4 t^2](k_r^*/k_c)-(1+\beta\Delta^2 t)\}}{\alpha\Delta^4 t}. \quad (48)$$

The exact solution presented here shows that only at $t=0$ is the real part of the local wave number k_r^* spatially uniform and equal to the carrier wave number k_c. For any small $t>0$ the WP acquires an inhomogeneous filling: any given value of the wavelength can be found within the WP. In the far-distance parts of the WP wings, very short local wavelengths appear for $t>0$, so that $k_r^*\to\pm\infty$ when $x\to\pm\infty$. The filling of the WP is not symmetric with respect to its center: any fixed point P from the right wing of the WP is characterized by shorter waves than the symmetric point P' at the left wing (see Fig. 2). The family of trajectories $k_r^*=\text{const}$ in (x,t) plane described by Eq. (48) is shown in Fig. 3. As is seen from this figure, for small times t sufficiently large values of $|k_r^*|$ propagate from the wings to the central part of the WP and after reaching the turning points where $dx/dt=0$ they propagate back to the wings, so that for $t\to\infty$ the velocity of propagation of a given value of k_r^* tends to the real group velocity $W^*=W(k_r^*+i\beta k_r^*/\alpha)$.

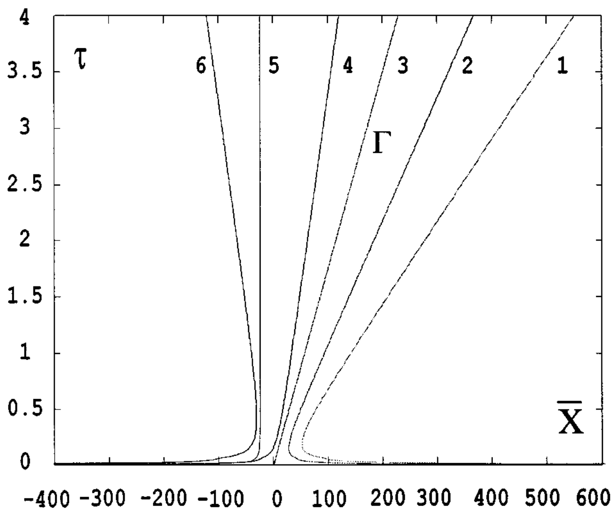

FIG. 3. The space-time trajectories along which the real part of the local wave number k_r^* is constant calculated for various values of the parameter $\kappa_r=k_r^*/k_c$ with $a=200$, $N=7$. The dimensionless coordinate and time are determined as $\overline{x}=\Delta x$ and $\tau=\beta k_c^2 t$. The straight line Γ shown for $\kappa_r=1$ is a Hamiltonian ray transferring the initial carrier wave number. The values of the parameter k_r^* for curves 1–6 are 2.4, 1.6, 1, 0.5, 0, and -0.5, respectively.

The straight-line trajectory Γ corresponding to $k_r^*=k_c$ is exceptional since it represents the genuine Hamiltonian ray,along which the constant value $k_r^*=k_c$ propagates with the real constant group velocity, i.e., $dX_H/dt=W(k^*)$ where the complex number $k^*=(1+i\beta/\alpha)k_c$ is constant. This ray is described by

$$x=X_H(t)=2(k_c/\alpha)[\beta/\Delta^2+(\alpha^2+\beta^2)t]. \quad (49)$$

On the straight line Γ both real and imaginary parts of the local wave number k^* *remain constant for all times* unlike other trajectories shown in Fig. 3, along which only k_r^* is constant while k_i^* varies. Any curvilinear trajectory, for which k_r^* differs from the carrier wave number k_c, approaches some straight-line Hamiltonian ray for large times $t\gg t_\alpha$ where

$$t_\alpha=\frac{|1-k_c/k_r^*|}{(\alpha^2+\beta^2)^{1/2}\Delta^2}. \quad (50)$$

(v) The trajectories $k_i^*=\text{const}$ are given by

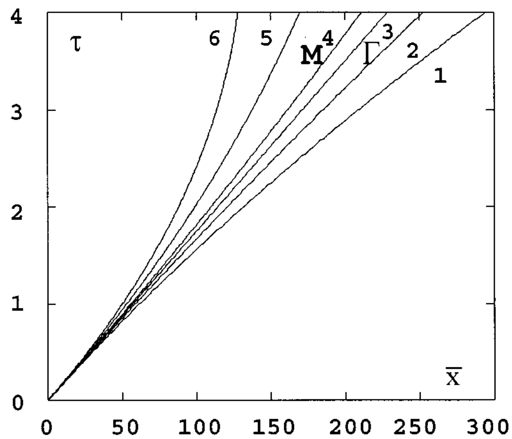

FIG. 4. The space-time trajectories along which the imaginary part of the local wave number k_i^* is constant calculated for various values of the parameter $\kappa_i = k_i^*/k_c$ with $a = 200$, $N = 7$. The dimensionless coordinate and time are defined as $\overline{x} = \Delta x$ and $\tau = \beta k_c^2 t$. The curve M corresponding to $\kappa_i = 0$ is the trajectory of the envelope's maximum. The straight line Γ corresponding to $\kappa_i = 0.005$ is the same Hamiltonian ray as in Fig. 3. The values of the parameter k_i^* for curves 1–6 are 0.024, 0.012, 0.005, 0, -0.012, and -0.024, respectively.

$$x = \hat{X}(t) = \frac{2\alpha k_c t + 2k_i^*[\Delta^{-2} + 2\beta t + (\alpha^2 + \beta^2)\Delta^2 t^2]}{1 + \beta\Delta^2 t}. \tag{51}$$

The family of curves $k_i^* = \text{const}$ is shown in Fig. 4. This family contains, in particular, the trajectory M of the spatial maximum of the WP envelope since $k_i^* = 0$ along M. The curve M intersects various trajectories $k_r^* = \text{const}$ shown in Fig. 3: the central wave number $k_r^*(x_m(t),t)$ decreases continuously with time and tends to zero as $t \to \infty$. Determining the coordinate of the envelope maximum from Eq. (51) with $k_i^* = 0$ and inserting this value of x in the left-hand side of Eq. (48) allows determining the central wave number $k_m(t) = k_r^*(x_m(t),t)$. This results in

$$k_m(t) = k_c/(1 + \beta\Delta^2 t). \tag{52}$$

This formula can be also obtained from the condition for maximum of the absolute value of the spectral function of the WP in k space [see the integrand in the right-hand side of Eq. (36)]. Formula (52) was given in [4] where the lowering of the central wave number with time due to the differential absorption was shown.

The exceptional straight-line trajectory Γ shown in Fig. 3 and described by Eq. (49) is also included in the family given by Eq. (51) slince along Γ the value of k_i^* is constant and equals $k_c\beta/\alpha$. Any trajectory for which k_i^* differs from $k_c\beta/\alpha$, for large times $t \gg t_\beta$, approaches the straight line Hamiltonian ray, whose slope is equal to the real group velocity $W(\alpha k_i^*/\beta + i k_i^*)$. The time t_β is determined by

$$t_\beta = \frac{|1 - \alpha k_i^*/\beta k_c|}{\beta\Delta^2}. \tag{53}$$

As is seen from Fig. 4, the positive values of k_i^*, i.e., the local values of the inverse attenuation length of the WP envelope in the positive x direction, propagate to the right wing of the WP with the velocities that increase with k_i^*. The negative values of k_i^* propagate to the left wing, excluding some initial range of the values $k_i^* < 0$, for which the trajectories $k_i^* = \text{const}$ possess the turning points. If the relative deviations of the quantities k_i^* and k_r^* from their values on the straight-line ray G, i.e., the numerators in Eqs. (53) and (50), are of one and the same order and if, in addition, $\beta \ll \alpha$, then the time t_β is much longer than the time t_α. This means that along the curvilinear trajectories transferring the constant values of k_r^* the difference of the real parts of the local wave number and the SP wave number becomes negligible much faster than the difference of their imaginary parts.

The obtained solution allows one to investigate two main stages of the WP evolution, the short-time (initial) stage and the long-time (asymptotic) stage. The initial stage is characterized by the instant appearance of all possible wavelengths within the WP envelope. The new local waves with the values of k_r^* different from the carrier wave number k_c appear at exponentially small wings of the envelope, propagate to the domain where $k_r^* \approx k_c$ until some critical time $t = t_{\text{cr}}$ and then go away from this domain as is seen in Fig. 3. It must be noted that we are dealing with a linear partial differential equation, therefore, spectral components cannot be created or destroyed. However, different spectral components are differently affected by attenuation, and thus the relative significance of various spectral components can vary in space and time. For $|k_r^*| \gg k_c$ the critical time t_{cr} is given by

$$t_{\text{cr}} \approx \Delta^{-2}(\alpha^2 + \beta^2)^{-1/2}. \tag{54}$$

The first stage of the WP evolution is provided mainly by the dispersive properties of a medium and this stage exists also in the absence of absorption ($\beta = 0$).

The asymptotic stage ($t \gg t_{\text{cr}}$) corresponds to the WP pattern that can be obtained by the SP method. In the limit $t \gg t_{\text{cr}}$ and $|x/t| < \text{const}$, the local complex wave number k^* and the local complex frequency ω^* tend to the SP values $\tilde{k}$ and $\tilde{\omega}$. In the case $\beta \ll \alpha$, which is of interest for studying the propagation of the whistler and helicon WP's, the critical time t_{cr} is of order of the time scale $1/(\alpha\Delta^2)$ of a wave, whose length is of order of the initial width of the WP. For $t \gg t_{\text{cr}}$ each small part of the WP can be characterized by the complex quantity $\tilde{k}$, which keeps a constant value along the ray $x/t = W(\tilde{k})$ whose slope $W(\tilde{k})$ is the real group velocity. Since along the ray $x/t = W(\tilde{k})$ the SP value of the eikonal is complex,

$$\tilde{S} = \tilde{k}x - \tilde{\omega}t = (\Omega + i\Lambda)t,$$

$$\Omega = W^2(\tilde{k})\alpha/4(\alpha^2 + \beta^2), \quad \Lambda = \beta\Omega/\alpha, \tag{55}$$

the time-dependent behavior of any local portion of the WP that could be observed in the frame of reference moving with the real group velocity $W(\tilde{k})$ should be seen as an exponentially decaying oscillation described by the leading factor $\exp(i\tilde{S})$ of the asymptotic solution given by Eq. (24). The nonexponential factor in Eq. (24) varies slowly along the straight line $x/t = W(\tilde{k})$ and behaves as $1/\sqrt{t}$. Since the damping rate Λ is proportional to the frequency Ω, this results in a nonuniform damping of various parts of the WP, so that the high-frequency wings are subjected to the strongest suppression in the process of propagation.

As is seen from the asymptotic formulas (47), for any fixed value of x, the SP wave number and the SP frequency tend to zero when $t \rightarrow \infty$. When the time t increases, the tail part of the signal reaching a given point $x = \text{const} > 0$ is characterized by continuous lowering of both the local group velocity $W = x/t$ and the local frequency that approaches the SP frequency $\tilde{\omega} = W^2/4\alpha$. For $x = \text{const} > 0$ the exact solution (38) considered as a function of time represents a temporal signal filled with the oscillations whose local frequency decreases with time. The form of the temporal envelope is not Gaussian and it is not symmetric even in the absence of absorption. The temporal WP described by the exact solution (38) for two various values of x are shown in Figs. 5(a) and 5(b). The lowering of the local frequency of the oscillations within the temporal envelope is demonstrated in Fig. 6.

Direct calculation of the local complex frequency ω^* from Eqs. (45) and (39)–(41) shows that both real and imaginary parts of ω^* *remain constant for all times* along the Hamiltonian ray Γ defined by Eq. (49). The relation between ω^* and k^* along Γ follows exactly the dispersion equation (33) as it would be for the case of the elementary exponential solution presented by Eq. (3). Unlike the local frequency on the ray Γ, the local frequency at the point of the maximum of the temporal envelope decreases with time along the maximum trajectory. This property is similar to the lowering of the local wave number at the maximum of the Gaussian spatial WP envelope; see Eq. (52).

V. DISCUSSION AND CONCLUSIONS

The main objective of this work is to examine the role and significance of the real group velocity in the WP propagation in an absorbing homogeneous medium. In an absorbing medium the group velocity vector **W** is, generally, complex and it represents some mathematical subject derived from a complex dispersion equation. In spite of the fact that such a complex vector does not admit a physical interpretation, it can be employed for calculations of some important physical characteristics of WP in an absorbing medium. Thus, the velocity of the spatial [4,9] and temporal [4,10] envelopes can be determined in terms of both real and imaginary parts of the vector **W** at the maximum. On the other hand, for an absorbing medium, one can point out the exponential waves of the form (3), for which vector **W** is real due to the special choice of the complex wave vector **k** [5,11,12]. As is shown in the present paper, such a real group velocity does have a physical meaning that comes to light when the asymptotic behavior of the WP is investigated.

An application of the SP method to calculating the asymptotic form of the Fourier integrals, describing the spatial one-dimensional WP in an absorbing media, leads to the condition for the real group velocity in the SP $k = \tilde{k}$. When the dispersion equation is complex, the SP wave number $\tilde{k}$ also is complex and it completely determines the asymptotic forms of the WP and of the Green function as well, Eqs. (25)

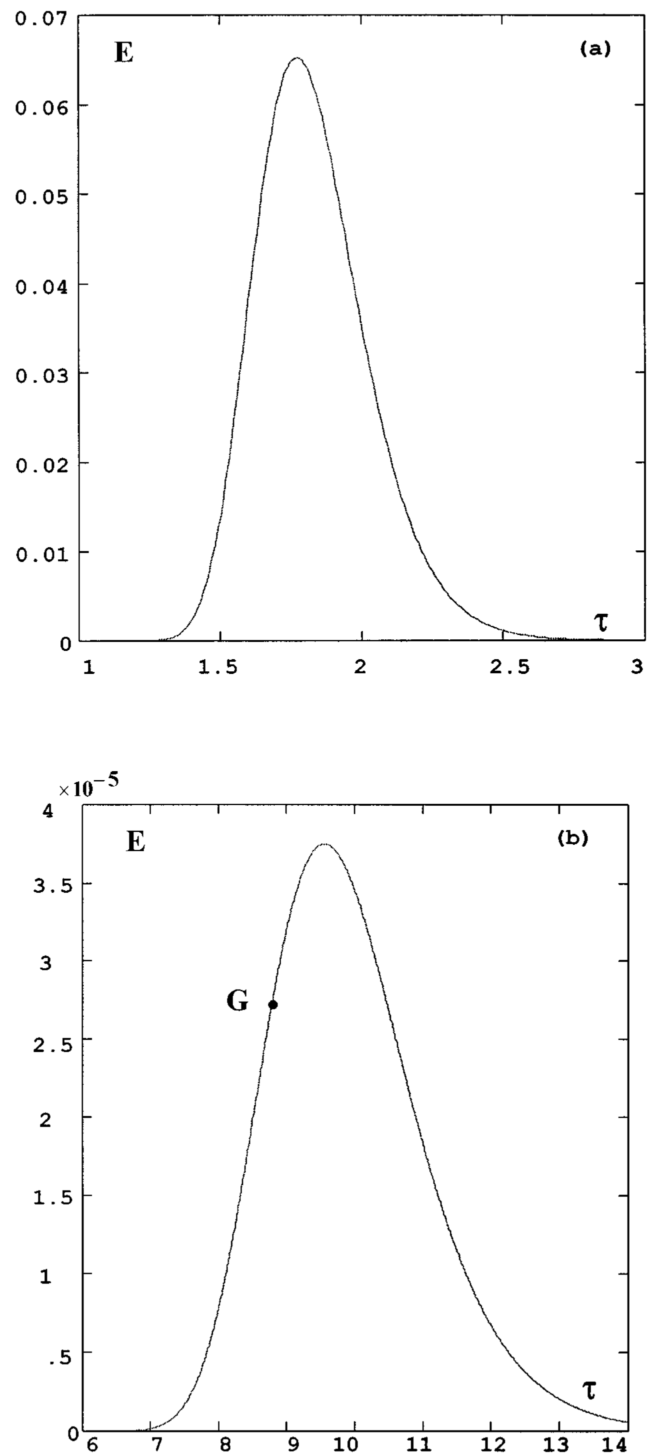

FIG. 5. The temporal envelope of the signal determined by Eqs. (38) and (39) with $a = 200$, $N = 7$ as a function of the dimensionless time $\tau = \beta k_c^2 t$ for various values of the dimensionless distance $\overline{x} = \Delta x$: (a) $\overline{x} = 100$, (b) $\overline{x} = 500$. In Fig. 5(b) G is the point in which the local frequency ω^* provides the real value of the local group velocity.

and (26), respectively. Employing the SP method for studying the asymptotic stage of the WP evolution is not restricted to the case of weak absorption and the SP method can be applied also to the systems for which the absorption effects dominate. For example, this method results in the correct

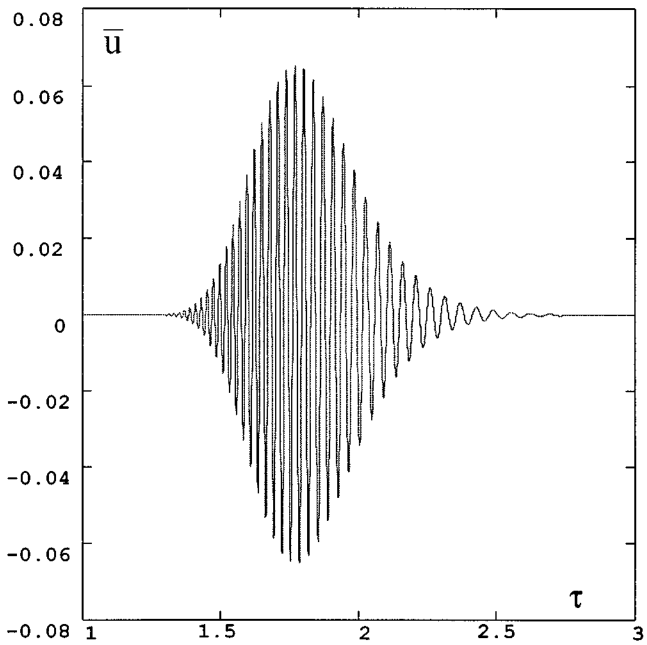

FIG. 6. The temporal oscillations within the temporal envelope calculated from the exact solution with $a=200$, $N=7$, and $\overline{x=\Delta x}=100$. The dimensionless time variable is defined as $\tau=\beta k_c^2 t$.

expression for the asymptotics of the Green function for the heat conduction equation, in which the SP wave number $\tilde{k}$ is purely imaginary.

The important feature of the asymptotic behavior of the WP demonstrated in Sec. III is the fact that despite the presence of absorption, the real group velocity appears as a local characteristic of any small section of a WP through the local complex SP wave number, which depending on the ratio $x/t=W$ only. This leads to the generalization of the concept of the group velocity $W(\tilde{k})$ established by Whitham [3] for nonabsorbing media as the velocity of propagation of the constant wave number $\tilde{k}$. This role of the real group velocity has been found to be universal and applicable to the absorbing media, unlike the sometimes conventional concept of the group velocity as the velocity of the envelope maximum [19], which fails in the presence of absorption.

The analysis of the exact solution for a Gaussian WP propagating in a medium with the quadratic complex law of dispersion performed in Sec. IV has shown quite nontrivial features of the trajectories in the (x,t) plane, on which the real part of the local wave number k_r^* remains constant. As is seen in Fig. 3, for $k_r^* \neq k_c$, where k_c is the initial carrier wave number, the trajectories $k_r^*=\text{const}$ possess turning points and for large times they approach the straight-line Hamiltonian rays characterized by the real values of the group velocity. It has been found that the trajectory Γ on which $k_r^*=k_c$ is a Hamiltonian ray. Thus, for all times the carrier wave number k_c propagates with constant real group velocity. This property is not related to the asymptotic behavior of the WP: here the real local group velocity is manifested from the very beginning $(t=0)$. This result is of crucial importance for the recognition of the role that the real group velocity may play in the propagation of WP.

The explanation of the existence of the straight-line trajectory Γ is as follows: as is seen from Eq. (11), for $t=0$ the filling of the Gaussian WP by the spatial oscillations is uniform: $k_r^*(x,t=0)=k_c$, while the imaginary part of the local wave number k_i^* changes monotonically from $-\infty$ at $x=-\infty$ to ∞ at $x=\infty$ passing through the zero value $k_i^*=0$ at the envelope maximum $(x=0)$. Therefore, in the domain $0<x<\infty$ one can find a point $x=\tilde{x}$, at which $k_i^*=\beta k_r^*/\alpha$. This relationship between k_r^* and k_i^* provides a real group velocity W at $x=\tilde{x}$. The obtained solution indicates that in a homogeneous medium the initial carrier wave number may propagate with the real group velocity at all times if the requirement $\text{Im}W=0$ is satisfied at some point of the WP profile at $t=0$. For arbitrary complex dispersion equation the point with $\text{Im}W=0$ may or may not exist within the initial WP profile. Thus, for a Gaussian WP propagating in a medium that is described by the heat conduction equation, the initial group velocity is complex for all values of x if $k_c \neq 0$. In this and similar cases the real group velocity is revealed only on the asymptotic stage of the WP evolution.

The results of the present work can be compared with those obtained by Muschietti and Dum [4], who applied the SP method to the analysis of the characteristics of the WP envelope's maximum (such as variations of the central velocity and the central wave number). For the whistler WP in a collisional plasma these features of the WP center are reproduced by the exact solution given in Sec. IV of the present paper; see Eqs. (44) and (52). According to Muschietti and Dum [4], studying the real trajectory of the envelope maximum raises the possibility for alternative ray tracing without employment of the rays belonging to the complex world. In this connection it is worthwhile to note that for an absorbing medium the real trajectory of the WP center is not a Hamiltonian ray. On the other hand, the rays along which the SP wave number is propagated, for example, the rays serving as asymptotic lines for the trajectories $k_r^*=\text{const}$, including the straight-line ray Γ (see Fig. 3 of the present paper), are genuine Hamiltonian rays belonging to real (x,t) world.

The velocity of the envelope peak is an important characteristic of the WP. However, in an absorbing medium a peak velocity is not a group velocity due to the fact that the wave group is marked by its wave number or frequency and these quantities change with time at the center of the WP. For large times the group of oscillations with the carrier wave number k_c can be found far away from the WP center. Such a behavior is displayed even for a weak absorption. This is seen from the comparison of the trajectory M of the WP peak and the Hamiltonian ray Γ (Fig. 4). The same phenomenon is reflected by the position of the point H in Fig. 1(c) showing the distance x (for a fixed t) where the carrier wave number can be observed.

To estimate the influence of damping on the typical whistler WP propagating in the Earth's magnetosphere we used the following parameters: the carrier wavelength $\lambda_c=2\pi/k_c=2000$ m, the width of the initial Gaussian envelope $2h=8900$ m, the electron plasma frequency $\omega_{pe}=1.8$ MHz (this value corresponds to the electron density $n_e=1000\ \text{cm}^{-3}$), the electron gyrofrequency $\omega_{ce}=0.26$ MHz (this value corresponds to a magnetic field $B=0.015$ G), and the collision frequency $\nu_e=1.3$ kHz. These parameters provide the values $N=7$, $a=200$ used in calculations of the characteristics of the WP shown in Figs. 1–6.

The distance between the spatial envelope maximum and the position of the carrier wave number marked by point H in Fig. 1(c) is close to 60 km and, according to Eq. (52), the lowering of the central wave number is 9%. For the temporal signal envelope shown in Fig. 5(b), which was calculated for the dimensional distance $x=1100$ km, the time shift between the envelope maximum and the point G corresponding to the carrier frequency $\Omega=7.1$ kHz is close to 0.002 s. Compared to the carrier frequency Ω, the frequency at the envelope maximum drops by 30%.

We may conclude that the real group velocity in absorbing media will appear as the important physical characteristic of the WP and signals, which are intended for the propagation of some prescribed carrier frequency. Such a situation occurs, for example, when a carrier frequency has to be detected by the narrow band receiver at a large distance from the source, and the relevant part of the propagating signal leaves the envelope peak due to the differential absorption.

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[1] J. W. S. Strutt (Lord Rayleigh), *The Theory of Sound*, Vol. 1, 2nd. ed. (Dover, New York, 1945).

[2] L. B. Felsen and N. Markuvitz, *Radiation and Scattering of Waves*, Vol. 1 (Prentice-Hall, Englewood Cliffs, NJ, 1973).

[3] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley and Sons, New York, 1974).

[4] L. Muschietti and C. T. Dum, Phys. Fluids B **5**, 1383 (1993).

[5] K. Suchy, Proc. IEEE **62**, 1571 (1974).

[6] J. A. Bennett, Proc. IEEE **62**, 1577 (1974).

[7] K. J. Budden, *The Propagation of Radio Waves* (Cambridge University Press, New York, 1985).

[8] A. Bravo-Ortega and A. H. Glasser, Phys. Fluids B **3**, 529 (1991).

[9] K. A. Connor and L. B. Felsen, Proc. IEEE **62**, 1586 (1974).

[10] M. Tanaka, M. Fujiwara, and H. Ikegami, Phys. Rev. A **34**, 4851 (1986).

[11] D. Censor, Int. J. Heat Mass Transf. **21**, 813 (1978).

[12] D. Censor and J. J. Gavan, IEEE Trans. Electromagn. Compat. **31**, 262 (1989).

[13] R. Tagg, W. S. Edwards, and H. L. Swinney, Phys. Rev. A **42**, 831 (1990).

[14] H. Pouverlein, Phys. Rev. **128**, 956 (1962).

[15] N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973).

[16] E. A. Kaner and V. G. Skobov, *Plasma Effects in Metals. Helicon and Alfven Waves* (Taylor & Francis, London, 1971).

[17] I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press, New York, 1980), formula No. (3.923).

[18] The Staff of the Bateman Manuscript Project (Director A. Erdélyi) *Tables of Integral Transforms*, Vol. 1 (McGraw-Hill, New York, 1954), formula No. (7.1.6).

[19] J. D. Jackson, *Classical Electrodynamics* (Wiley and Sons, New York, 1962).

[20] A. Friedman, *Partial Differential Equations of Parabolic Type* (Prentice-Hall, Englewood Cliffs, NJ, 1964).