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Nonuniversality of weak synchronization in chaotic systems

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We show that the separate properties of weak synchronization (WS) and strong synchronization (SS), reported recently by Pyragas [Phys. Rev. E **54**, R4508 (1996)], in unidirectionally coupled chaotic systems, are not generally distinct properties of such systems. In particular, we find analytically for the tent map and numerically for some parameters of the circle map that the transitions to WS and SS coincide. [S1063-651X(97)50610-6]

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Chaotic systems, by definition, are characterized by strong sensitivity to the initial conditions. Thus, in a general situation, one cannot synchronize two chaotic systems, since in a practical situation it is impossible to start the evolution of the two systems with *exactly* the same initial conditions. However, one can synchronize subsystems of a chaotic system that have a chaotic output, but are intrinsically stable. This was shown by Pecora and Carroll [1]. More specifically, Pecora and Carroll studied dynamical systems of the type $\dot{u}=g(u,w)$, $\dot{w}=h(u,w)$ and showed that variable $\dot{w}'$ governed by $\dot{w}'=h(u,w')$ can synchronize with w if the sub-Liapunov exponents of the driven subsystem w' are all negative (implying that the subsystem w' is stable). The sub-Liapunov exponents they defined depend on the Jacobian matrix of the w subsystem, taking derivatives with respect to w only.

In a recent paper, Pyragas [2] studied one-dimensional chaotic systems, governed by a function f, coupled unidirectionally in the following way:

$$x_{n+1}=f(x_n), \quad y_{n+1}=f(y_n)-k[f(y_n)-f(x_n)],$$

$$z_{n+1}=f(z_n)-k[f(z_n)-f(x_n)]. \tag{1}$$

He found numerically that in a specific system, namely, the logistic map, synchronization between the variables x, y, and z, as k is increased, occurs in two stages. In the first stage, y synchronizes with z and not with x, starting at a critical value k_w. He called this *weak synchronization* (WS). The second stage of synchronization starts at $k=k_s$, with $k_s>k_w$, and is characterized by the synchronization of x, y, and z, which he called *strong synchronization* (SS). Note that in the above equation the case $k=1$ represents synchronization in a trivial way, since this leads to $y_{n+1}=z_{n+1}=x_{n+1}$. So, for this kind of coupling k is taken in the interval $[0,1)$.

The conditions under which WS and SS occur were determined by two Liapunov exponents, that is, the conditional Liapunov exponent,

$$\lambda^R=\ln(1-k)+\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\ln|f'(y_n)|, \tag{2}$$

defining the stability of the invariant manifold $y=z$, and the transverse Liapunov exponent of the invariant manifold $x=y$,

$$\lambda_0=\ln(1-k)+\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\ln|f'(x_n)|. \tag{3}$$

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In the logistic map, Pyragas found that λ^R becomes zero at two characteristic values of the coupling strength, k_w and k_s, corresponding to the threshold of WS and SS, respectively. In the region of weak synchronization ($k_w<k<k_s$), $\lambda^R<0$ and $\lambda_0>0$. Strong synchronization occurs for $k>k_s$, where these two Liapunov exponents coincide, i.e., $\lambda_0=\lambda^R$.

We studied the phenomenon of chaotic synchronization in other one-dimensional maps, coupled also according to Eq. (1). We found that the phenomenon of weak synchronization is not always found in such systems.

We start by showing what is the relationship between the exponents λ^R and λ_0, defined by Pyragas, and the Liapunov exponents of the three-dimensional system (with dynamical variables x, y, and z) governed by Eq. (1). We will call this the global system, which has the following Jacobian matrix at a single position along the orbit:

$$J_n=\begin{pmatrix} f'(x_n) & 0 & 0 \\ kf'(x_n) & [1-k]f'(y_n) & 0 \\ kf'(x_n) & 0 & [1-k]f'(z_n) \end{pmatrix}. \quad (4)$$

The Liapunov exponent of this system is found by calculating the eigenvalues of the matrix that consist of the product of the Jacobian matrices along a given orbit. It turns out that, because of the symmetry of this particular matrix, the product J of the Jacobian matrices J_n is of the type

$$J=\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}, \quad (5)$$

where a_{11}, a_{21}, a_{22}, a_{31}, and a_{33} are in principle different from zero. The eigenvalues of this matrix are

$$\Lambda_1=a_{11}=\prod_{n=1}^{N} f'(x_n), \quad (6)$$

$$\Lambda_2=a_{22}=\prod_{n=1}^{N} [1-k]f'(y_n), \quad (7)$$

$$\Lambda_3=a_{33}=\prod_{n=1}^{N} [1-k]f'(z_n). \quad (8)$$

Consequently, in this way of coupling, the eigenvalues of the system are easily found, and each one is a function of a single variable. The Liapunov exponents are therefore

$$\lambda_1=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} \ln|f'(x_n)|, \quad (9)$$

$$\lambda_2=\ln(1-k)+\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} \ln|f'(y_n)|, \quad (10)$$

$$\lambda_3=\ln(1-k)+\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} \ln|f'(z_n)|. \quad (11)$$

If the initial values of y and z are in the same basin of attraction, then $\lambda_3=\lambda_2$, because the parameters of the maps are the same. The Liapunov exponent λ^R is equal to one of the Liapunov exponents of the global system, namely, it is equal to λ_2. Thus, this Liapunov exponent has a clear physical importance even when y and z are not synchronized. On the other hand, λ_0 is not a Liapunov exponent of the global system in the region where x and y are not synchronized (weak synchronization). However, λ_0 and λ_1 are related to each other via a simple additive term, namely, $\lambda_0=\ln(1-k)+\lambda_1$. It is obvious that in the region of SS, $\lambda_0=\lambda^R=\lambda_2$.

We found in Ref. [3] that the loss of synchronization between subsystems of a dynamical system coupled according to Pecora and Carroll's coupling is characterized by the Liapunov exponents of the global system. That is, when one of such Liapunov exponents crosses zero (in the positive direction) the two identical subsystems will lose synchrony. When this occurs, a transition from chaos to hyperchaos [4] takes place. Here we find a similar phenomenon. The transition from SS to WS corresponds to the transition from chaos to hyperchaos in the global system.

Our next step is to show that WS does not necessarily precede SS. We show this analytically in the tent map, and later numerically in the circle map.

The tent map is defined as

$$f(x)=\begin{cases} ax, & \text{if } 0\leqslant x\leqslant 1/2, \\ a(1-x), & \text{if } 1/2\leqslant x\leqslant 1. \end{cases} \quad (12)$$

The tent map has a period-one orbit when $0\leqslant a<1$, with fixed point $x^*=0$. For $1<a<2$, the map has a chaotic orbit. For $a>2$, the orbit diverges. (Note that, to avoid divergencies, when $0\leqslant a<2$ the initial conditions must be in the interval $[0,1]$). Since $|f'(x_n)|=a$ in any point of the orbit, we have, using Eqs. (9)–(11)

$$\lambda_1=\ln a, \quad (13)$$

$$\lambda_2=\lambda_3=\ln(1-k)+\ln a. \quad (14)$$

For a fixed a, we find that λ_2 decreases monotonically as k is increased from $k=0$. Consequently, there is no region of WS in this map. For this map, synchronization between x and y and z occurs simultaneously, and only what is called SS is seen. The transition where SS occurs is determined by $\lambda_2<0$, which gives $k_s=1-1/a$.

Next, we consider coupled circle maps according to Eq. (1), where

$$f(x)=x+\omega-\frac{b}{2\pi}\sin(2\pi x). \quad (15)$$

Here we find regions of the parameter space where WS is not observed. We show in Fig. 1(a) an example for this, where $b=6$ and $\omega=0.44$. The solid line in that figure represents λ^R, which is equal to λ_2, and the dashed line represents λ_0, which is equal to $\ln(1-k)+\lambda_1$. We do not find a region of k where $\lambda^R<0$ and $\lambda_0>0$, and consequently no WS is seen.

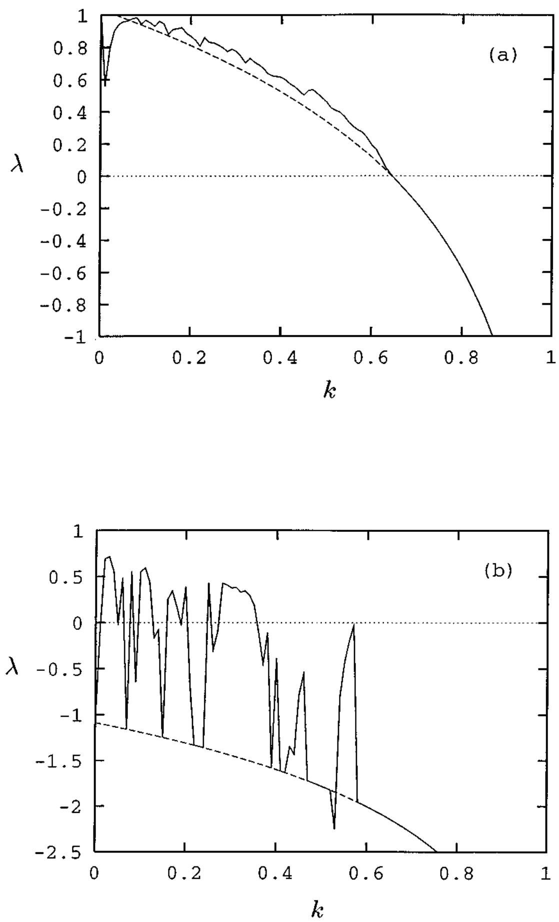

FIG. 1. Liapunov exponents (in Pyragas' notation) λ^R (solid line) and λ_0 (dashed line) for coupled circle maps, with (a) $b=6$ and $\omega=0.44$, and (b) $b=4$ and $\omega=0.4$. We used $N=30\ 000$ in Eqs. (2) and (3), and neglected a transient of 3000 iterations. The initial conditions were $x=0.1$, $y=0.2$, and $z=0.3$.

For the circle map (and also in the logistic map) we observed interesting phenomena when the driving variable x is in one of the periodic windows of the chaotic band. There, we see regions of k where y and z are chaotic ($\lambda_2=\lambda^R>0$) even when x is periodic ($\lambda_1<0$ and $\lambda_0<0$). This is shown in Fig. 1(b), where $b=4$ and $\omega=0.4$.

In Ref. [2], Pyragas also studied the coupling of Rossler's and Lorenz's systems according to

$$\dot{x}_1=-\alpha[x_2+x_3],$$

$$\dot{x}_2=\alpha[x_1+0.2x_2], \tag{16}$$

$$\dot{x}_3=\alpha[0.2+x_3(x_1-5.7)],$$

$$\dot{y}_1=10(-y_1+y_2),$$

$$\dot{y}_2=28y_1-y_2-y_1y_3+kx_2, \tag{17}$$

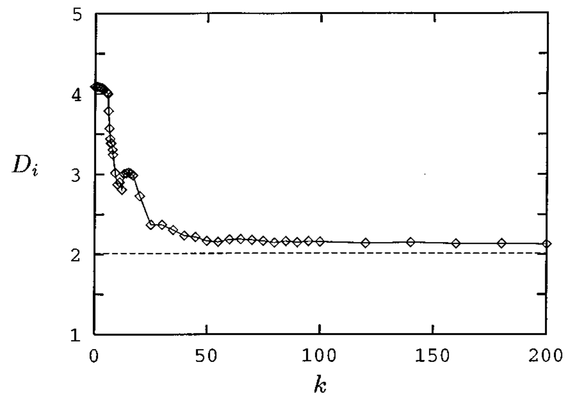

FIG. 2. Information dimension D_i as a function of k for the global attractor [Eqs. (16) and (17)] (solid line) and for the driving system [Eq. (16)] (dashed line). The error bars for D_i are smaller than the diamond symbol. Since as k increases, the variables y_i change faster than x_i, we decrease the integration time step Δt according to $\Delta t=0.02/(k+1)$. The initial conditions used were $x_1=0.1$, $x_2=0.2$, $x_3=0.3$, $y_1=0.4$, $y_2=0.5$, and $y_3=0.6$.

$$\dot{y}_3=y_1y_2-8/3y_3,$$

where $\alpha=6$. Here, we also can understand the phenomenon of weak synchronization reported by Pyragas for this coupled system, by considering it as a single global system of six variables. We calculated all the Liapunov exponents of the global system, and found that at $k\approx6.6$ one of the Liapunov exponents changes sign. Beyond this critical value of k, synchronization between two copies of the Lorenz system [Eq. (17)] with different initial conditions (in the same basin of attraction) will occur. That is, WS will be seen. No other change of sign was found in the Liapunov exponents as we increased k to $k=200$. Next, we calculated the information dimension D_i of the attractor of the global system, assuming that the Kaplan and Yorke [5] conjecture holds. Contrary to the numerical results reported in Ref. [2], we find numerically that the dimension of the global attractor does not converge to the dimension of the driving system [Eq. (16)]. The convergence of these two dimensions was considered in Ref. [2] as the characterization of SS in this system. As Fig. 2 shows, for $k\gtrsim50$, D_i for the global attractor (solid line) is approximately constant, but different from the information dimension of the driving system (dashed line). For the global system we find that $D_i=2.16$ when $k\gtrsim50$ and for the driving system $D_i=2.01$ (with the last digit being uncertain in both cases). Consequently, although we observed the property of WS in the coupled system governed by Eqs. (16) and (17), we have not identified a regime of SS, as characterized in Ref. [2].

In summary, we have found that the property of weak synchronization in maps coupled according to Eq. (1) is only a particular property of some systems. We have found that it does not hold in at least two systems, namely, the circle map and the tent map. We have also clarified the relationship between λ^R and λ_0 and the Liapunov exponents of the global system. Finally, we have verified that strong synchronization as defined in Ref. [2] is not observed in the coupled system governed by Eqs. (16) and (17).

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