Relaxation of the distribution function tails for gases with power-law interaction potentials

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The relaxation of rarefied gases of particles with the power-law interaction potentials $U=\alpha/r^s$, where $1 \leqslant s<4$, is considered. The formation and evolution of the distribution function tails are investigated on the basis of the one-dimensional kinetic Landau–Fokker-Planck equation. For long times, the constructed asymptotic solutions have a propagating-wave appearance in the high velocity region. The analytical solutions are expressed explicitly in terms of the error function. The analytical consideration is accomplished by numerical calculations. The obtained analytical results are in a good agreement with the numerical simulation results. [S1063-651X(97)03712-4]

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I. INTRODUCTION AND STATEMENT OF THE PROBLEM

The relaxation of the initial distribution function to the equilibrium state is a classical problem of rarefied gas dynamics that is employed, particularly, as a test for fusion numerical simulations [1–3]. The knowledge of the asymptotic behavior of the solution of the kinetic equation is useful for many plasma physics problems [4,5]. Among these are the problems that consider plasma-wave interactions for wave phase velocities larger than the thermal velocity (for example, the lower hybrid and the electron cyclotron heating) [6–8]. The electron acceleration process by a dc electrical field and the formation of runaway electron tails are also widely studied in the laboratory and in the space plasma applications [9–11].

In the present work we study the relaxation of a space homogeneous gas consisting of one sort of particle. The treatment is based on the Landau–Fokker-Planck–type (LFP type) equation, which is a model of Boltzmann's equation for arbitrary interaction potentials, and briefly considered in Sec. I. Starting from the one-dimensional LFP type equation in Sec. II, we solve the initial value problem for the power-law interaction potentials $U=\alpha/r^s$, $1 \leqslant s<4$ and study the asymptotic solutions in the high energy region for long times. The analytical calculations are carried out on a physical level of precision. All asymptotic results are confirmed by numerical calculations with high accuracy in Sec. III. The conclusion follows in Sec. IV.

The kinetic Boltzmann equation is the basic equation for a rarefied gas dynamics model [12]. In the absence of particle and energy sources as well as their sinks, the spatially uniform Boltzmann equation for the distribution function $f(\mathbf{v},t)$ is

$$\frac{\partial f}{\partial t}=\hat{J}[f,f] = \int d\mathbf{w}\, d\mu d\phi\, u\sigma(u,u)[f(\mathbf{v}')f(\mathbf{w}') - f(\mathbf{v})f(\mathbf{w})], \quad t \geqslant 0. \tag{1}$$

The differential scattering cross section $\sigma(u,\mu)$ is a function of the relative speed $u>0$ and $\mu=\cos\theta \in [-1,1]$. The construction of the cross section $\sigma(u,\mu)$ with the given interaction potential $U(r)$ is a well-known problem of classical mechanics [13,14]. It should be noted that experimental data concerning the real intermolecular potentials are very far from being complete, therefore, in the theory of the Boltzmann equation, the cross section $\sigma(u,\mu)$ is usually assumed to be a known function.

For charged particles, in the case of scattering at small angles $\theta \leqslant \theta_0 \ll 1$, Boltzmann's equation is reduced to the Fokker-Planck equation [15,16], which in the general case has the form [12]

$$\frac{\partial f}{\partial t}=B\frac{\partial}{\partial v_i}\left\{\int d\mathbf{w}\frac{u^2\delta_{ij}-u_iu_j}{u^3}\left(\frac{\partial}{\partial v_j}-\frac{\partial}{\partial w_j}\right)f(\mathbf{v})f(\mathbf{w})\right\}, \tag{2}$$

where

$$B=\frac{8\pi^5}{m^2}\int dk k^3 U_k^2, \quad U_k=\frac{1}{(2\pi)^3}\int d\mathbf{r}U(r)e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3}$$

For the Coulomb interaction the divergence of the first integral in Eq. (3) is connected with the Rutherford cross section singularity at $\mu=1$. To make the integration over k reasonable it is necessary to set finite limits. As a result of some physical motivations, in Eq. (3) $B=2\pi e^4 L/m^2$, where L is the so-called Coulomb's logarithm.

In this paper we use the model of Boltzmann's equation [17,18], which approximates the integral operator $\hat{J}[f,f]$ (1) for arbitrary potentials by an integrodifferential operator of the LFP type. Equation (2) cannot be a generalized form of Boltzmann's equation for arbitrary potential because the first

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integral in Eq. (3) diverges even for the ''step'' potential: $U(r)=U_0$ for $r<r_0$ and $U(r)=0$ for $r>r_0$.

The physical idea underlying the construction of a model collision operator is as follows. Note that in Boltzmann's tequation the only quantity depending on the molecular interaction law is the differential cross section $\sigma(u,\cos\theta)$. We compare the real gas of particles with its analog having another cross section: $\tilde{\sigma}(u,\cos\theta)=0$ for $\theta>\theta_0$ where $\theta\ll 1$ is the scattering at small angles. Hence in the model Boltzmann equation the exact collision integral can be replaced by the approximation $\hat{J}[f,f]\approx\hat{I}[f,f][1+O(\theta_0)]$, as $\theta_0\to 0$. However, the ''cut-off'' cross section $\tilde{\sigma}(u,u)$ is not very similar to the cross section $\sigma(u,\mu)$ because for real intermolecular potentials the contributions from scattering at large and small angles have approximately the same order of magnitude. Nevertheless, the last approximation could be used for arbitrary cross sections, if the fundamental criterion of the chosen approximation in the proposed model is the number of identical moments of the exact and the approximate collision integrals. Really, the most important macroscopic characteristics of a gas are the first moments of the distribution function. Then, the problem consists in an adjustment of the new scattering cross section $\tilde{\sigma}(u,\mu)$ in order to achieve the coincidence of the macroscopic characteristics of the original with that of the model gases:

$$\int d\mathbf{v}\phi(\mathbf{v})\hat{J}[f,f]=\int d\mathbf{v}\phi(\mathbf{v})\hat{I}[f,f], \tag{4}$$

where $\phi(\mathbf{v})$ is an arbitrary function. When this problem is solved, then instead of Boltzmann's integral equation, the simplified collision integral of the LFP type can be considered:

$$\frac{\partial f}{\partial t}=\hat{I}[f,f]=\frac{1}{8}\frac{\partial}{\partial v_i}\left\{\int d\mathbf{w}\, u\sigma(u)(u^2\delta_{ij}-u_iu_j)\times\left(\frac{\partial}{\partial v_j}-\frac{\partial}{\partial w_j}\right)f(\mathbf{v})f(\mathbf{w})\right\}. \tag{5}$$

Here

$$\sigma(u)=\int d\mu\, d\phi\tilde{\sigma}(u,\mu)(1-\mu)=\frac{1}{2}\int d\mu\, d\phi\sigma(u,\mu)(1-\mu^2). \tag{6}$$

The moments of the exact collision integral $\hat{J}[f,f]$ [Eq. (1)] with its model $\hat{I}[f,f]$ [Eq. (5)] coincide up to the tensor third order moment as well as up to the fourth order scalar moment irrespectively of the interaction potential form. That is, the equality (4) is valid for the functions $\phi(\mathbf{v})=\{v_iv_k, v_iv_kv_l,\ i,k,l=1,2,3;\ v^4\}$. The same conservation laws of the particle density, the moment and the energy, as well as the Boltzmann H theorem are valid for the model equation (5). It leads to the correct moment equations up to Grad's twentieth-moment approximation [14]. By inserting the ''cut-off'' Rutherford cross section $\sigma(u,\theta)=(e^2/mu^2)^2\sin^{-4}(\theta/2)\,\eta[\theta-\theta_{\min}]$ into Eq. (6) we get $\sigma(u)\approx 8\pi(2e^2/mu^2)^2L$, where $L=\ln\theta_{\min}^{-1}$, as $\theta_{\min}\to 0$, and one can obtain the well known classical LFP equation for plasmas [12,13]. Finally, for the so-called Maxwell's molecules [1,14] ($s=4$, $U\sim\alpha/r^4$) the exact solution of the approximation equation (5) is the solution of the exact Boltzmann equation (1) [19].

The points enumerated above are on the credit side for the considered model. However, the intermediate potentials of the modeled gas may considerably affect the various local properties of the solution of the model equations and of Boltzmann's equation. Indeed, an equation of LFP type contains an operator of the diffusion type, which leads to instantaneous smoothing of discontinuities in the initial conditions, and for Boltzmann's operator with a short-range intermolecular potential such discontinuities are preserved, since there is some number of particles not experiencing collisions for any finite interval. This distinction emphasizes that the natural sphere for the application of the model equation (5) is gases with potentials of an infinite radius of action, since for these potentials Boltzmann's equation itself must also possess some smoothing properties. That is why, in this paper, we study the distribution function relaxation tails using Eq. (5) for the long-distance soft potentials $U\sim\alpha/r^s$, $1\leqslant s<4$.

II. ASYMPTOTIC SOLUTIONS

We restrict ourselves to the case of the isotropic distribution function $f(v,t)=f(|\mathbf{v}|,t)$ and consider the initial-value problem for the finite initial conditions. At the instant $t=0$, the initial function $f_0(v)=f(v,0)$ is located in the thermal velocity region $v\sim v_{\rm th}$. Our investigation is concentrated on the formation and evolution of the distribution function tails for long times, as $|\mathbf{v}|\to\infty$.

For the isotropic distribution function, Eq. (5) can be written as

$$\frac{\partial f}{\partial t}=\frac{1}{v^2}\frac{\partial}{\partial v}\left\{\frac{1}{v}\int_0^\infty dwQ(v,w)\left[wf(w)\frac{\partial f(v)}{\partial v}-vf(v)\frac{\partial f(w)}{\partial w}\right]\right\}, \tag{7}$$

where the symmetric kernel $Q(v,w)$ is

$$Q(v,w)=\frac{\pi}{8}v^3w^3\int_{-1}^1 d\mu(1-\mu^2)u\sigma(u),$$

$$u^2=v^2+w^2-2vw\mu.$$

The above equation should be completed by the conservation laws

$$\rho=4\pi\int_0^\infty dvv^2f(v,t)=\text{const},$$

$$T=\frac{4\pi m}{3\rho}\int_0^\infty dvv^4f(v,t)=\text{const},\quad t\geqslant 0, \tag{8}$$

where ρ is the particle density and the temperature T is expressed in units of energy. The unique equilibrium solution of the problem is the Maxwell distribution function

$$f_M(v)=\rho(2\pi)^{-3/2}v_{\rm th}^{-3}\exp\left[-\frac{v^2}{2v_{\rm th}^2}\right],\quad v_{\rm th}=\sqrt{T/m}.$$

We construct the asymptotic solution of Eq. (7) for a long period when the relaxation in the thermal velocity region is practically finished. Therefore, in the high velocity region $v\gg v_{\rm th}$ the nonlinear equation (7) can be approximated by the linear equation

$$\frac{\partial f}{\partial t}=\frac{\rho}{8}\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^3\sigma(v)\left(\frac{T}{m}\frac{\partial f}{\partial v}+vf\right)\right].$$

Here we take into account the fact that in the high velocity region, the kernel $Q(v,w)$ reduces to $Q(v,w)\to\pi w^3v^4$, $\sigma(v)/6$, as $v\to\infty$. Furthermore, for the power-law potentials, we employ the cross section $\sigma(u,\mu)=g_s(\mu)u^{-4/s}$ [14]. By inserting this expression in Eq. (7), as $v\gg v_{\rm th}$, we obtain for the last equation

$$\frac{\partial f}{\partial t}=\frac{\rho g_s}{8}\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^{3-4/s}\left(\frac{T}{m}\frac{\partial f}{\partial v}+vf\right)\right],\tag{9}$$

where the notation

$$q_s=2\pi\int_{-1}^{1}d\mu g_s(\mu)(1-\mu^2)\tag{10}$$

is used. For instance, for the Coulomb potential, $s=1$, formula (10) yields $g_1=32\pi e^4L/m$. Then, from Eq. (9) the well known linear LFP equation for plasmas follows. Further we shall consider the equation (9) for the arbitrary interaction potentials $U=e^2/r^s$, $1\leqslant s<4$.

For convenience we transform Eq. (9) in a suitable way by introducing the new dimensionless variables for the speed, time, and distribution function:

$$x=\left[\frac{v}{v_{\rm th}}\right]^{(4+s)/2s}X^{(4+s)/4s},\quad \tau=t\frac{\rho g_s}{8}\frac{4-s}{s}\left[\frac{X}{v_{\rm th}}\right]^{(4-s)/2s},$$

$$f(v,t)=f_M(v)u(x,\tau),\tag{11}$$

where $X=2s(4-s)/)(4+s)^2$, and $f_M(v)$ is the Maxwell distribution. By inserting Eq. (11) in Eq. (9) we obtain for the distribution function $u(x,\tau)$

$$\frac{\partial u}{\partial\tau}+\frac{x^{1-p}\partial u}{p\ \ \partial x}=\frac{1}{2}\frac{\partial^2u}{\partial x^2},\ p=2\frac{4-s}{4+s},\quad 1\leqslant s<4.\tag{12}$$

The type of the last equation gives us an idea about the wave character of a solution. Really, it is worth recalling that we are interested in the relaxation of the distribution function tails in the high energy region $x\gg1$. Otherwise, we consider the period τ when the relaxation process is practically finished in the thermal velocity region $x<1$, i.e., the period when $u(x,\tau)\approx1$. Hence we solve the problem in the superthermal velocity region $x\gg1$ in which the slow establishing of the equilibrium solution $u_M(x)=1$ occurs, and $u(x,\tau)\to0$ as $x\to\infty$. The condition $x\gg1$ can be taken into account by the new variable $\tilde{x}=x/x_0$ and $\tilde{\tau}=\tau/x_0^p$, where $x_0\gg1$ is some characteristic scale of the problem under investigation. Then, the small factor $x_0^{p-2}\ll1$ appears before the highest (the second) derivative on the right hand side of Eq. (12). From this it can be concluded that the equation changes its type and works like a transport operator in the high velocity region $x\to\infty$. Having this in mind, we introduce in (12), for simplicity, the formal parameter ε before the second derivative. The result it

$$\frac{\partial u}{\partial\tau}+\frac{x^{1-p}}{p}\frac{\partial u}{\partial x}=\frac{\varepsilon}{2}\frac{\partial^2u}{\partial x^2}.\tag{13}$$

Now, if we let $\varepsilon\approx0$, Eq. (13) collapses into a first order equation, and the equilibrium solution $u_M(x)$ is simply transported over the characteristics of this equation. The typical solution has the form of a step function:

$$u(x,\tau)\approx\eta[\tau^{1/p}-x],\quad p=2\frac{4-s}{4+s},\quad 1\leqslant s<4.\tag{14}$$

The approximate solution (14) reflects correctly the asymptotic law under which the wave front moves

$$x_f(\tau)\sim\tau^{1/p}$$

but does not describe the wave front structure.

In order to analyze the front structure, we shall transform Eq. (13) using the previously obtained information about the wave front law. By setting

$$z=\frac{x-\tau^{1/p}}{\sqrt{\varepsilon}},\quad u(x,\tau)=\phi(z,\tau),$$

we can rewrite Eq. (13) for the new function $\phi(z,\tau)$ as

$$\frac{\partial\phi}{\partial\tau}=\frac{1}{2}\frac{\partial^2\phi}{\partial z^2}-\frac{1}{p}\frac{\tau^{1-p}-1}{\sqrt{\varepsilon}}\left[\left(1+\sqrt{\varepsilon}\frac{z}{\tau^{1/p}}\right)^{(1-p)}-1\right]\frac{\partial\phi}{\partial z}.\tag{15}$$

Now let ε tend to zero but z remain finite. Then, under this conditional limit, Eq. (15) takes the form

$$\frac{\partial\phi}{\varepsilon\,\tau}=\frac{1}{2}\left(\frac{\partial^2\phi}{\partial z^2}-\gamma\frac{z}{\tau}\frac{\partial\phi}{\partial z}\right),\quad \gamma=2\frac{1-p}{p}.\tag{16}$$

The boundary conditions for the function ϕ are the following: $\phi\to1$, as $z\to-\infty$ (behind the wave front), and $\phi\to0$, as $z\to\infty$ (before the wave front). The function $\phi(z,\tau)$ can be constructed as a solution of an initial value problem for time $\tau>1$ with the corresponding initial condition $\phi(z,1)=\eta(-z)$ [compare with function (14)]. By the proper changing of variables, Eq. (16) can be reduced to the heat equation [20]. After that, the solution of Eq. (16) can be obtained in the self-conserved form:

$$\phi(z,\tau)=\Phi\left(z\left(\frac{(1-\gamma)}{2(\tau-\tau^\gamma)}\right)^{1/2}\right),\quad \tau>1,$$

$$\Phi(z)=\frac{1}{\sqrt{\pi}}\int_z^\infty dy\ \exp(-y^2).$$

We now return to the variable x and suppose again ε to be equal to 1. After this, one an find the quasistationary asymptotic solution of Eq. (12)

$$u(x,\tau)\approx\Phi\left(\frac{x-\tau^{1/p}}{\sqrt{2}}\left(\frac{1-\gamma}{\tau-\tau^{\gamma}}\right)^{1/2}\right),\quad \tau>1,\quad \gamma=2\frac{1-p}{p}. \tag{17}$$

In order to understand when solution (17) is valid, let us formulate again what was previously assumed. We consider the kinetic equation in the high velocity region $x\gg 1$ for a time period that is much greater than the so-called collision time $\tau\gg 1$ and in the vicinity of the wave front point $x_f\sim\tau^{1/p}$: $(x-\tau^{1/p})\ll\tau^{1/p}$. The last inequality justifies the transition from Eq. (15) to (16) under the condition $\varepsilon=1$ and gives us the following restrictions. The obtained solution (17) is inapplicable, first, for the cold particles within the velocity interval $0\leqslant x\ll x_f$ and, second, for the particles outside the velocity region $x\geqslant 2x_f$. Practically speaking, both these restrictions have a small interest because with a good accuracy, it may be reasonably supposed that $u(x,\tau)\approx 1$ in the first region $x\ll x_f(\tau)$, and $u(x,\tau)\approx 0$ in the second region $x\geqslant 2x_f(\tau)$.

We can simplify formula (17) taking into account the condition $\tau\gg 1$. Since the final result will depend on γ, we consider three cases for the expression within lightface large parentheses in formula (17). For $\gamma<1$ the expression $(1-\gamma)(\tau-\tau^{\gamma})^{-1}$ tends to $(1-\gamma)\tau^{-1}$, and for $\gamma>1$ it tends to $(\gamma-1)\tau^{-\gamma}$. The case of $\gamma=1$ is distinctive of others. Note that $\gamma=1$ corresponds to $s=2$. To derive the formula for this limiting case, we rewrite the expressions as follows:

$$(1-\gamma)(\tau-\tau^{\gamma})^{-1}=(1-\gamma)\left(\tau^{\gamma}\sum_{m=1}^{\infty}\frac{[(1-\gamma)\ln\tau]^m}{m!}\right)^{-1}$$

$$=\left(\tau\gamma\sum_{m=1}^{\infty}\frac{(1-\gamma)^{m-1}\ln^m\tau}{m!}\right)^{-1}.$$

For $\gamma=1$ only the first term is retained in the last expression since the coefficients of the sum vanish for all $m\geqslant 2$.

Thus, having assumed the condition $\tau\gg 1$, we obtain the simplified asymptotic solution $u(x,\tau)$ that will vary as a function of γ:

$$u(x,\tau)\approx\begin{cases}\Phi\left(\dfrac{(x-\tau^{1/p})}{\sqrt{2}}\left(\dfrac{1-\gamma}{\tau}\right)^{1/2}\right), & \gamma<1,\\ \Phi\left(\dfrac{(x-\tau^{1/p})}{\sqrt{2}}\left(\dfrac{1}{\tau\ln\tau}\right)^{1/2}\right), & \gamma=1,\\ \Phi\left(\dfrac{(x-\tau^{1/p})}{\sqrt{2}}\left(\dfrac{\gamma-1}{\tau^{\gamma}}\right)^{1/2}\right), & \gamma>1.\end{cases}$$

Finally, we formulate the results obtained above for the variables v, t and for the distribution function $f(v,t)$. Considering the LFP type equation (7) for the soft power-law potentials $U\sim\alpha/r^s$, $1\leqslant s<4$, we have constructed the approximate asymptotic (as $v\to\infty$, and $t\to\infty$) solution

$$f_s(v,t)\approx\rho\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left(-\frac{mv^2}{2T}\right)$$

$$\times u\left(v\left(\frac{m}{T}\right)^{1/2},t\frac{pg_s}{8}\left(\frac{m}{T}\right)^{(4-s)/2s}\right). \tag{18}$$

The constant value g_s is defined in Eq. (10). The function $u(v,t)$ has the form of a propagated wave front that moves under the law

$$v_f(t)=\left(\frac{4-s}{s}t\right)^{s/(4-s)}. \tag{19}$$

The function $u(v,t)$ is descried by three cases that are dependent on the exponent s:

$$u(v,t)=\begin{cases}\Phi\left(2\dfrac{\sqrt{s(2-s)}}{4+s}v_fV^{(4+s)/2s}\right), & 1\leqslant s<2,\\ \Phi\left(\frac{1}{3}(\ln v_f)^{-1/2}v_fV^{3/2}\right), & s=2,\\ \Phi\left(\left(\dfrac{2(s-2)}{4-s}\right)^{1/2}\left[\dfrac{2s(4-s)}{(4+s)^2}\right]^{(4-s)/2s}v_f^{(4-s)/s}V^{(4+s)/2s}\right), & 2<s<4,\end{cases} \tag{20}$$

where

$$V=\frac{v-v_f(t)}{v_f(t)}.$$

The applicability conditions for the solution (20) will take the following form of the enhanced inequalities:

$$v\gg 1,\quad v_f(t)\gg 1,\quad |v-v_f(t)|\ll v_f(t). \tag{21}$$

Noting that $u[v_v(t),t]=1/2$, we, as usual, define the front width as

$$\Delta_f(t)=2\left|\frac{u(v,t)}{\partial u(v,t)/\partial v}\right|_{v=v_f(t)}=\left|\frac{\partial u(v,t)}{\partial v}\right|^{-1}_{v=v_f}.$$

Therefore, for the cases considered above we obtain

$$\Delta_f(t)=\begin{cases}\sqrt{\pi s/(2-s)}, & 1\leqslant s<2,\\ \sqrt{\pi\ \ln v_f(t)}\ , & s=2,\\ \left(\dfrac{\pi s}{s-2}\right)^{1/2}\left[\dfrac{2s(4-s)}{(4+s)^2}\right]^{(s-2)/s}[v_f(t)]^{2(s-2)/s}, & 2<s<4.\end{cases}\tag{22}$$

As can be seen, the front width $\Delta_f(t)$ is substantially dependent on the exponent s in the interaction potential. Thus, $\Delta_f(t)$ is a constant value for $1\leqslant s<2$. Therefore, in this case the solution $u(v,t)$ has a character of a wave with a stable profile that does not spread for all $t>0$. This fact, for $s=1$, was discovered in the numerical calculation [17,18] and a corresponding analytical solution was constructed. Starting from the value $s=2$ the front width begins to grow in time but has a weak logarithmic dependence: $\Delta_f\sim\sqrt{\ln t}$. The behavior of the solution still has the wave type. The front propagation velocity $v_f(t)$ remains sufficiently larger than the velocity of its spreading:

$$\frac{\partial\Delta_f/\partial t}{\partial v_f/\partial t}=\frac{d\Delta_f}{dv_f}=\frac{1}{2\ln v_f(t)}\frac{\Delta_f(t)}{v_f(t)}\ ,\quad s=2.$$

In the case $2<s\leqslant 4$ the velocity of the wave front propagation $\partial v_f/\partial t$ and the velocity of its dispersion $\partial\Delta_f/\partial t$ are approximately equal:

$$\frac{\partial\Delta_f/\partial t}{\partial v_f/\partial t}=\frac{d\Delta_f}{dv_f}=\left(2-\frac{4}{s}\right)\frac{\Delta_f}{v_f}\ ,\quad 2\leqslant s<4.$$

Obviously, in the case of Maxwell's molecules, $s=4$, the behavior of the distribution function tail does not have the real wavelike character.

The results of the numerical calculations are presented in the following section. They illustrate the obtained analytical results.

III. NUMERICAL CALCULATIONS

The purpose of the numerical calculations is to compare the evolution of the same initial distribution function in a gas of particles with different power-law interaction potentials. We consider Eq. (7):

$$\frac{\partial f}{\partial t}=\frac{1}{2v^2}\frac{\partial}{\partial v}\left\{\frac{1}{v}S[f,v]\right\},\tag{23}$$

where

$$S[f,v]=\int_0^\infty dw Q(v,w)\left[wf(w)\frac{\partial f(v)}{\partial v}-vf(v)\frac{\partial f(w)}{\partial w}\right].$$

For the case under consideration, the symmetrical kernel $Q(v,w)$ can be written as follows [18]:

$$Q(v,w)=\frac{a(v,w)(v+w)^{n+4}+b(v,w)|v-w|^{n+4}}{(n+2)(n+4)(n+6)}\ ,$$

with

$$a(v,w)=(n+4)[vw-(v^2+w^2)],$$

$$b(v,w)=(n+4)[vw+(v^2+w^2)],\quad n=(s-4)/s.$$

Positive values of n correspond to the stiff interaction potentials ($s>4$), and negative values of n correspond to the soft interaction potentials ($1\leqslant s<4$). The special classes of the particle interaction, which are particularly explored in our numerical calculations, should be mentioned. These are charged particles, $s=1(n=-3)$, and Maxwell's molecules, $s=4$ $(n=0)$. For the chosen dimensionless variables the density and the energy conservation laws are

$$\rho=\int_0^\infty dv\ v^2f(v,t)=1,\quad \mathcal{E}=\int_0^\infty dv\ v^4f(v,t)=1,\quad t\geqslant 0,\tag{24}$$

respectively. The thermal velocity is equal to unity, $v_{\rm th}=1$, and the equilibrium solution of Eq. (23) is

$$f_M(v)=\frac{4}{\pi^{1/2}}\left(\frac{3}{2}\right)^{3/2}\exp\left(-\frac{3}{2}v^2\right).$$

At the initial instant, the distribution function is located in the thermal velocity region and has a δ-function type: $f(v,0)=\delta(v-1)/v^2$.

To compose the difference scheme we replace the infinite velocity interval $[0,\infty)$ by the finite segment $[0,L]$, which is chosen so as to take into account the high-energy particles. As a rule, it is sufficient to take $L\approx(7-8)v_{\rm th}$. In the considered domain we introduce the space-time mesh $\{v_{i+1}=v_i+h_{i+1},\ i=1,2,\ldots,M-1,\ v_1=0,\ v_M=L;t^{k+1}=t^k+\tau,\ k=0,1,\ldots,t^0=0\}$ and define the mesh functions f_i^k, $S_i^k[f]$. The following notation will be used: $f_{i+1/2}=0.5(f_{i+1}+f_i)$. We approximate the integral by the trapezoid formula and the derivatives by the central differences. Applying the integrointerpolation method to Eq. (23), we obtain the implicit difference scheme

$$\frac{f_i^k-f_i^{k-1}}{\tau}=\frac{1}{v_i^2h_{i+1/2}}\left[\frac{S_{i+1/2}}{v_{i+1/2}}-\frac{S_{i-1/2}^k}{v_{i+1/2}}\right].\tag{25}$$

The boundary conditions are $S_{3/2}=0$ at the point $v=0$ and $f_M=0$ at the point $v=L$. The initial distribution is approximated on the mesh in the usual way, that is,

$$f(v_i,0)=\begin{cases}2/(v_{i+1}-v_{i-1}) & \text{if } v_i=1,\\ 0 & \text{otherwise.}\end{cases}$$

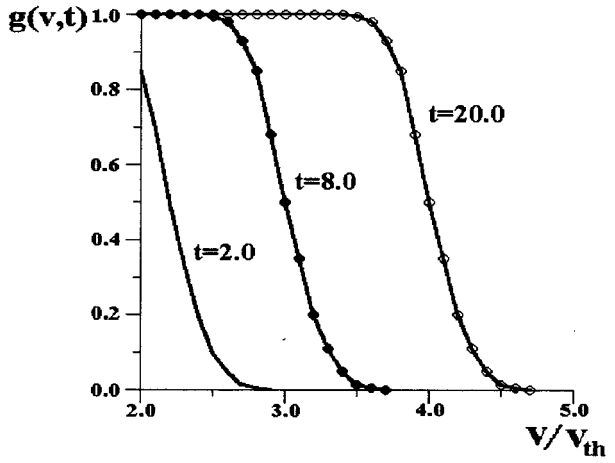

FIG. 1. The graph of the distribution function $g(v,t) = f(v,t)/f_M(v)$ normalized to the Maxwell distribution in the velocity region $2<v/v_{\text{th}}<5$ for different time instants t (arbitrary units). Time normalized to the electron-electron collision time is defined in Eq. (11). The case corresponds to the Coulomb interaction $n=-3(s=1)$, $U=\alpha/r$.

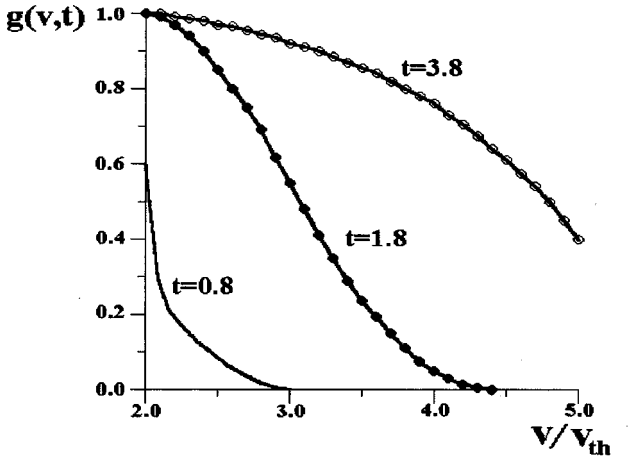

FIG. 2. The graph of the distribution function $g(v,t) = f(v,t)/f_M(v)$ normalized to the Maxwell distribution in the velocity region $2<v/v_{\text{th}}<5$ for different time instants t (arbitrary units). Time normalized to the corresponding collision time is defined in Eq. (11). The case corresponds to the value $n=-1(s=2)$, $U=\alpha/r^2$.

This approximation makes the number of particles and kinetic energy, as well, equal to unity. The difference scheme is constructed on a symmetric pattern, and has a second order approximation with respect to the velocity space [21]. The completely conservative difference scheme used here [22] allows us to make calculations without numerical error accumulation, except for machine errors. It must be emphasized that the error in the numerical analog of the conservation laws (24) is of the same order as the rounding machine error.

Now we briefly describe the results of numerical simulations. An equation of the LFP type contains an operator of the diffusion type, which leads to instantaneous smoothing of discontinuities of the initial conditions. Very rapidly, the solution acquires a quasiequilibrium form in the thermal velocity region $(0\leqslant v\leqslant 2)$ at the instant t_0 that corresponds to the so-called collision time. The characteristic time t_0 weakly depends on various values of the exponent s in potentials $U\sim\alpha/r^s$. In this region the distribution functions are close to each other throughout the entire relaxation process for different values s. As is expected, the main difference is observed in the region of the distribution tails for $v>2$. For better representation of the numerical results in the high velocity region, we introduce the function $g(v,t) = f(v,t)/f_M(v)$, which is the analog of the function $u(v,t)$ in Eq. (18). Figures 1–3 show graphs of the functions $g(v,t)$ in the velocity region $2<v<5$ for different time instants and for various parameters s. All figures demonstrate a wave spreading of $g(v,t)$ into the region of high velocities. For charged particles with the Coulomb interaction $(s=1, n=-3, U=\alpha/r)$, the function $g(v,t)$ is shown in Fig. 1. The solution has the character of a propagated wave with a stable profile. For the potentials with $1\leqslant s\leqslant 2$, the tail relaxation proceeds more slowly than the relaxation of the distribution function core. For the values $2<s\leqslant 3$ the evolution of the distribution function tail still maintains the wave character. It is shown in Fig. 2 that the wave front width grows slowly with time for the case $s=2$, $n=-1$, $U=\alpha/r^2$. For the stiff potentials $(s\geqslant 4)$ the characteristic times of the distribution core relaxation and of the distribution tail relaxation are practically the same. In this case, the wave propagation velocity and the speed of the wave diffusing are almost indistinguishable (Fig. 3). Practically, the relaxation of the distribution function in the thermal and superthermal velocity regions occurs simultaneously.

IV. CONCLUSION

The relaxation process of a space uniform isotropic rarefied gas compose of one sort of particle is considered (electrical neutrality is supposed). The paper is exclusively concerned with the asymptotic behavior of the distribution

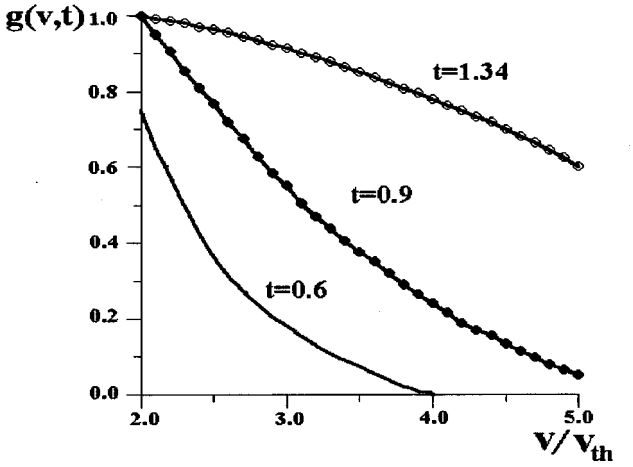

FIG. 3. The graph of the distribution function $g(v,t) = f(v,t)f_M(v)$ normalized to the Maxwell distribution in the velocity region $2<v/v_{\text{th}}<5$ fop different time instants t (arbitrary units). Time normalized to the corresponding collision time is defined in Eq. (11). The case corresponds to the Maxwellian molecules $n=0, (s=4)$, $U=\alpha/r^4$.

function tails, as $v \to \infty$, $t \to \infty$.

In the present work, the relaxation to the equilibrium state is considered on the basis of the nonlinear kinetic LFP type equation (5), which is the model of Boltzmann's equation. The model is based on the replacement of the exact collisional integral by an integrodifferential equation of the LFP type. The fundamental criterion of approximation chosen in the proposed model is the number of identical moments of the exact and approximate collision integrals. This criterion is generally accepted for the construction of kinetic models. The model equation (5) possesses the basic properties of the Boltzmann equation. Nevertheless, the natural sphere of applications of this equation is in gases with interaction potentials of an infinite radius of action.

The formation and evolution of the distribution function tails are investigated for the arbitrary power-law potentials $U=\alpha/r^s$ where $1 \leqslant s \leqslant 4$ ($s=1$ corresponds to the Coulomb interaction of charged particles). The initial distribution function is located in the thermal velocity region. The constructed approximate asymptotic solutions $f_s(v,t)$ [Eq. (18)] have a type of a propagated wave to the high velocity region. The wave front moves under the law (19). The applicability conditions (21) of the solutions (18) are given. The obtained analytical solutions are expressed explicitly through the error function. The analytical expressions for the wave front structure (20) and for the wave front width (22) are dependent on the interaction potential. Here, depending on whether s is less than or larger than 2, the type of the distribution tail has a strong pronounced wave character. Particularly, in the case of the potential $U=\alpha/r^s$ with exponent $1 \leqslant s < 2$, the solution $u(v,t)$ has a stable wave profile for the velocity region $v \gg v_{\rm th}$ during the relaxation process. And for $s \leqslant 4$ the distribution function relaxation in the thermal and in the superthermal region occurs almost simultaneously.

The completely conservative finite difference scheme (25) provides calculations without error accumulation. The error in the conservation laws (24) is of the same order as the rounding machine error. Numerical simulation results presented here are in good agreement with the obtained analytical asymptotic solutions.

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