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Linear response theory applied to stochastic resonance in models of ensembles of oscillators

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We apply linear response theory to a parallel ensemble of stochastic resonators. We show that for a large number of elements the system can be used to process broadband signals without frequency distortions. Both conventional stochastic resonance and aperiodic stochastic resonance are studied. [S1063-651X(97)50407-7]

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The phenomenon of stochastic resonance (SR) [1] is a bright example of the constructive role that noise might play in the nonlinear world: the response of a wide class of nonlinear systems to a weak periodic input signal can be optimized by including a certain noise level. The practical significance of this effect finds applications in a wide variety of physical and biological systems as has been underlined by a series of meetings [2] and by an increasing number of publications [3].

Often, the input signal is a single frequency, however, recently input signals with more complicated structure became of interest. For example, SR for both quasiperiodic [4] and narrow-band noisy signals has been considered [5,6]. Recently aperiodic stochastic resonance (ASR) has been introduced by Collins *et al.* [7] using the input-output cross-correlation function rather than the output signal-to-noise ratio (SNR) as a measure.

On the other hand, it has been shown that SR can be significantly enhanced if, instead of a single stochastic resonator, an array of coupled [8–11] or uncoupled [12] resonators is taken. In the case of an uncoupled parallel array of resonators convergent on a summing center, *stochastic resonance without tuning* has been observed [13]. The characteristic maximum in the SNR curve disappears as the shape becomes broadened with an increasing number of elements. Thus the collective response of such a summing network is optimized for any arbitrary noise level larger than some small value. Recently ASR and SR without tuning have been discussed in terms of ''stochastic linearization'' [14].

Previous theoretical studies have shown that single element, single frequency, or conventional, SR can be correctly described in terms of the linear response theory (LRT) [5,15] for weak signals that can be also noisy [16]. In the present Rapid Communication we apply LRT to both ASR and a parallel array. In contrast to previous studies [7,13,14], where one single realization of the aperiodic input signal was used, we assume a stationary stochastic process and make ensemble averages.

We start with ASR. Consider that a nonlinear stochastic system with coordinate $x(t)$ has susceptibility $\chi(\omega,D)$, where ω is the frequency and D the noise intensity. The function $\chi(\omega,D)$ contains all the information on the response of the system to a weak input signal, and its dependence on D should show a maximum indicating SR. As the input of the system, we take a weak noisy signal $s(t)$, which is a *stationary Gaussian process*. Let also both $s(t)$ and $x(t)$ be zero valued.

We aim to calculate cross-correlation measures between the input and the output of the system. An appropriate measure is the coherence function [17,10,18] $\Gamma(\omega)$, obtained from $s(t)$ and $x(t)$ for a given frequency:

$$\Gamma^2(\omega)=\frac{|G_{sx}(\omega)|^2}{G_{ss}(\omega)G_{xx}(\omega)}. \tag{1}$$

In Eq.(1), $G_{sx}(\omega)$ is the cross-spectral density, $G_{ss}(\omega)$ and $G_{xx}(\omega)$ are the spectral densities of the input and the output, respectively. The covariance $C_0=\langle s,x\rangle$ and the correlation coefficient $C_1=C_0/\sqrt{\langle x^2\rangle\langle s^2\rangle}$, used previously [7,14], can easily be obtained from the spectral densities as $C_0=\int_0^\infty \mathrm{Re}G_{sx}(\omega)d\omega$ and $C_1=C_0[\int_0^\infty G_{ss}(\omega)d\omega\int_0^\infty G_{xx}(\omega)d\omega]^{-1/2}$.

For a weak Gaussian signal $s(t)$ the spectral density at the output and the input-output cross-spectrum are defined according to LRT as

$$G_{xx}(\omega)=G_{xx}^{(0)}(\omega)+|\chi(\omega)|^2G_{ss}(\omega), \tag{2}$$

$$G_{sx}(\omega)=\chi(\omega)G_{ss}(\omega), \tag{3}$$

where $G_{xx}^{(0)}(\omega)$ is the spectral density of the system in the absence of the input signal. Substituting Eq. (2) into Eq. (1) we obtain $\Gamma(\omega)$ as

$$\Gamma^2(\omega)=1-\frac{G_{xx}^{(0)}(\omega)}{G_{xx}^{(0)}(\omega)+|\chi(\omega)|^2G_{ss}(\omega)}. \tag{4}$$

As a particular example, we consider here an overdamped symmetric double-well oscillator with white noise. The input signal $s(t)$ is a Gaussian colored noise with correlation time $\tau=1/\gamma$ and standard deviation Q. The model is thus described by the stochastic differential equation (SDE):

$$\dot{x}=x-x^3+\sqrt{2D}\xi(t)+s(t), \tag{5}$$

where $\langle\xi(0)\xi(t)\rangle=\delta(t)$ and $G_{ss}(\omega)=\gamma Q/(\gamma^2+\omega^2)$. We use here a simple but reasonable single exponent approximation. Accounting for only the intrawell motion [5], the unperturbed spectral density $G_{xx}^{(0)}(\omega)$ and the susceptibility are

$$G_{xx}^{(0)}(\omega)=\frac{\langle x_0^2\rangle\lambda_m}{\lambda_m^2+\omega^2}, \tag{6}$$

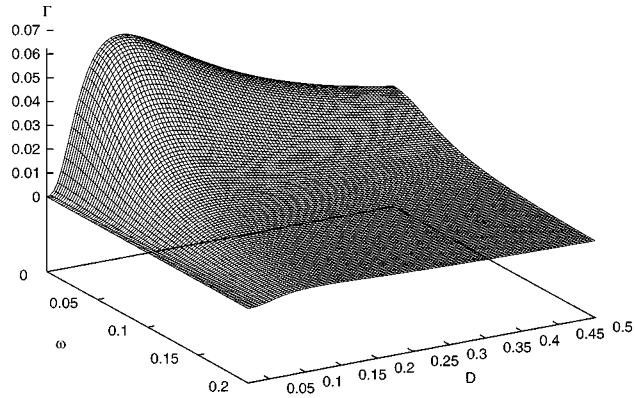

FIG. 1. The coherence function, Eq. (8), for ASR given by Eq. (5) with $\gamma=0.05$ and $Q=10^{-3}$.

$$\chi(\omega)=\frac{1}{D}\frac{\lambda_m\langle x_0^2\rangle}{\lambda_m^2+\omega^2}(\lambda_m-\mathrm{i}\omega). \quad (7)$$

In Eq. (6), $\lambda_m=(\sqrt{2}/\pi)\exp(-1/4D)$ is the smallest nonvanishing eigenvalue of the corresponding Fokker-Planck operator [5] and refers to the Kramers rate [19], and $\langle x_0^2\rangle$ is the stationary value of the second moment, both for the unperturbed system. Finally for the coherence function, we obtain

$$\Gamma^2(\omega)=1-\left(1+\frac{\lambda_m\langle x_0^2\rangle}{D^2}\frac{Q\gamma}{(\gamma^2+\omega^2)}\right)^{-1}. \quad (8)$$

The dependence of Γ on the frequency and noise intensity is presented in Fig. 1, where the phenomenon of ASR can be clearly seen: for any frequency the coherence function possesses a maximum at an optimal noise level. The maximum is most pronounced at low frequencies, since the signal is low frequency. The correlation coefficient,

$$C_1=\left[\left(\frac{\lambda_m}{\gamma+\lambda_m}\right)\left(1-\frac{D^2(\gamma+\lambda_m)}{D^2(\gamma+\lambda_m)+Q\lambda_m\langle x_0^2\rangle}\right)\right]^{1/2}, \quad (9)$$

is shown in Fig. 2. We note very reasonable correspondence between theoretical and numerical results. Therefore for weak Gaussian signals ASR can be described in terms of LRT in the same way as conventional SR.

Let us now turn to the ensemble of N uncoupled SR elements as considered in Refs. [12,13]. Each element is characterized by its state variable $x_k(t)$ and susceptibility $\chi_k(\omega)$ and has its own statistically independent internal noise with intensity D. The inputs of the elements are subjected to the same weak signal $s(t)$, and their outputs are summed:

$$x_M(t)=\frac{1}{N}\sum_{k=1}^{N}x_k(t). \quad (10)$$

The spectral density of the summed output $G_{MM}(\omega)$ is

$$G_{MM}(\omega)=\frac{1}{N^2}\left[\sum_{k=1}^{N}G_{kk}(\omega)+\sum_{k=1}^{N}\sum_{\substack{m=1\\k\neq m}}^{N}G_{km}(\omega)\right], \quad (11)$$

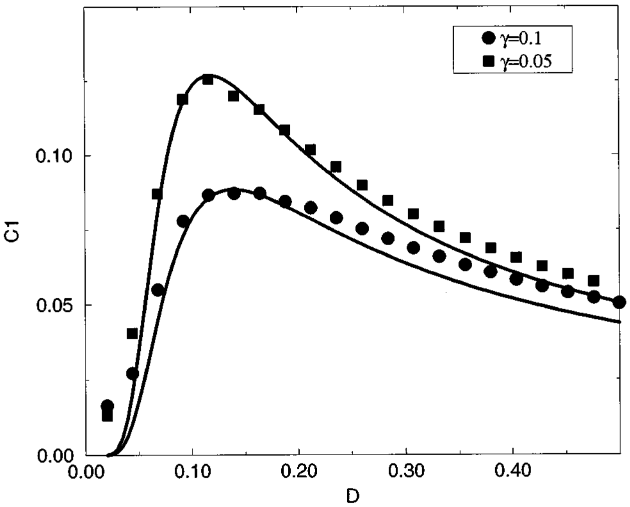

FIG. 2. The correlation coefficient as a function of noise intensity for two values of the signal's correlation time γ^{-1}, and for $Q=10^{-3}$. Results of the numerical simulations are shown by the symbols while theoretical curves, Eq. (9), are shown by solid lines.

where G_{kk} is the spectral density of kth element, and G_{km} is the cross-spectral density of the kth and mth elements. Note that in the absence of the input signal the cross-spectrum disappears from Eq. (11), so that all cross correlations in the ensemble are due to $s(t)$ only. In the approximation of LRT the spectral density of the kth element is

$$G_{kk}(\omega)=G_{kk}^{(0)}+|\chi_k(\omega)|^2G_{ss}(\omega), \quad (12)$$

where $G_{kk}^{(0)}(\omega)$ is the spectral density of the unperturbed kth element. For the cross spectra we obtain [17]

$$G_{km}(\omega)=\chi_k^*(\omega)\chi_m(\omega)G_{ss}(\omega), \quad (13)$$

where $*$ means complex conjugate. For the cross spectrum $G_{sM}(\omega)$ between $s(t)$ and the summed output $x_M(t)$ we obtain

$$G_{sM}(\omega)=\frac{G_{ss}(\omega)}{N}\sum_{k=1}^{N}\chi_k(\omega). \quad (14)$$

Now we are able to calculate all cross-correlation measures. In order to show explicitly the behavior of the ensemble with an increase of the number of elements, we consider the simplest case of *identical elements* with $\chi(\omega)\equiv\chi_k(\omega)$ and $G_{xx}^{(0)}(\omega)\equiv G_{kk}^{(0)}(\omega)$, for which we obtain

$$G_{MM}(\omega)=\frac{1}{N}G_{xx}^{(0)}(\omega)+|\chi(\omega)|^2G_{ss}(\omega), \quad (15)$$

$$G_{sM}(\omega)=\chi(\omega)G_{ss}(\omega). \quad (16)$$

From Eq. (15) we can immediately understand the behavior of the summed output with increases of N. In the limit of large N, the first term in Eq. (15), which refers to internal fluctuations within the elements, disappears, and the whole ensemble behaves as a linear system with the transfer function $\chi(\omega)$. For the coherence function between the input and the summed output we get

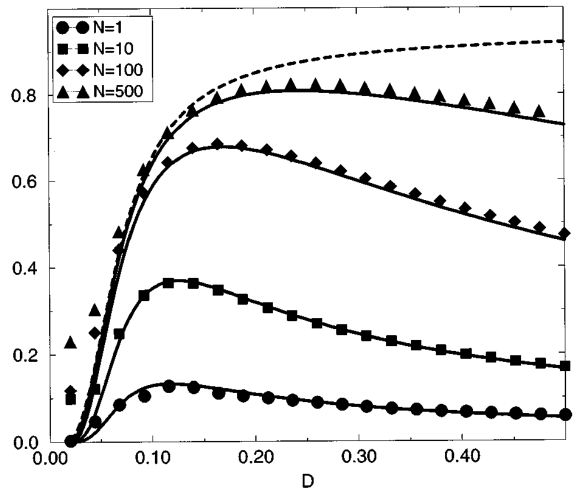

FIG. 3. The correlation coefficient as a function of noise intensity for the ensemble of overdamped bistable oscillators for different numbers of elements for $\gamma=0.05$, $Q=10^{-3}$. Theoretical curves from Eq. (20) are shown by solid lines. The limiting value, Eq. (21), is shown by the dashed line.

$$\Gamma^2(\omega)=\frac{|\chi(\omega)|^2 G_{ss}(\omega)}{G_{xx}^{(0)}(\omega)/N+|\chi(\omega)|^2 G_{ss}(\omega)}. \tag{17}$$

For large N the coherence function tends to 1, i.e., to the most coherent state, as must be always true for an equivalent linear system. Moreover, in Eq. (17), only $G_{xx}^{(0)}(\omega)$ and $\chi(\omega)$ depend on the noise intensity. Clearly, as N becomes large, the noise dependence of the numerator-to-denominator ratio nearly cancel. This explains the broadening of the typical SR characteristic and appearance SR without tuning.

Let us now consider conventional SR, where $s(t)$ is the sum of a weak Gaussian noise $n(t)$ and a weak periodic function $A\sin(\Omega t+\phi)$. We will average over a random phase ϕ in order to ensure that the process is stationary [5]. The appropriate measure in this case is the ratio of the SNR at the input to the SNR at the output. The SNR at the input is simply $\mathrm{SNR}_{\mathrm{in}}=A^2/G_{nn}(\Omega)$, where $G_{nn}(\omega)$ is the spectral density of the noisy part of the signal. The output SNR can be easily obtained from Eq. (15) yielding for the ratio

$$\eta=\frac{\mathrm{SNR}_{\mathrm{out}}}{\mathrm{SNR}_{\mathrm{in}}}=1-\frac{G_{xx}^{(0)}(\Omega)}{G_{xx}^{(0)}(\Omega)+N|\chi(\Omega)|^2 G_{nn}(\Omega)}. \tag{18}$$

This ratio is less than 1 [16], unless N tends to infinity. Thus for large N the dependence of the SNR at the summed output on the noise intensity disappears in the same way as for ASR. But it might be useful for experimentalists to estimate the number of elements N in the ensemble that are needed to obtain a given value of η. From Eq. (18) we obtain

$$N=\frac{G_{xx}^{(0)}(\Omega)}{|\chi(\Omega)|^2 G_{nn}(\Omega)}\frac{\eta}{1-\eta}, \tag{19}$$

which is closely akin to Eq. (4) in Ref. [13]. We again note that in the limit $N\to\infty$, $\mathrm{SNR}_{\mathrm{out}}\to\mathrm{SNR}_{\mathrm{in}}$ as it should for a linear system.

As is well known, SR is often more pronounced for low-frequency signals. This feature is determined by the susceptibility $\chi(\omega)$. In practice it means that the low-frequency domain of the signal will be processed better than the high-frequency one and as a result we will get frequency distortions at the output. In this view Eq. (17) is of practical importance, because for large N the frequency dependence disappears in the same way as the dependence on the noise intensity. Therefore a parallel array of SR elements can be used for processing of broadband signals.

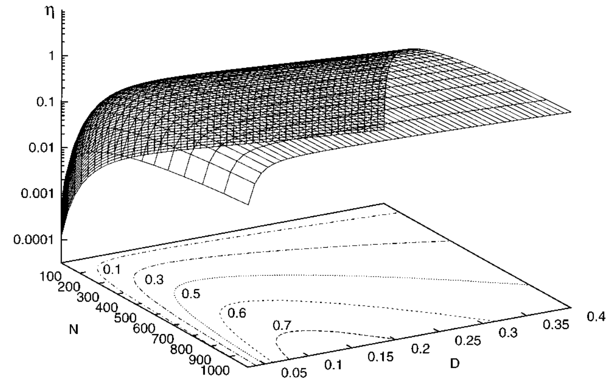

FIG. 4. The ratio of SNRs, $\eta(N,D)$, Eq. (18), for the ensemble of overdamped bistable oscillators; $Q=10^{-3}$, $\gamma=1.0$, $A=0.1$, $\Omega=0.1$. The contour lines for indicated values of η are given by Eq. (19).

Now we apply the general theory described above to the particular example of an ensemble of identical bistable noisy elements. Each element is described by the SDE Eq. (5). For the correlation coefficient C_1, we obtain

$$C_1=C_1^{\infty}\left[1-\frac{D^2(\gamma+\lambda_m)}{D^2(\gamma+\lambda_m)+NQ\lambda_m\langle x_0^2\rangle}\right]^{1/2}, \tag{20}$$

$$C_1^{\infty}=\left(\frac{\lambda_m}{\gamma+\lambda_m}\right)^{1/2}. \tag{21}$$

In the limit $N\to\infty$, C_1 tends to its limiting value C_1^{∞}, which still, however, depends on the noise intensity. But this dependence vanishes for slow input signals, $\gamma\ll\lambda_m$. The results are shown in Fig. 3. We again note good correspondence between theory and numerical simulations.

The results for conventional SR are summarized in Fig. 4. The input signal in this case is the sum of a periodic function and a random noise: $s(t)=A\sin(\Omega t+\phi)+n(t)$, where $n(t)$ is a Gaussian stationary stochastic process with spectral density $G_{nn}(\omega)=Q\gamma/(\omega^2+\gamma^2)$. The input-output SNR ratio, η, from Eq. (18), is shown as a function of both noise intensity and number of elements. The contour lines show that the dependence $N(D)$ for a given value of the ratio η possesses a minimum that determines the lowest possible number of elements necessary to obtain a given value of η.

In conclusion, we have presented a general theory based on LRT for ensembles of stochastic resonators. This theory applies to both aperiodic and conventional SR. We have shown that the collective response to a weak input signal of an ensemble of stochastic resonators, acting in parallel, is closely akin to that of an equivalent linear system, only in the limit of large number of elements. The theory also predicts that parallel arrays of SR elements can be used to process broadband signals without distortions at the output. We underline that the presented theory can be applied to a wide range of systems provided that the susceptibility of an element of the system is known.

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