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Complete and exact solutions of a class of nonlinear diffusion equations and problem of velocity selection

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In this Rapid Communication complete and exact solutions of a class of nonlinear diffusion equations have been presented. The exact solutions give a tutorial explanation about the mechanism of velocity selection. The marginal stability hypothesis is extended to predict velocity selection for the equations. It has been shown that this class of equations can be transformed to the heat equation via nonlinear transformations. Numerical experiments have been performed to test the theoretical prediction. [S1063-651X(97)51411-5]

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In exploration of dissipative systems far from equilibrium [1,2], one of the challenging problems concerns the selection mechanism: what kinds of evolving velocity and emerging pattern would be selected in a kinetic process when the system is suddenly quenched into an unstable state. This nonequilibrium problem shares a common ground with physical kinetics, chemical reaction and living phenomena [3–13]. The difficulties with this problem originate from the specific nonlinearities in dissipative systems. When a nonlinear system loses its stability, a set of possible states of the system would appear. In practice, however, one and only one state is realized. Therefore one needs to find a criterion that allows one to judge which state would be finally selected in a dissipative process. Although the search for such a selection mechanism has a long history in studies of crystal growth [3], recent interest and developments in this topic have been directly stimulated by rigorous mathematical studies done in the 1970's about a specific kind of nonlinear diffusion equations [14–16].

In Aroson and Weinberger's work on the Fisher-type nonlinear diffusion equation [14]

$$\partial\phi/\partial t-\partial^2\phi/\partial x^2=f(\phi)\,, \tag{1}$$

$$f(\phi)\in C^1[0,1];\quad f(\phi)>0,\forall\phi\in(0,1);$$

$$f(0)=f(1)=0;\quad f'(0)>0;f'(1)<0\,,$$

it has been rigorously proved, based on the comparison principle on parabolic operators, that there exists a distinct selection mechanism in this Fisher-type equation. In plain physical language the main results obtained by Aroson and Weinberger are the following: (i) There exists a unique critical velocity C^* for Eq. (1), such that for any $C>C^*$ Eq. (1) admits a kink-type solitary wave solution; (ii) the system is extremely insensitive to the initial conditions; during the dynamic process all irrelevant modes of initial perturbations quickly dissipate, while relevant modes survive and are amplified; (iii) almost any physical disturbance of a nonlinear diffusion system governed by Eq. (1) develops into a kink-type solitary wave with a critical velocity C^*; the front with speed C^* is an attractive state of the time evolution, which is automatically selected by the system itself; (iv) this critical velocity can be estimated by $2\sqrt{f'(0)}\leqslant C^*\leqslant 2\sqrt{\mathrm{Sup}_{\phi\in(0,1)}[f(\phi)/\phi]}$. If $f(\phi)$ is a concave function, the estimation actually gives an exact expression for the critical velocity. All of these points are physically significant, in particular the selection of the asymptotic speed C^*. This characteristic quantity C^*, in a sense, plays a similar role for describing kinetic dissipative systems far from equilibrium as the critical exponents do for phase transitions in equilibrium.

A conventional wisdom is to predict the selected velocity and other basic quantities without solving any tough nonlinear equations. Inspired by the mathematical studies, physicists have proposed a few scenarios [17–21] regarding the selection mechanism on some nonlinear equations. Dee and Langer [17], based on heuristic arguments, made a step towards obtaining a criterion of the marginal stability. According to their arguments in simple cases the selected velocity and wave number can be determined via analyzing the dispersion relationship of the corresponding equations, and the selected velocity is basically represented by the smallest phase velocity. Since then several different scenarios including the structure stability principle [20], and the variational principle [21], have been proposed in addition to the extensive discussions about marginal stability [18,19].

As nonlinearity is a subtle subject, finding such a selection principle depends to a large extent on a detailed knowledge of the corresponding equations. A completely and exactly solvable example of a nonlinear partial differential equation which could be related to the selection mechanics is indispensable. First of all, such an example would shed light on where and how a selection event emerges. Second, it could be used to either illustrate or to judge the validity of a proposed selection principle. Third, it would stimulate the development of a more general principle of selection for a wider class of equations. The purpose of this Rapid Communication is to offer such examples of nonlinear diffusion equations, and to discuss the problem of velocity selection.

We begin with the following nonlinear diffusion equation studied by Sattinger [16]:

$$\partial\phi/\partial t-\partial^2\phi/\partial x^2=g(\partial\phi/\partial x,\phi)\,, \tag{2}$$

where $g[(\partial\phi/\partial x),\phi]$ is a smooth function of $\partial\phi/\partial x$ and ϕ. Equation (2) is a generalization of Eq. (1). By defining the weighted function and weighted norms, Sattinger proved a general stability theorem regarding wave front solutions of Eq. (2). As a particular example of Eq. (2), let us consider the following modified Fisher equation

$$\frac{\partial\phi}{\partial t}-\frac{\partial^2\phi}{\partial x^2}-\frac{m}{1-\phi}\left(\frac{\partial\phi}{\partial x}\right)^2=\phi(1-\phi), \tag{3}$$

where m is a parameter. When $m=0$, Eq. (3) becomes the standard Fisher equation. The third term on the left hand side of Eq. (3) represents a modification to the diffusion. It will be seen shortly that by adding this term into the standard Fisher equation, Eq. (3) acquires an important inherent symmetry. This equation has applications to real systems such as bacteria colony growth [10]. If one looks for a traveling wave solution, $\phi=\phi(x-ct)\equiv\phi(\xi)$, Eq. (3) becomes

$$-c\frac{d\phi}{d\xi}-\frac{d^2\phi}{d\xi^2}-\frac{m}{1-\phi}\left(\frac{d\phi}{d\xi}\right)^2=\phi(1-\phi). \tag{4}$$

In a mechanical analogy, this ordinary differential equation describes the motion of a particle in the potential $-\int\phi(1-\phi)d\phi$ with the resistance forces proportional to the velocity as well as to the velocity squared. The state $\phi=0$ is unstable, whereas the state $\phi=1$ is stable.

By letting $\phi=1-u$, Eq. (3) becomes

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+\frac{m}{u}\left(\frac{\partial u}{\partial x}\right)^2=u(u-1). \tag{5}$$

One of the crucial characteristics of this nonlinear partial differential equation is that the basic form of the differential operator on the left hand side of Eq. (5) remains invariant under the operation of any power transformation. Actually if we make the transformation $u=v^{1/\alpha}$ for Eq. (5) where α is a real number, then Eq. (5) becomes

$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}+\frac{(\alpha+m-1)/\alpha}{v}\left(\frac{\partial v}{\partial x}\right)^2=\alpha v(v^{1/\alpha}-1). \tag{6}$$

This means that after performing the power transformation, the differential operator on the left hand side of Eq. (6) shows the same algebraic structure as that of Eq. (5). It is this symmetry observation that suggests that Eq. (5) can be exactly linearized for the case $m=2$ by using the following transformation:

$$u=v^{-1}. \tag{7}$$

By virtue of transformation (7), Eq. (5) is reduced to a linear diffusion equation [22]

$$\partial v/\partial t-\partial^2 v/\partial x^2=v-1. \tag{8}$$

In the case of separable variables, Eq. (8) can be exactly solved. Therefore the most general solution of Eq. (3) for the case $m=2$ is

$$\phi=1-1/\left(1+\sum_i A_{i1}\exp\{\sqrt{\omega_i-1}[x+(\omega_i/\sqrt{\omega_i-1})t]\}+A_{i2}\exp\{-\sqrt{\omega_i-1}[x-(\omega_i/\sqrt{\omega_i-1})t]\}\right), \tag{9}$$

where A_{i1} and A_{i2} are integration constants and ω_i is an arbitrary constant which represents an eigenvalue of Eq. (8). In order to guarantee that ϕ is a bounded real number, the conditions $A_{i1}\geqslant 0$ and $A_{i2}\geqslant 0$ are required.

This general solution provides a great deal of information about Eq. (3). First of all, it is easy to see from Eq. (9) that

$$\phi=1-\frac{1}{1+e^{\sqrt{\omega-1}x+\omega t}+e^{-\sqrt{\omega-1}x+\omega t}} \tag{10}$$

is a special solution of Eq. (3). When $\omega>1$, this solution expresses a pair of wave fronts which propagate with a velocity

$$c=\frac{\omega}{\sqrt{\omega-1}} \tag{11}$$

in the opposite directions to each other. In this way the system evolves to the stable state $\phi=1$ from the unstable state $\phi=0$. Figure 1 shows a computer simulation of this wave phenomenon. The numerical calculations illustrate that the wave propagates with a definite velocity $c=2$. However, when $\omega<0$, solution (10) indicates that any disturbance will decay rapidly; so there will be no wave propagation. Furthermore, for any pair of values $\omega_1>1$ and $\omega_2>1$, the following

$$\phi_1=1-\frac{1}{1+A_1\exp[-\sqrt{\omega_1-1}(x-c_1t)]},\quad c_1\equiv\frac{\omega_1}{\sqrt{\omega_1-1}}, \tag{12}$$

$$\phi_2=1-\frac{1}{1+A_2\exp[-\sqrt{\omega_2-1}(x-c_2t)]},\quad c_2\equiv\frac{\omega_2}{\sqrt{\omega_2-1}}, \tag{13}$$

and

$$\phi_3=1-1/\{1+A_1\exp[-\sqrt{\omega_1-1}(x-c_1t)] +A_2\exp[-\sqrt{\omega_2-1}(x-c_2t)]\} \tag{14}$$

are the exact wave front solutions of Eq. (3). Solution (14), ϕ_3, contains the two eigenmodes of Eq. (8), which could be regarded as a ‘‘nonlinear combination’’ of solutions (12) and (13). A natural question is this: which exact solution is more probable, ϕ_1, ϕ_2, or ϕ_3? The answer, however, is very clear. If $c_1<c_2$, the last term in the denominator of Eq. (14) will be much smaller than the other terms in the asymptotic limit; therefore solution (14) will quickly evolve into solution (12). The role of wave fronts with a faster velocity will be gradually diminished in the kinetic process. In other words, the slower the wave travels, the longer the wave survives. Similarly, the general solution (9) contains the multi-eigenmodes of Eq. (8), which may be considered as a ‘‘non-

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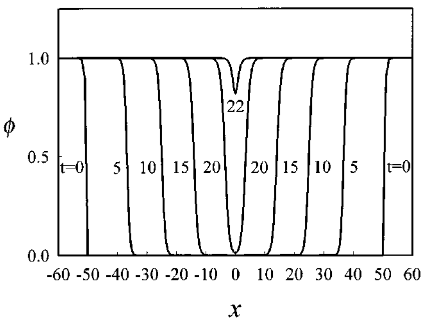

FIG. 1. Numerical simulation of Eq. (3) for $m=2$, exhibiting a pictorial representation of the exact solution (10). The two fronts propagate with velocity $C=2$ in the opposite directions, and gradually disappear after collision.

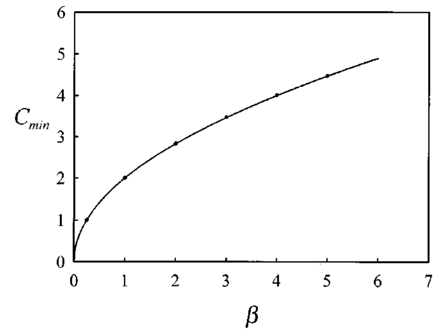

FIG. 3. The numerical calculations on the selected velocity C_{min} versus the parameter β. The solid line is the theoretical prediction by Eq. (19), and the filled circles represent the numerical results on Eq. (17).

linear combination'' of multisolitary wave solutions; each with a different velocity. The same arguments applied to the general solution (9) lead to the conclusion that only the front with the smallest velocity will survive in the asymptotic process. By using expression (11), this smallest velocity is $c_{min}=2$. Therefore the front with the smallest velocity is naturally selected in the kinetic process. This example illustrates some basic points of the selection process and suggests that the selection mechanism is the competition between the nonlinear eigenmodes. When the system is suddenly driven to an unstable state, a large amount of nonlinear modes are excited. The number of these modes may be innumerably infinite, and each of them has a different decaying rate. These modes interact with each other in a dissipative environment. The eigenmodes with a larger decaying rate are depressed. Finally only the mode with the smallest decaying rate survives in a nonlinear limit. It is this lowest decay mode that determines the selected velocity. This picture on the velocity selection is, in fact, consistent with the corresponding conclusions in the mathematical literature [14–16] and the considerations of marginal stability [17,18].

Another interesting point for Eq. (8) is that if $v=1+\exp(t)w(x,t)$, Eq. (8) will become the heat equation $\partial w/\partial t-(\partial^2 w/\partial x^2)=0$. This shows that for the case $m=2$, the modified Fisher equation (3) is transformed into the heat equation by the following transformation:

$$\phi=\exp(t)w(x,t)/1+[\exp(t)w(x,t)]. \tag{15}$$

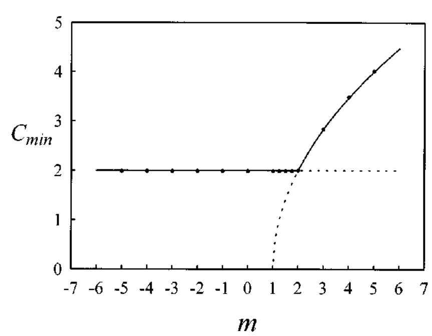

FIG. 2. The numerical calculations on the selected velocity C_{min} vs the parameter m. The solid line is the theoretical prediction by Eq. (16), and the filled circles represent the numerical results on Eq. (3).

Therefore all the classical solutions of the modified Fisher equation can be exactly recovered. This situation is similar to that of the Burgers equation which can be transformed to the heat equation by the Cole-Hopf transformation [22]. This is another example of a nonlinear partial differential equation which can be exactly and completely solved.

When $m\neq 2$, to the best of our knowledge, no exact and complete solutions of Eq. (3) can be found. Although the above exact solvable example and the proposed selection mechanism support the basic points of the marginal stability hypothesis, the correct selected velocity cannot be predicted successfully by simply using the conventional marginal stability hypothesis [17–19]. The reason is that there is a nonlinear derivative term $m/(1-\phi)(\partial\phi/\partial x)^2$ in Eq. (3). Thus a supplementary version of the marginal stability hypothesis should be made. Generally, asymptotic behavior of a wave front is dominated by both stable and unstable states; and a selected velocity is determined by a balance between asymptotic behavior of the wave on both sides. Let us write two dispersion relationships for Eq. (3) for both the unstable state $\phi=0$ and the stable state $\phi=1$. The first is $\lambda_1=k^2+1$ (around $\phi=0$), by which the corresponding minimum phase velocity is $|c_{1min}|=2$. The second is $\lambda_2=(1-m)k^2-1$ (around $\phi=1$), by which the corresponding minimum phase velocity is $|c_{2min}|=2\sqrt{m-1}$, where $m>1$. The question is: which velocity is reliable, $|c_{1min}|=2$ or $|c_{2min}|=2\sqrt{m-1}$? Note that both the c_{1min} and c_{2min} represent the stability limits of the wave front on the opposite sides. In order to guarantee the global stability of the wave front, the low limit of the velocity should be the larger of the two velocities, $|c_{1min}|$ and $|c_{2min}|$. Therefore the naturally selected velocity, i.e., the marginal velocity, is

$$C_{min}=\mathrm{Max}\{|c_{1min}|,|c_{2min}|\}. \tag{16}$$

In short, the selected velocity is $C_{min}=2$ when $m<2$; and $C_{min}=2\sqrt{m-1}$ when $m>2$. A numerical experiment has been performed to check this prediction. The results are shown in Fig. 2. It is clear that the numerical calculations are in excellent agreement with the theoretical prediction.

Next we would like to discuss a general example of Eq. (2):

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+\frac{\beta+1}{u}\left(\frac{\partial u}{\partial x}\right)^2=u(u^{\beta}-1), \tag{17}$$

where the parameter $\beta>0$ is an arbitrary real number. This class of nonlinear diffusion equations can be considered as a generalization of Eq. (5). Armed by the above detailed discussion and analysis, our task now becomes straightforward and swift. First of all, it is easy to see that by letting $u=v^{1/\beta}$, Eq. (17) is transformed into

$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}=\beta(v-1). \tag{18}$$

So one immediately gets the complete and exact solutions of Eq. (17) for any value of the parameter β,

$$u=\left\{\frac{1}{1+\sum_i A_{i1}\exp\left[\sqrt{\beta(\omega_i-1)}\left(x+\frac{\omega_i\sqrt{\beta}}{\sqrt{\omega_i-1}}t\right)\right]+A_{i2}\exp\left[-\sqrt{\beta(\omega_i-1)}\left(x-\frac{\omega_i\sqrt{\beta}}{\sqrt{\omega_i-1}}t\right)\right]}\right\}^{1/\beta}. \tag{19}$$

The structure of solutions of Eq. (17) is similar to that of Eq. (3). Carrying out a discussion parallel to that for Eq. (9), we reach to the same conclusions that all the fast nonlinear eigenmodes are depressed by the slowest eigenmode during a dissipative process, and finally the asymptotic behavior of the system is dominated only by this slowest eigenmode. According to Eq. (19), each nonlinear propagating mode has a phase velocity $C=\omega\sqrt{\beta}/\sqrt{\omega-1}$. Then the slowest velocity, or the selected velocity is

$$C_{min}=2\sqrt{\beta}. \tag{20}$$

Figure 3 demonstrates that our numerical simulations on Eq. (17) which show the perfectly same velocities for the different values of β as predicted by Eq. (20). These general results obviously support the proposed selection mechanism again and are more convincing. Furthermore a noticeable fact is that by using the transformation $u=[1/1+\exp(\beta t)w(x,t)]^{1/\beta}$, Eq. (17) becomes the linear heat equation.

We have also considered a more general case of Eq. (5):

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+\frac{m}{u}\left(\frac{\partial u}{\partial x}\right)^2=u(u^\beta-1), \tag{21}$$

where $m\neq\beta+1$. In this case no complete and exact solution could be found. Using the extended marginal stability criterion discussed above, the selected velocity is

$$C_{min}=\mathrm{Max}\{2\sqrt{\beta},2\sqrt{m-1}\}. \tag{22}$$

It is encouraging that for the different values of β and m our numerical calculations [23] are once again in excellent agreement with the prediction by Eq. (22).

In summary, the complete and exact solutions of a class of nonlinear diffusion equations have been found based on a symmetry observation, which are the first and perhaps the simplest examples that explicitly illustrate the fundamental points of selection mechanism and offers a concrete explanation for the abstract conclusions on the nonlinear diffusion equations related to the selection mechanism. It has been shown that these equations can be exactly mapped into the linear heat equation. This fact indicates that there exists an inherent relationship among this class of equations, the heat equation and the Burgers equation. Furthermore an alternative scenario on the marginal stability hypothesis has been proposed, which is capable of predicting the selected velocity for the general case of the equations. One element of this scenario is to emphasize the significance of asymptotic analysis on stability on both the stable and unstable sides of a wave front; the other is that a selected velocity should be chosen in such a way that would guarantee the necessary conditions on stability on the both sides. In our opinion the extended marginal stability hypothesis should be valid for a wider class of nonlinear diffusion equations.

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[1] M. C. Cross *et al.*, Rev. Mod. Phys. **65**, 851 (1993).
[2] K. Showalter, Nonlinear Science Today **4**, 1 (1995).
[3] J. S. Langer, Rev. Mod. Phys. **52**, 1 (1980).
[4] G. Ahlers and D. S. Cannell, Phys. Rev. Lett. **50**, 1583 (1983).
[5] M. Lucke *et al.*, Phys. Rev. Lett. **52**, 625 (1984).
[6] J. Fineberg and V. Steinberg, Phys. Rev. Lett. **58**, 1332 (1987).
[7] X. Y. Wang, Commun. Theor. Phys. **2**, 1307 (1983); Phys. Rev. A **32**, 3126 (1985); Z. M. Sun *et al.*, J. Phys. C **21** L771 (1988); X. Y. Wang *et al.*, Phys. Rev. A **41**, 7056 (1990).
[8] X. Y. Wang *et al.*, Phys. Rev. Lett. **76**, 640 (1996); J. F. Li *et al.*, Phys. Rev. B **52**, R13 075 (1995).
[9] S. Barolo and A. Dorsey, Phys. Rev. Lett. **77**, 4442 (1996).
[10] M. J. Grimson *et al.*, Phys. Rev. E **49**, 1680 (1994).
[11] T. Kyu and J. H. Lee, Phys. Rev. Lett. **76**, 3746 (1996).
[12] A. Maree and A. Panfilov, Phys. Rev. Lett. **78**, 1819 (1997).
[13] J. Mai *et al.*, Phys. Rev. Lett. **77**, 4462 (1996).
[14] D. G. Aronson and H. F. Weinberger, in *Partial Differential Equations and Related Topics*, edited by J. A. Goldstein (Springer, Heidelberg, 1975); Adv. Math. **30**, 33 (1978).
[15] P. Fife, *Mathematical Aspects of Reaction and Diffusing System*, Vol. 28 of Lecture Notes in Biomathematics, edited by S. Levin (Springer, New York, 1979).
[16] D. H. Sattinger, Adv. Math. **22**, 312 (1976).
[17] G. T. Dee *et al.*, Phys. Rev. Lett. **50**, 383 (1983).
[18] E. Ben-Jacob *et al.*, Physica D **14**, 348 (1985).
[19] W. van Saarloos, Phys. Rev. A **37**, 211 (1988); **39**, 6367 (1989).
[20] G. C. Paquette *et al.*, Phys. Rev. Lett. **72**, 76 (1994); L. Y. Chen *et al.*, Phys. Rev. E **50**, 3701 (1994).
[21] R. D. Benguria and M. C. Depassier, Phys. Rev. Lett. **73**, 2272 (1994); Commun. Math. Phys. **175**, 221 (1996); Phys. Rev. Lett. **77**, 1171 (1996).
[22] J. M. Burgers, Adv. Appl. Mech. **1**, 171 (1948).
[23] X-Y. Wang *et al.* (unpublished).