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Non-Markovian persistence and nonequilibrium critical dynamics

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The persistence exponent θ for the global order parameter $M(t)$ of a system quenched from the disordered phase to its critical point describes the probability, $p(t)\sim t^{-\theta}$, that $M(t)$ does not change sign in the time interval t following the quench. We calculate θ to $O(\epsilon^2)$ for model A of Hohenberg and Halperin [Rev. Mod. Phys. **49**, 435 (1977)] (and to order ϵ for model C) and show that at this order $M(t)$ is a non-Markov process. Consequently, to our knowledge, θ is a new exponent. The calculation is performed by expanding around a Markov process, using a simplified version of the perturbation theory recently introduced by Majumdar and Sire [Phys. Rev. Lett. **77**, 1420 (1996)]. [S1063-651X(97)50707-0]

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The ''persistence exponent'' θ, which characterizes the decay of the probability that a stochastic variable exceeds a threshold value (typically its mean value) throughout a time interval, has attracted a great deal of recent interest [1–11]. One of the most surprising properties of this exponent is that its value is highly nontrivial even in simple systems. For example, θ is irrational for the $q>2$ Potts model in one dimension [6] (where the fraction of spins that have not changed their state in the time t after a quench to $T=0$ decays as $t^{-\theta}$) and is apparently not a simple fraction for the diffusion equation [9,10] (where the fraction of space where the diffusion field has always exceeded its mean decays as $t^{-\theta}$).

A recent study of nonequilibrium model A critical dynamics [12], where a system coarsens at its critical point starting from a disordered initial condition, looked at the probability $P(t_1,t_2)$ that the global magnetization does not change sign during the interval $t_1<t<t_2$ [11]. The persistence exponent for this system is defined by $P(t_1,t_2)\sim(t_1/t_2)^\theta$ in the limit $t_2/t_1\to\infty$. Explicit results were obtained for the one-dimensional (1D) Ising model, the $n\to\infty$ limit of the $O(n)$ model, and to order $\epsilon=4-d$ near dimension $d=4$. For these systems it was found that the value of θ was related to the dynamic critical exponent z, the static critical exponent η, and ''nonequilibrium'' exponent λ [which describes the decay of the autocorrelation with the initial condition, $\langle\phi(\mathbf{x},t)\phi(\mathbf{x},0)\rangle\sim t^{-\lambda/z}$] by the scaling relation $\theta z=\lambda-d+1-\eta/2$. This relation may be derived from the assumption that the dynamics is Markovian, which is indeed the case for all of the cases considered in that paper.

From a consideration of the structure of the diagrams that appear at order ϵ^2 (and higher order), however, it was argued that the dynamics of the global order parameter should not be Markovian to all orders, implying that the exponent θ does not obey exactly that ''Markovian scaling relation'' [11]. Thus, to our knowledge, θ is a new exponent. Monte Carlo simulations in two dimensions indeed suggest weak violation of the Markov scaling relation [11].

In this paper we present an explicit calculation of the non-Markovian properties of the global order parameter. The nonequilibrium magnetization-magnetization correlation function is calculated to order ϵ^2, and this is then used to calculate θ to the same order, using a perturbative method proposed by Majumdar and Sire (MS) [8], valid in the vicinity of a Markov process. The Markov scaling relation is shown explicitly to be violated at order ϵ^2, supporting our claim that θ is a new independent exponent.

Before discussing the calculation of θ, however, we provide first a simpler, and more transparent, formulation of the perturbation theory than that given in MS. In particular the final result, Eq. (14), does not appear explicitly in MS [13].

Let $y(t)$ be a Gaussian stochastic process with zero mean, whose correlation function obeys dynamical scaling, i.e., $\langle y(t_1)y(t_2)\rangle=t_1^\alpha\Phi(t_1/t_2)$. Let $T=\ln t$ and $x(T)=y(t)/\langle y^2(t)\rangle^{1/2}$. Then $x(t)$ is a Gaussian *stationary* process with zero mean, i.e., its correlation function is translationally invariant, $\langle x(T_1)x(T_2)\rangle=A(T_2-T_1)$. Notice that $A(0)=1$ by construction, a convention that we shall adopt throughout this paper (in contrast to that of Ref. [8]). If the persistence probability of y decays algebraically in t, then the persistence probability of $x(T)$ decays as $\sim\exp(-\theta T)$ for $T\to\infty$.

The persistence probability may be expressed as the ratio of two path integrals, as follows [8]:

$$P(x(T')>0;0<T'<T)=\frac{\int_{x>0}Dx(T)\exp(-S)}{\int Dx(T)\exp(-S)},\qquad(1)$$

where

$$S=\frac{1}{2}\int_0^T dT_1\int_0^T dT_2\, x(T_1)G(T_1,T_2)x(T_2).\qquad(2)$$

Here $G(T_1,T_2)$ is the matrix inverse of the correlation matrix $\langle x(T_1)x(T_2)\rangle\equiv A(T_2-T_1)$. Notice that G is not simply a function of T_2-T_1 (unless we impose periodic boundary conditions).

In MS this path-integral formalism was used to map the Markov process onto a quantum harmonic oscillator in imaginary time, developing the perturbation theory in the formalism of quantum mechanics. We shall merely use path integrals as a convenient notation, performing all our calculations within the natural framework of stochastic processes.

Let $x^0(T)$ be a stationary Gaussian Markov process, i.e., one defined by

$$\frac{dx^0}{dT} = -\mu x^0 + \xi(T), \tag{3}$$

where ξ is a Gaussian white noise, with $\langle \xi(T)\xi(T')\rangle = 2\mu\delta(T-T')$. The noise strength has been chosen so that the autocorrelation function is $A^0(T) = \exp(-\mu T)$.

Suppose the process $x(T)$ is perturbatively close to a Markov process, in the sense that $G = G^0 + \epsilon g$. Then we can expand the exponentials in the path integrals in Eq. (1) and reexponentiate, so that to $O(\epsilon)$ the numerator becomes

$$\int_{\mathcal{C}} Dx(T) e^{-S} = \int_{\mathcal{C}} Dx(T) \exp\biggl(-S^0 - \frac{\epsilon}{2}\int_0^T dT_1 \int_0^T dT_2 \times g(T_1,T_2) A^0_{\mathcal{C}}(T_1,T_2) + O(\epsilon^2)\biggr), \tag{4}$$

where the subscript $\mathcal{C}$ represents the constraint $x(T')>0$ $(0<T'<T)$ on the paths in the integral in the numerator of Eq. (1), and

$$A^0_{\mathcal{C}}(T_1,T_2) \equiv \frac{\int_{\mathcal{C}} Dx(T) x(T_1) x(T_2) e^{-S^0}}{\int_{\mathcal{C}} Dx(T) e^{-S^0}} \tag{5}$$

is the correlation function for the Markov process, averaged (and normalized) only over the paths consistent with the constraint $\mathcal{C}$. The denominator in Eq. (1) is given by an identical expression, except that $A^0_{\mathcal{C}}$ is replaced by A^0, the unconstrained correlation function.

By virtue of the constraint, $A^0_{\mathcal{C}}$ will not be strictly translationally invariant for finite T. In the limit $T\to\infty$, however, the double time integral in Eq. (4) reduces to T times an infinite integral over the relative time $T_2 - T_1$, with $A^0_{\mathcal{C}}(T_1,T_2)$ replaced by its stationary limit $A^0_{\mathcal{C}}(T_2-T_1)$. Similarly, g will be translationally invariant in this regime, giving

$$\int_0^T dT_1 \int_0^T dT_2 g(T_1,T_2) A^0_{\mathcal{C}}(T_1,T_2) \to T\int_{-\infty}^{\infty} (d\omega/2\pi)\tilde{g}(\omega)\tilde{A}^0_{\mathcal{C}}(\omega), \tag{6}$$

where we have used the translational invariance to write the final result in Fourier space [15]. Note that the zeroth-order result $\int_{x>0} Dx(T)\exp(-S^0)/\int Dx(T)\exp(-S^0)$ is just the persistence probability of the stationary Gaussian Markov process $x^0(T)$, which decays as $\exp(-\mu T)$ as $T\to\infty$.

Using (1), (4) and (6), we find that the persistence exponent may be written in the form

$$\theta \equiv \lim_{T\to\infty} -\frac{1}{T}\ln[P(x(T')>0;0<T'<T)] = \mu + \epsilon\int_0^\infty \frac{d\omega}{2\pi}\tilde{g}(\omega)[\tilde{A}^0_{\mathcal{C}}(\omega) - \tilde{A}^0(\omega)] + O(\epsilon^2). \tag{7}$$

where the term in $\tilde{A}^0(\omega)$ is the $O(\epsilon)$ contribution from the denominator in Eq. (1), and we have exploited the $\omega\to-\omega$ symmetry of the integrand.

We now calculate $A^0_{\mathcal{C}}(T)$. The conditional probability $Q(x,T|x_0,0)$ for the stationary Markov process may be obtained directly from Eq. (3):

$$Q(x,T|x_0,0) = \left[\frac{1}{2\pi(1-e^{-2\mu T})}\right]^{1/2} \exp\left[-\frac{(x-x_0 e^{-\mu T})^2}{2(1-e^{-2\mu T})}\right]. \tag{8}$$

The conditional probability $Q^+(x_2,T_2|x_1,T_1)$ that the process goes to (x_2,T_2), given that it started from (x_1,T_1), without x ever being negative is given by the method of images:

$$Q^+(2|1) = Q(x_2,T_2|x_1,T_1) - Q(x_2,T_2|-x_1,T_1), \tag{9}$$

where we have adopted an obvious shorthand notation for the arguments of Q^+.

To calculate the joint probability $P^+(x_1,T_1;x_2,T_2)$ that the process passes through x_1 at T_1 and x_2 at T_2, averaged only over paths where $x(T)$ is always positive, we consider a path starting at (x_i,T_i) and finishing at (x_f,T_f), passing through (x_1,T_1) and (x_2,T_2) without ever crossing the origin. Then the required stationary limit is

$$P^+(x_1,T_1;x_2,T_2) = \lim_{T_i\to-\infty, T_f\to\infty} \frac{Q^+(f;2;1|i)}{Q^+(f|i)}. \tag{10}$$

The Markov property means that we can decompose $Q^+(f;2;1|i) = Q^+(f|2)Q^+(2|1)Q^+(1|i)$. Using Eqs. (8) and (9) in Eq. (10), we find

$$P^+(x_1,0;x_2,T) = \frac{2}{\pi}(1-e^{-2\mu T})^{-1/2} x_1 x_2 e^{\mu T} \times \exp\left[-\frac{(x_1^2+x_2^2)}{2(1-e^{-2\mu T})}\right]\sinh\left(\frac{x_1 x_2}{2\sinh\mu T}\right). \tag{11}$$

It is now straightforward to evaluate the autocorrelation function:

$$A^0_{\mathcal{C}}(T) = \int_0^\infty dx_1 \int_0^\infty dx_2 x_1 x_2 P^+(x_1,0;x_2,T) \tag{12}$$

$$= \frac{2}{\pi}[3(1-e^{-2\mu T})^{1/2} + (e^{\mu T} + 2e^{-\mu T})\sin^{-1} e^{-\mu T}]. \tag{13}$$

Equation (7) for θ can now be expressed as a real-time integral as follows. We first write $A(T) = A^0(T) + \epsilon a(T)$, and we note that in Fourier space $[\tilde{A}(\omega)]^{-1} = \tilde{G}(\omega) = \tilde{G}^0(\omega) + \epsilon\tilde{g}(\omega)$. Using $A^0 = \exp(-\mu T)$ gives $\tilde{g}(\omega) = -\tilde{a}(\omega)(\omega^2+\mu^2)^2/4\mu^2$. Inserting this in (7), and transforming to real time, gives

$$\theta = \mu - \frac{\epsilon}{4\mu^2}\int_0^\infty dT a(T)\left(\mu^2 - \frac{d^2}{dT^2}\right)^2 [A^0_{\mathcal{C}}(T) - A^0(T)] = \mu\left\{1 - \epsilon\frac{2\mu}{\pi}\int_0^\infty a(T)[1-\exp(-2\mu T)]^{-3/2} dT\right\}. \tag{14}$$

The final result is remarkably compact. Since $\epsilon a(T)$ is just the perturbation to the Markov correlator $A^0(T)=e^{-\mu T}$, the normalization $A(T)=1$ forces $a(0)=0$. This is sufficient to converge the integral in Eq. (14) provided $a(T)$ vanishes more rapidly than $T^{1/2}$. Equation (14) has recently been used to calculate persistence exponents for interface growth in a class of generalized Edwards-Wilkinson models [14].

As was remarked earlier, the problem of nonequilibrium critical dynamics is Markovian to first order in $\epsilon=4-d$. In the thermodynamic limit the global order parameter is Gaussian because, at time t, it is the sum of $[L/\xi(t)]^d$ (essentially) statistically independent contributions, where L is the system size and $\xi\sim t^{1/z}$ is the length scale over which critical correlations have been established. Corrections to the Gaussian distribution can be expressed in terms of higher cumulants of the normalized total magnetization $M(t)/\langle M^2(t)\rangle^{1/2}$. Using the translational invariance of the system with respect to space it is easy to show that for large L the $2N$-point cumulant is smaller by a factor $(t^{1/z}/L)^{(N-1)d}$ than the Gaussian part of the $2N$-point correlation function. The perturbative approach discussed in the first part of this paper can therefore be applied. To calculate the lowest non-Markovian term in θ, we need to calculate the autocorrelation function of the total magnetization $M(t)$ to order ϵ^2, i.e., we need to calculate the autocorrelation function $A(t_1,t_2)=\langle M(t_1)M(t_2)\rangle/\langle M^2(t_1)\rangle^{1/2}\langle M^2(t_2)\rangle^{1/2}$, which in the scaling regime depends only on the ratio t_2/t_1. The necessary techniques of dynamical field theory, incorporating the extra renormalization associated with the random initial condition (and responsible for the nonequilibrium exponent λ), have been developed by Janssen *et al.* [16,17].

Models A and C of Hohenberg and Halperin [12] are defined by Langevin equations for a nonconserved n-component vector order-parameter field $\vec{s}(\mathbf{r},t)$ and (for model C) a noncritical conserved density $m(\mathbf{r},t)$:

$$\partial_t\vec{s}=-\frac{\delta H}{\delta\vec{s}}+\vec{\zeta}, \tag{15}$$

$$\partial_t m=\rho\nabla^2\frac{\delta H}{\delta m}+\eta \tag{16}$$

with the Hamiltonian

$$H[\vec{s},m]=\int d^dr\left[\frac{\tau}{2}\vec{s}^2+\frac{1}{2}(\nabla\vec{s})^2+\frac{g}{4!}(\vec{s}^2)^2+\frac{1}{2}m^2+\frac{\gamma}{2}m\vec{s}^2-h_m m\right]. \tag{17}$$

The external field h_m is to be adjusted such that $\langle m\rangle=0$. The Langevin noises $\vec{\zeta}$ and η are Gaussian random forces with zero mean and correlators $\langle\zeta_i(\mathbf{r},t)\zeta_j(\mathbf{r}',t')\rangle=2\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, $\langle\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle=-2\rho\nabla^2\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, while the initial conditions $\vec{s}_0(\mathbf{r})$, $m_0(\mathbf{r})$ are Gaussian random variables with distribution $P[\vec{s}_0,m_0]\propto\exp(-H_0[\vec{s}_0,m_0])$, where $H_0[\vec{s}_0,m_0]=\int d^dr[\tau_0\vec{s}_0^2/2+m_0^2/2c_0]$.

We first consider "model A" dynamics (where for $n>1$ "persistence" is associated with a given component of the order parameter). For model A, Eq. (16) is discarded, and the terms in m are omitted from Eq. (17). The calculation of the autocorrelation function is straightforward in principle [16,17], but algebraically tedious, and the final result is [with $T=\ln(t_2/t_1)$]

$$A(T)=e^{-\mu T}\left[1-\frac{3(n+2)}{4(n+8)^2}\epsilon^2F_A(e^T)+O(\epsilon^3)\right], \tag{18}$$

where $\mu=(\lambda-d+1-\eta/2)/z$ from the one-loop calculation [11] (equivalent to the "Markov scaling relation"), and

$$\begin{aligned}F_A(x)=&-\ln\frac{4}{3}[2\ln(2x)+(x-1)\ln(x-1)-(x+1)\ln(x+1)]-2(\ln2)^2-\frac{\pi^2}{6}+4\ln2-(x-1)\ln\left(\frac{x-1}{2x}\right)\\&+(x+1)\ln\left(\frac{x+1}{2x}\right)+(x-1)\ln\left(\frac{x-1}{2x}\right)\ln\left(\frac{3x-1}{2x}\right)-(x+1)\ln\left(\frac{x+1}{2x}\right)\ln\left(\frac{3x+1}{2x}\right)-(3x+1)\ln\left(\frac{3x+1}{2x}\right)\\&+(3x-1)\ln\left(\frac{3x-1}{2x}\right)-\frac{(x-1)}{2}\left[\ln\left(\frac{3x-1}{2x}\right)\right]^2+\frac{(x+1)}{2}\left[\ln\left(\frac{3x+1}{2x}\right)\right]^2-(x-1)\mathrm{Li}_2\left(\frac{x-1}{2x}\right)+(x+1)\mathrm{Li}_2\left(\frac{x+1}{2x}\right)\\&-2(x+1)\mathrm{Li}_2\left(\frac{x+1}{4x}\right)+2(x-1)\mathrm{Li}_2\left(\frac{x-1}{4x}\right)+(x+1)\mathrm{Li}_2\left(\frac{2x}{3x+1}\right)-(x-1)\mathrm{Li}_2\left(\frac{2x}{3x-1}\right),\end{aligned}$$

and $\mathrm{Li}_2(x)\equiv-\int_0^x dt\ln(1-t)/t$ is the dilogarithm function. The function $F_A(e^T)$ is a bounded, monotonically increasing, function of T in $(0,\infty)$. It vanishes as $T\ln T$ for $T\to0$ [satisfying the requirement for convergence at $T=0$ of the integral in Eq. (14)], while $F(\infty)=0.057622\ldots$.

The non-Markov nature of the process $M(t)$ at order ϵ^2 follows from the fact that, at this order, $A(T)$ is no longer a simple exponential. Substituting $a(T)=A(T)-e^{-\mu T}$ from Eq. (18) into Eq. (14), using $\mu=(1/2)+O(\epsilon)$, we find (after some algebra)

$$\theta=\mu\left\{1+\frac{3(n+2)}{4(n+8)^2}\epsilon^2\alpha\right\}, \tag{19}$$

where

$$\alpha = 4(\sqrt{2}-2\sqrt{3}+\sqrt{6})+8\sqrt{2}\ln 2-4(\sqrt{2}-1)\ln 3 \\ -2(1+2\sqrt{2})\ln(3+2\sqrt{2})-14\ln(5+2\sqrt{6}) \\ +10\ln(7+4\sqrt{3})+8\sqrt{2}\ln[(4+\sqrt{2}-\sqrt{6})/(4-\sqrt{2} \\ -\sqrt{6})]-4\sqrt{2}\ln[(2\sqrt{3}-2+\sqrt{2})/(2\sqrt{3}-2-\sqrt{2})] \\ =0.271577604975\ldots .$$

This result can be compared with recent simulation data for the Ising model in two [11,19,20] and three [19] dimensions. For $d=2$, using $\lambda=1.585\pm0.006$ [21], and $\eta=1/4$ (exact) gives $\mu z=0.460\pm0.006$. Ignoring non-Markov corrections, one would obtain $\theta z=\mu z$, smaller than the measured value $\theta z=0.505\pm0.020$ (the finite-size scaling method used in [11] naturally determines the combination θz [20]). The non-Markov correction factor in Eq. (19) is, for $n=1$, $(1+0.0075438\ldots,\epsilon^2)\simeq1.030$ for $\epsilon=2$. The "improved" estimate for θz becomes 0.474 ± 0.006, closer to, but still somewhat smaller than, the numerical estimate.

For $d=3$, one has $z=2.032\pm0.004$, $\lambda=2.789\pm0.006$ [21], and $\eta=0.032\pm0.003$, giving $\mu=0.380\pm0.003$. Multiplying by the non-Markov correction factor for $\epsilon=1$, i.e., 1.0075, gives $\theta=0.383\pm0.003$, compared to the numerical result $\theta\simeq0.41$ [19]. A direct expansion to order ϵ^2, using the known expansions for z, λ, and η, gives (specializing to $n=1$) $\theta=1/2-\epsilon/12+(\alpha-2\ln 3)\epsilon^2/72-2\epsilon^2/81+O(\epsilon^3)$, i.e., $\theta\simeq0.365$ for $d=3$, slightly lower than that obtained using the best numerical estimates of z, λ, and η and only using the ϵ expansion for the non-Markov correction.

A similar approach can be applied to "model C," defined by the full set of equations (15)–(17). In this case, one obtains non-Markovian corrections already at order ϵ. The autocorrelation function is given by (for $n=1$)

$$A(T)=\exp(-\mu T)\left[1-\frac{\epsilon}{6}F_C(e^T)+O(\epsilon^2)\right], \tag{20}$$

$$F_C(x)=\ln 2-\frac{x-1}{2}\ln(x-1)-\frac{x+1}{2}\ln(x+1) \\ +x\ln x-\frac{x-1}{2x}. \tag{21}$$

Again, $F_C(e^T)$ vanishes like $T\ln T$ for $T\to0$, while $F_C(\infty)=\ln 2-1/2$. Inserting $a(T)=A(T)-\exp(-\mu T)$ from Eq. (20) into Eq. (14) gives

$$\theta=\mu\left[1+\frac{7-4\sqrt{2}}{12}\epsilon+O(\epsilon^2)\right], \tag{22}$$

where $\mu=(\lambda-d+1-\eta/2)/z$ as before, but now the dynamical exponents z and λ take their model-C values [12,18].

In summary, we have computed to order ϵ^2 the persistence exponent θ for the global order parameter $M(t)$ of models A and C. At this order, the dynamics of $M(t)$ are non-Markovian, and θ is a new exponent, not related to the usual static and dynamic exponents. The calculation was performed by expanding around a Markov process, using a simplified form of the perturbation theory introduced by Majumdar and Sire.

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[1] B. Derrida, A. J. Bray, and C. Godrèche, J. Phys. A **27**, L357 (1994).
[2] A. J. Bray, B. Derrida, and C. Godrèche, Europhys. Lett. **27**, 175 (1994).
[3] D. Stauffer, J. Phys. A **27**, 5029 (1994).
[4] E. Ben-Naim, P. L. Krapivsky, and S. Redner, Phys. Rev. E **50**, 2474 (1994).
[5] B. Derrida, P. M. C. de Oliveira, and D. Stauffer, Physica A **224**, 604 (1996).
[6] B. Derrida, V. Hakim, and V. Pasquier, Phys. Rev. Lett. **75**, 751 (1995).
[7] E. Ben-Naim, L. Frachebourg, and P. L. Krapivsky, Phys. Rev. E **53**, 3078 (1996).
[8] S. N. Majumdar and C. Sire, Phys. Rev. Lett. **77**, 1420 (1996).
[9] S. N. Majumdar, C. Sire, A. J. Bray, and S. Cornell, Phys. Rev. Lett. **77**, 2867 (1996).
[10] B. Derrida, V. Hakim, and R. Zeitak, Phys. Rev. Lett. **77**, 2871 (1996).
[11] S. N. Majumdar, A. J. Bray, S. J. Cornell, and C. Sire, Phys. Rev. Lett. **77**, 3704 (1996).
[12] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[13] The result (14) can be deduced after some algebra from Eqs. (4), (5), and (7) of Ref. [8].
[14] J. Krug, H. Kallabis, S. N. Majumdar, S. Cornell, A. J. Bray, and C. Sire (unpublished).
[15] Note that $A_C^0(\infty)>0$ [see Eq. (13)], so $\tilde{A}_C^0(\omega)$ implicitly contains a term proportional to $\delta(\omega)$.
[16] H. K. Janssen, B. Schaub, and B. Schmittman, Z. Phys. **73**, 539 (1989).
[17] H. K. Janssen, in *From Phase Transitions to Chaos, Topics in Modern Statistical Physics*, edited by G. Györgyi, I. Kondor, L. Sasvári, and T. Tél (World Scientific, Singapore, 1992), pp. 68–91.
[18] K. Oerding and H. K. Janssen, J. Phys. A **26**, 3369 (1993).
[19] D. Stauffer, Int. J. Mod. Phys. C **7**, 753 (1996).
[20] L. Schülke and B. Zheng (unpublished) have recently measured θ directly for the Ising model in $d=2$. Their result $\theta=0.237\pm0.003$ is consistent with $\theta z=0.505\pm0.020$ [11] for z in the accepted range $2.13-2.17$.
[21] P. Grassberger, Physica A **214**, 547 (1995).