Momentum relaxation of a charged particle by small-angle Coulomb collisions

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The theory on Coulomb collisions in plasmas has been previously presented [K. Nanbu, Phys. Rev. E **55**, 4642 (1997)]. The theory is based on the relaxation equation for the scattering angle and the model equation for the probability density function of the scattering angle. The validity of both equations was ascertained by numerical simulations in the above paper. It is shown that the first equation, which describes the relaxation of the momentum of a particle, can be derived analytically.
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It would be much better if important equations in physics could be verified analytically instead of numerically. In this Brief Report I analytically derive Eq. (9) in my previous paper [1]. It is the key equation in the theory of cumulative small-angle collisions in plasmas. Here all symbols and equations in the previous paper are used without explanation. Equation (9) in Ref. [1] can be rewritten as

$$\langle \cos\chi_N \rangle = e^{-s}. \tag{1}$$

Since the momentum of a particle in the initial direction is proportional to $\cos\chi_N$, Eq. (1) represents the relaxation of the mean momentum in this direction.

Equation (1) can be derived by use of the equations in the Appendix of Ref. [1]. The scattering angles θ_1, $\theta_2,\ldots$ in small-angle collisions are mutually independent random variables whose probability density functions are identical. Let $h(\theta)$ be a function. By use of Eq. (A9) in Ref. [1] the expectation $\langle \cos\chi_N \rangle$ is given by

$$\langle \cos\chi_N \rangle = \int \cdots \int J(\theta_1,\theta_2,\ldots,\theta_N) h(\theta_1) \times h(\theta_2)\cdots h(\theta_N)\, d\theta_1 d\theta_2 \cdots d\theta_N, \tag{2}$$

where

$$J(\theta_1,\theta_2,\cdots,\theta_N) = \frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} G_i^{(N)} \times [A^{(N-1)}A^{(N-2)}\cdots A^{(1)}]_{i3}\, d\varphi_1 d\varphi_2 \cdots d\varphi_N. \tag{3}$$

The integrand of Eq. (3) is a linear function of the elements of vector $G_i^{(N)}$ and matrices $A^{(1)}$, $A^{(2)}$,..., $A^{(N-1)}$; the product of two elements belonging to the same matrix does not appear. Also, φ_N is included only in $G_i^{(N)}$ and φ_k is only in $A^{(k)}$, where $k=1,2,\ldots,N-1$. Taking these into consideration, we can rewrite Eq. (3) as

$$J(\theta_1,\theta_2,\ldots,\theta_N) = \overline{G_i^{(N)}}[\overline{A^{(N-1)}}\,\overline{A^{(N-2)}}\cdots\overline{A^{(1)}}]_{i3}, \tag{4}$$

where

$$\overline{G_i^{(N)}} = \frac{1}{2\pi}\int_0^{2\pi} G_i^{(N)} d\varphi_N = (0,0,\cos\theta_N),$$

$$\overline{A^{(k)}} = \frac{1}{2\pi}\int_0^{2\pi} A^{(k)} d\varphi_k = \begin{pmatrix} \frac{1}{2}(1+\cos\theta_k) & 0 & 0 \\ 0 & \frac{1}{2}(1+\cos\theta_k) & 0 \\ 0 & 0 & \cos\theta_k \end{pmatrix}.$$

Equation (4) now takes the form

$$J(\theta_1,\theta_2,\ldots,\theta_N) = \prod_{k=1}^{N} \cos\theta_k. \tag{5}$$

Substitution of Eq. (5) into Eq. (2) yields

$$\langle \cos\chi_N \rangle = \langle \cos\theta_1 \rangle^N, \tag{6}$$

where $\langle \cos\theta_1 \rangle = \int h(\theta_1)\cos\theta_1 d\theta_1$ is the expectation of $\cos\theta_1$ and use is made of $\langle \cos\theta_1 \rangle = \langle \cos\theta_2 \rangle = \cdots = \langle \cos\theta_N \rangle$.

Note that Eq. (6) has been derived by no use of the assumption that $\theta_1,\theta_2,\ldots,\theta_N \ll 1$. Introducing this assumption, we then have

$$\langle \cos\chi_N \rangle \simeq \left(1 - \frac{1}{2}\langle \theta_1^2 \rangle\right)^N \simeq \left[\left(1+\frac{1}{x}\right)^x\right]^{-s}, \tag{7}$$

where $x = -2/\langle \theta_1^2 \rangle$ and $s = \langle \theta_1^2 \rangle N/2$. Taking the limit of $\langle \theta_1^2 \rangle \to 0$ under the condition that s is fixed, we then have Eq. (1).

[1] K. Nanbu, Phys. Rev. E **55**, 4642 (1997).

