Influence of the shape of small scatterers upon their resonance features

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Scattering of waves by scatterers (inhomogeneities), whose sizes are much smaller than the incident wavelength in the surrounding dielectric medium, has been considered in this paper. The expressions for the scattered field are obtained by applying the localized perturbation method. The scatterers are in an unbounded medium or under the interface between two dielectric half-spaces. The features of resonances for arbitrary configuration of the scatterers are studied for three-dimensional (3D), two-dimensional, and one-dimensional cases. The resonance frequency and the resonance linewidth have been demonstrated to be dependent on volume characteristics of a scatterer. As a consequence, the fields scattered by inhomogeneities with different shapes may have the same form of the resonance line. If scatterers are three-dimensional the resonance line is sharp; for the 2D case it is noticeably wider, and in the 1D case the resonance is absent. In the 3D and 2D cases, the resonance frequency and the width of the resonance line are oscillating functions of the distance between the scatterer and the interface. The influence of time dispersion of the dielectric permittivity of a scatterer upon wave scattering features is considered. [S1063-651X(97)06111-4]

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The problem of wave scattering from an arbitrarily shaped scatterer (inhomogeneity) can be solved only approximately. The popular Born approximation is used when the scattered field is small in comparison with the incident one. However, this method is not applicable when parameters of the scatterers differ noticeably from parameters of the surrounding medium. If the characteristic size of the scatterer L is significantly less than wavelength λ in the surrounding medium (in all or in some directions) then the localized perturbation method (LPM) proposed by Lax [1] and I. Lifšic [2] can be utilized. This method has been used, for example, for solution of problems of acoustic wave scattering by a defect in a crystal lattice [1–4], of the scattering of an electron by an impurity [5,6], for the solution of the Helmholtz equation solution in investigations of resonance phenomena investigation (see Ref. [7] and its references).

The LPM can be applied for the investigation of the wave scattering from inhomogeneities when the surrounding medium is also inhomogeneous. We will first consider the most common case—an unbounded space. The basic equation of this scattering problem is

$$\hat{M}\left\{\frac{\partial}{\partial \mathbf{r}},\frac{\partial}{\partial t}\right\}E(\mathbf{r},t)+U(\mathbf{r}_n)E(\mathbf{r},t)=I(\mathbf{r},t), \tag{1}$$

where $\hat{M}\{\partial/\partial\mathbf{r},\partial/\partial t\}$ is an arbitrary operator-function, $\mathbf{r}$ is a radius vector, t is the time, $E(\mathbf{r},t)$ is the unknown field, $U(\mathbf{r})$ is the function describing the configuration of an inhomogeneity, $\mathbf{r}_3=\mathbf{r}(x,y,z)$ for a three-dimensional (3D) inhomogeneity, $\mathbf{r}_2=\mathbf{r}(x,y)$ for a two-dimensional (2D) inhomogeneity, and $\mathbf{r}_1=z$ for a one-dimensional (1D) inhomogeneity. In the 3D case the inhomogeneity is at the point $\mathbf{r}_3=\mathbf{0}$; in the 2D case the inhomogeneity axis is the axis OX; and in the 1D case the inhomogeneity lies in the plane $z=0$; $I(\mathbf{r},t)$ is a source (current)—a known function of coordinates and time.

Equation (1) can be a partial differential equation, an integral equation, or a finite-difference equation, or an equation of mixed type. It describes many physical phenomena in different areas of physics. All the equations are assumed to be linear.

Let $I(\mathbf{r},t)$ and $E(\mathbf{r},t)$ be $I(\mathbf{r},t)=I_\omega(\mathbf{r})e^{-i\omega t}$ and $E(\mathbf{r},t)=E_\omega(\mathbf{r})e^{-i\omega t}$, respectively. Therefore Eq. (1) can be transformed into

$$\hat{M}\left\{\frac{\partial}{\partial \mathbf{r}},-i\omega\right\}E_\omega(\mathbf{r})+U(\mathbf{r}_n)E_\omega(\mathbf{r})=I_\omega(\mathbf{r}). \tag{2}$$

It is impossible to solve Eq. (2) exactly, but it can be solved approximately if the inequality $\lambda \ll L$ is used.

For the sake of definiteness let us consider the 3D case. The value $U(\mathbf{r})$ rapidly decreases over a distance L, and has a sharp maximum at the point $\mathbf{r}=0$. Therefore we have the following approximate equality:

$$U(\mathbf{r})E(\mathbf{r})\approx U(\mathbf{r})E(0). \tag{3}$$

To obtain this, we neglect the region $r>L$, where $U(\mathbf{r})$ is very small.

After substitution of Eq. (3) into Eq. (2), we obtain the equation

$$\hat{M}\left\{\frac{\partial}{\partial \mathbf{r}}\right\}E(\mathbf{r})+U(\mathbf{r})E(0)=I(\mathbf{r}). \tag{4}$$

Further, we will omit for simplicity the argument $-i\omega$ and the index ω. The solution of Eq. (4) has been found with the help of the Fourier method. The reduction of Eq. (2) to Eq. (4) is the essence of the localized perturbation method (LPM).

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Let us determine the inverse Fourier transform in the following form

$$f_q=\frac{1}{(2\pi)^3}\int f(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r}. \tag{5}$$

The equation for E_q is derived from Eq. (4):

$$M\{i\mathbf{q}\}E_q+E(0)U_q=I_q, \tag{6}$$

from which we find E_q and $E(\mathbf{r})$,

$$E(\mathbf{r})=\int\frac{I_q}{M\{i\mathbf{q}\}}e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{q}-E(0)\int\frac{U_q}{M\{i\mathbf{q}\}}e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{q}. \tag{7}$$

Assuming $\mathbf{r}=0$ in Eq. (7), we obtain $E(0)$; after substituting $E(0)$ into Eq. (7), we find $E(\mathbf{r})$.

The first term in Eq. (7) determines the field in a homogeneous space; the second term describes the scattered field, which is of principal interest. We will further consider the scattered field $E(\mathbf{r})$ only. The calculations for the 2D and 1D cases are identical to those for the 3D case.

In all instances the scattered field has the following form:

$$E_n(\mathbf{r})=-\int\frac{U_{q_n}}{D_{q_n'}}\frac{E_0(\mathbf{q}_n')}{M\{i\mathbf{q}\}}e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{q}, \tag{8}$$

where

$$\mathbf{q}_n'=\mathbf{q}-\mathbf{q}_n,\quad E_0(\mathbf{q}_n')=\int\frac{I_q}{M\{i\mathbf{q}\}}d\mathbf{q}_n,$$

$$D(\mathbf{q}_n')=1+\int\frac{U_{q_n}}{M\{i\mathbf{q}\}}d\mathbf{q}_n. \tag{9}$$

Expression (8) is a generalization of the result due to Lifšic [2].

Let the equation

$$D_n(\omega,q_n')=0 \tag{10}$$

have a real solution. This means that there is a resonance frequency in the 3D case, and eigenwaves in the 2D and 1D cases. In the Born approximation, $D_n=1$, and thus the resonance frequency and the eigenproper waves cannot be obtained. The field determined by Eq. (8) has been calculated in the far zone with the help of the residue theorem and the stationary phase method.

Let us consider the equation

$$\hat{M}(i\mathbf{q})=0, \tag{11}$$

where the unknown value is q_z. This equation is assumed to have a real positive solution. The vector $\mathbf{q}_\perp$ has components q_x and q_y. $\mathbf{q}_\perp''$ is the solution of the equation,

$$\frac{\partial q_z(\mathbf{q}_\perp'')}{\partial\mathbf{q}_\perp''}=-\frac{\mathbf{r}_\perp}{z}, \tag{12}$$

where the vector $\mathbf{r}_\perp$ has only x and y components.

The scattered field can be determined by the expression

$$E_n=-\frac{4\pi^2E_0(\mathbf{q}_\perp'')U_{q_\perp''}}{I''^2M_{q_z}(\mathbf{q}_\perp'')D(\mathbf{q}_\perp'')\cos\theta}\frac{\exp\{i[q_z(\mathbf{q}_\perp'')z+\mathbf{q}_\perp''\mathbf{r}_\perp]\}}{r}, \tag{13}$$

where

$$I''=\frac{\partial^2q_z(\mathbf{q}_\perp'')}{\partial q_x''^2}\frac{\partial^2q_z(\mathbf{q}_\perp'')}{\partial q_y''^2}-\left(\frac{\partial^2q_z(\mathbf{q}_\perp'')}{\partial q_x''\partial q_y''}\right)^2,$$

$$M_{q_z}(\mathbf{q}_\perp'')=\frac{\partial M_{q_z}[q_z(\mathbf{q}_\perp'')\mathbf{q}_\perp'']}{dq_z},\quad\cos\theta=\frac{z}{r}.$$

We apply the generalized theory to the field, described by the wave equation

$$\hat{M}\left\{\frac{\partial}{\partial\mathbf{r}}\right\}=\Delta+\frac{\omega^2}{c^2}\varepsilon,\quad U_3(\mathbf{r})=\frac{\omega^2}{c^2}\varepsilon' f_3(\mathbf{r}). \tag{14}$$

Δ is the Laplace operator, c is the light velocity in vacuum, ε is the dielectric permittivity of the space outside the inhomogeneity, ε' (we assume that $\varepsilon'\gg\varepsilon$) is the dielectric permittivity of the inhomogeneity, and $f_3(\mathbf{r})$ is a dimensionless function, describing the shape of the inhomogeneity.

Let a plane electromagnetic wave $E_0e^{i\mathbf{kr}}$ be incident upon inhomogeneity.

In the 3D case, for the scattered field E, we have

$$E_n=-\frac{4\pi A_2\omega^2L^3\varepsilon'\sqrt{\varepsilon}}{c^2}\frac{E_0}{1-\frac{\omega^2L^2\varepsilon'}{c^2}\left[A_1+\frac{iA_2\omega L\sqrt{\varepsilon}}{c}\right]}\times\frac{\exp\left(i\frac{\omega}{c}\sqrt{\varepsilon}r\right)}{r}, \tag{15}$$

where

$$A_1=\int_0^\infty v\overline{f(v)}dv,\quad A_2=\tfrac{1}{4}\int_0^\infty v^2\overline{f(v)}dv,$$

$$\overline{f}\left(\frac{r}{L}\right)=\frac{1}{4\pi}\int_\Omega f(\mathbf{r})d\Omega.$$

L in this case is the effective size of the inhomogeneity; Ω is the solid angle. A formula of type (15) is presented in [6,7] for scatterers of a special shape. The resonance takes place when $\varepsilon'>0$.

The scattered field E_n has a resonance structure with the following resonance frequency ω_r and a relative width of the resonance γ:

$$\omega_r=\frac{c}{\sqrt{A_1}L\sqrt{\varepsilon'}},\quad\gamma=\frac{\pi}{2}\frac{A_2}{A_1}\frac{\omega\sqrt{\varepsilon}L}{c}\sim\left(\frac{\varepsilon}{\varepsilon'}\right)^{1/2}\ll1. \tag{16}$$

One can see that $\gamma\sim L/\lambda\!\!\!^{-}_\varepsilon\ll1$ (where $\lambda\!\!\!^{-}_\varepsilon=c/\omega\sqrt{\varepsilon}$), and therefore the resonance line is narrow. The resonance condition corresponds qualitatively to the relation $L \sim \lambda_{\varepsilon'}$, and is satisfied when the size of the inhomogeneity is comparable with the wavelength inside the inhomogeneity. This is the well-known electromagnetic spatial resonance [8]. The imaginary part of the denominator describes the energy outgoing from the inhomogeneity into the surrounding space. This energy is transformed into the energy of eigenproper waves in the surrounding space. It must be noted that the part of the energy being transformed is small when the dielectric permittivity of the inhomogeneity ε' and of the surrounding space ε differ significantly, and also if the area of the surface of the inhomogeneity is small. It must also be taken into account that the resonance frequency ω_r is defined by ε' only while the width of the resonance γ is defined by both ε' and ε. This follows from the physical sense of the resonance phenomenon.

The resonance frequency and the width of the resonance depend only on the integrals A_1 and A_2, i.e., depend weakly on the shape of the inhomogeneity. The adaptability conditions of the LPM enable only the main resonance to be determined. This means that scatterers with different shapes may have identical resonance lines. The 3D inhomogeneities scatter isotropically; the 2D and 1D inhomogeneities scatter anisotropically.

In the 2D case we obtain

$$E_n = \frac{\sqrt{2\pi^3} E_0 A \varepsilon' L^2 \dfrac{\omega^2}{c^2}}{1 - \dfrac{A\varepsilon' L^2 \omega^2}{c^2}\left[\ln \dfrac{c}{\sqrt{\varepsilon} L\omega \sin\theta} + i\,\dfrac{\pi}{2}\right]} \times \frac{\exp[i(q_x r' + i k_x x)]}{\sqrt{q_x r'}},$$

$$\sin\theta = \frac{r_\perp}{r}, \quad A = \int v \overline{f(v)}\, dv,$$

$$\overline{f}\left(\frac{r_\perp}{L}\right) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathbf{r}_\perp)\, d\phi. \tag{17}$$

The resonance curve in the 2D case is wider than in the 3D case, because the surface area of an inhomogeneity infinite in one dimension, and therefore the energy transfer is also larger.

In the 1D case the scattered field is

$$E_n = \frac{i\pi A\omega\varepsilon' L E_0 \exp[i(qz + \mathbf{k}_\perp \cdot \mathbf{r}_\perp)]}{c\sqrt{\varepsilon}\cos\theta\left[1 - \dfrac{i\pi A\omega L\varepsilon'}{c\sqrt{\varepsilon}\cos\theta}\right]}, \tag{18}$$

where

$$q = \frac{\omega}{c}\sqrt{\varepsilon}\cos\theta.$$

One can see from Eq. (18) that there is no resonance because the energy transfer from the inhomogeneity is very large.

The problem of wave scattering from an inhomogeneity under an interface between two dielectric media is of significant interest. Let us consider such a problem.

Let a plane electromagnetic wave $E_0 e^{i\mathbf{k}\cdot\mathbf{r}}$ be incident upon a flat interface $z=0$ from the half-space $z>0$ with a dielectric permittivity ε_+ to the half-space $z<0$ with dielectric permittivity ε_-. The inhomogeneity is at the point $\mathbf{r}_1(z<0)$. The basic system and boundary conditions are

$$\Delta E_+(\mathbf{r}) + \frac{\omega^2}{C^2}\varepsilon_+ E_+(\mathbf{r}) = 0 \quad \text{if } z>0,$$

$$\Delta E_-(\mathbf{r}) + \frac{\omega^2}{C^2}\varepsilon_- E_-(\mathbf{r}) + U(\mathbf{r}_n - \mathbf{r}_{\ln}) E_-(\mathbf{r}) = 0 \quad \text{if } z>0, \tag{19}$$

$$E_+(z=0) = E_-(z=0), \quad \frac{\partial E_+(z=0)}{\partial z} = \frac{\partial E_-(z=0)}{\partial z}. \tag{20}$$

The calculation of the scattered field is analogous to the calculations considered above, so we present the final results only.

3D case

$$E_n = -\frac{4\pi A_2 \omega^2 L^3 \varepsilon' \sqrt{\varepsilon_+}\cos\phi\, Q(\theta) P(\phi)}{c^2}\,\frac{E_{03}}{1 - \dfrac{\omega^2 L^2 \varepsilon'}{c^2}\left[A_1 + iA_2\dfrac{\omega L\sqrt{\varepsilon_-}}{c} - \dfrac{\pi^2 A_2 R(0) L}{2z_1}\exp\left(-\dfrac{2i\omega\sqrt{\varepsilon_-} z_1}{c}\right)\right]} \times \frac{\exp\left(i\,\dfrac{\omega}{c}[\sqrt{\varepsilon_+}|\mathbf{r}-\mathbf{r}_1| + S_3]\right)}{|\mathbf{r}-\mathbf{r}_1|}, \tag{21}$$

where θ is the angle of incidence, $Q(\theta)$ is the transmission coefficient in the situation when the wave is incident upon a flat interface from the half-space $z>0$, and $P(\phi)$ is the same when the wave is incident from the half-space $z<0$,

$$Q(\theta)=\frac{2\sqrt{\varepsilon_+}\cos\theta}{\sqrt{\varepsilon_+}\cos\theta+\sqrt{\varepsilon_- -\varepsilon_+\sin^2\theta}},$$

$$P(\phi)=\frac{2\sqrt{\varepsilon_- -\varepsilon_+\sin^2\phi}}{\varepsilon_+\cos\phi+\sqrt{\varepsilon_- -\varepsilon_+\sin^2\phi}},$$

$$\cos\ \phi=\frac{z-z_1}{\sqrt{(\mathbf{r}_\perp-\mathbf{r}_{1\perp})^2+(z-z_1)^2}},$$

$$S_3=\frac{\omega}{c}\left[\left(\sqrt{\varepsilon_- -\varepsilon_+\sin^2\phi}-\sqrt{\varepsilon_+}\cos\phi-\varepsilon_+\cos\ \theta\right)z_1-\mathbf{k}_\perp\mathbf{r}_\perp\right],$$

where $\mathbf{k}_\perp$ is is the wave vector, lying in the plane parallel to the plane XOY.

$$R(\theta)=\frac{\sqrt{\varepsilon_+}\cos\theta-\sqrt{\varepsilon_- -\varepsilon_+\sin^2\theta}}{\sqrt{\varepsilon_+}\cos\theta+\sqrt{\varepsilon_- -\varepsilon_+\sin^2\theta}},$$

$$R(0)=\frac{\sqrt{\varepsilon_+}-\sqrt{\varepsilon_-}}{\sqrt{\varepsilon_+}+\sqrt{\varepsilon_-}},$$

where $R(\theta)$ is the reflection coefficient for the wave incident from the half-space $z>0$.

The first and second terms of the denominator in square brackets of Eq. (21) are the same as in the case of a uniform medium. The third term represents an interference of waves reflected from the interface and from the inhomogeneity. It is an oscillating function of z_1 with period π/q_- $[q_\pm=(\omega/c)\sqrt{\varepsilon_\pm}]$, and it decreases proportionally to z_1^{-1}. We assume that the inequalities $L\ll\lambda_\varepsilon\ll z_1$ are satisfied. The above-mentioned third term is very small in comparison to the first and second terms; the ratios of the third term to the first and second terms are of order L/z_1 and L/λ_ε, respectively.

2D case

$$E_n=\frac{\sqrt{2}\sqrt{\pi^3}E_0A\varepsilon' L^2\dfrac{\omega^2}{c^2}Q(\theta)P(\phi)}{1-\dfrac{A\varepsilon' L^2\omega^2}{c^2}\left[\ln\dfrac{c}{\sqrt{\varepsilon_-}L\omega\sin\theta}+i\dfrac{\pi}{2}+\dfrac{\exp\left(i\dfrac{\pi}{4}\right)\sqrt{\pi^3}R(q_x)\exp(-2iq_xz_1)}{\sqrt{q_xz_1}}\right]}\frac{\exp[i(q_xr'+ik_xx+S_2)]}{\sqrt{q_xr'}}, \qquad (22)$$

where

$$P(\phi)=\frac{2\nu_x\cos\phi}{\sqrt{\varepsilon_+-\varepsilon_--\nu_x^2\cos^2\ \phi}+\nu_x\cos\phi},\quad \nu_x=\frac{c}{\omega}q_x,\quad q_x=\sqrt{q_-^2-k_x^2},$$

$$R(q_x)=\frac{\cos\phi\sqrt{q_+^2-q_x^2}-\sqrt{q_-^2-q_x^2-(q_+^2-q_x^2)^2\sin^2\phi}}{\cos\phi\sqrt{q_+^2-q_x^2}+\sqrt{q_-^2-q_x^2-(q_+^2-q_x^2)^2\sin^2\phi}},$$

$$r'=\sqrt{(y-y_1)^2+(z-z_1)^2},$$

$$S_2=[q_x(z_1\cos\phi+y_1\sin\phi)+z_1\sqrt{q_x^2-k_y^2}+k_yy_1].$$

The structure of the scattered field in the 2D case is the same as in the 3D case, but the third term in the denominator of Eq. (22) is significantly larger than in Eq. (21). The ratios of the third to the first and second terms are of the order of $[1/\ln(\lambda_\varepsilon/L)]\sqrt{L/\lambda_\varepsilon}$ and $\sqrt{L/\lambda_\varepsilon}$, respectively. This is caused by an inhomogeneity infinite in one direction, and therefore by the increased interference.

1D case

$$E_n=\frac{i\pi A\omega\varepsilon' LE_0P(\theta)Q(\theta)\exp[i(q_+z+\mathbf{k}_\perp\mathbf{r}_\perp-q_-z_1)]}{c\sqrt{\varepsilon_- -\varepsilon_+\sin^2\theta}\left\{1-\dfrac{i\pi A\omega L\varepsilon'\left[1-R(\theta)\exp\left(\dfrac{-2i\omega\sqrt{\varepsilon_-}z_1}{c}\right)\right]}{c\sqrt{\varepsilon_- -\varepsilon_+\sin^2\theta}}\right\}}, \qquad (23)$$

$$A=\int f(v)dv.$$

There is no resonance in this case. The results presented can be generalized to more complex equations, for example,

$$\frac{\partial^2 E}{\partial z^2}+\hat{Q}\left\{\frac{\partial}{\partial \mathbf{r}_\perp}\right\}E+U(\mathbf{r})E(\mathbf{r})=0, \tag{24}$$

where $\hat{Q}$ is an arbitrary function of $\partial/\partial\mathbf{r}_\perp$. The solution can be obtained as described above.

The LPM can be utilized for the investigation of wave scattering by inhomogeneities when the surrounding medium is also inhomogeneous. Let us consider the equation

$$\hat{M}\left\{\mathbf{r},\frac{\partial}{\partial\mathbf{r}}\right\}E(\mathbf{r})+U(\mathbf{r})E(\mathbf{r})=I(\mathbf{r}), \tag{25}$$

and let us suppose that the distance l over which $\hat{M}\{\mathbf{r},(\partial/\partial\mathbf{r})\}$ varies is much more largely than the distance L. We will also consider that the Green's function $g(\mathbf{r},\mathbf{r}')$ for Eq. (26) is known:

$$\hat{M}\left\{\mathbf{r},\frac{\partial}{\partial\mathbf{r}}\right\}g(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'). \tag{26}$$

By utilizing the relation $U(\mathbf{r})E(\mathbf{r})\approx U(\mathbf{r})E(0)$, we obtain the equation

$$\hat{M}\left\{\mathbf{r},\frac{\partial}{\partial\mathbf{r}}\right\}E(\mathbf{r})+U(\mathbf{r})E(0)=I(\mathbf{r}). \tag{27}$$

The solution of this equation is

$$E(\mathbf{r})=\int g(\mathbf{r},\mathbf{r}')I(\mathbf{r}')d\mathbf{r}'-E(0)\int g(\mathbf{r},\mathbf{r}')U(\mathbf{r}')d\mathbf{r}. \tag{28}$$

Assuming $\mathbf{r}=0$ in Eq. (28), we can find $E(0)$; after substituting $E(0)$ into Eq. (28), we obtain

$$E(\mathbf{r})=\frac{\int g(0,\mathbf{r}')I(\mathbf{r}')d\mathbf{r}'\int U(\mathbf{r}')d\mathbf{r}'}{1+\int g(0,\mathbf{r}')U(\mathbf{r}')d\mathbf{r}'}\,g(\mathbf{r},0). \tag{29}$$

Relation (29) is a generalization of formula (8). If $\hat{M}=\Delta+k^2\varepsilon$ and $U(\mathbf{r})=\varepsilon' f(\mathbf{r})$ (see above), the denominator of Eq. (29) has a form identical to that of Eq. (15) with the resonance linewidth dependent on $\varepsilon(0)$, i.e., on the position of the inhomogeneity. The resonance frequency does not depend on $\varepsilon(0)$.

We have not considered the time dispersion of ε' (the dependence on ω). However, this dependence could be significant. Let ε' be given by the expression that is valid when the scatterer is composed from linear oscillators:

$$\varepsilon'=\varepsilon_0\left(1+\frac{\Omega^2}{\omega_0^2-\omega^2}\right), \tag{30}$$

where ω_0 is the fundamental frequency of the oscillator, and Ω is the Langmuir frequency of the gas of oscillators. We consider the 3D case.

We have two resonance frequencies determined by the formula

$$\omega_r^2=\frac{\omega_0^2+\omega_{0r}^2}{2}\pm\left[\frac{(\omega_0^2-\omega_{0r}^2)^2}{4}-\omega_{0r}^2\,\Omega^2\right]^{1/2}, \tag{31}$$

where $\omega_{r0}^2=(c^2/A_1L^2e_0)$ is the square of the resonance frequency, when the time dispersion is absent. We see that the resonance frequency depends on the scatterer's characteristic size, on the oscillator's fundamental frequency, and on the Langmuir frequency. The resonance takes place if ω_r^2 is positive and real, i.e., if $|\omega_0^2-\omega_{r0}^2|>2\Omega^2\omega_{r0}^2$.

In conclusion, the scattered field possesses the following important features: If the surrounding medium is infinite, the scattered field in the 3D and 2D cases has some resonance. The resonance line for a 3D inhomogeneity is wide. The resonance frequency is of order $\omega_r\sim c/L\sqrt{\varepsilon'}$, and the relative width $\gamma\sim\sqrt{\varepsilon/\varepsilon'}\ll 1$. The resonance frequency depends on ε', and the resonance width depends on ε' and ε. The scattered field is isotropic.

In the 2D case the resonance frequency is of the same order as in the 3D case, but the resonance line is wide and is of order $[1/\ln(\lambda_\varepsilon/L)]$. The scattered field is anisotropic. In the 1D case the resonance is absent.

If the surrounding medium contains two half-spaces, the resonance frequency in the 3D and 2D cases is an oscillating function of z_1 with the spatial period $\lambda_\varepsilon/2$. These results are valid for an arbitrary shape of a scatterer (inhomogeneity).

[1] M. Lax, Phys. Rev. **94**, 1391 (1964).
[2] I. M. Lifšic, Nuovo Cimento Suppl. **3**, 716 (1956).
[3] A. A. Maradudin, *Theoretical and Experimental Aspects of the Effects of Point Defects and Disorder on the Vibrations of Crystals* (Academic, New York, 1966).
[4] J. A. Reissland, *The Physics of Phonons* (Wiley, New York, 1973).
[5] G. F. Koster and J. C. Slatter, Phys. Rev. **94**, 1392 (1954); **95**, 1167 (1954).
[6] G. F. Koster, Phys. Rev. **95**, 1436 (1954).
[7] Ad Lagendijk and B. A. van Tiggelen, Phys. Rep. **270**, No. 3 (1966).
[8] H. C. van de Hulst, *Light Scattering by Small Particles* (Dover, New York, 1981).