Spatial complex behavior in nonchaotic flow systems

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The convective instability induces the phenomenon of spatial sensitivity to the boundary conditions. We introduce a quantity, the "spatial" Lyapunov exponent, to characterize this kind of complex behavior, in nonchaotic but convectively unstable flow systems. Then we establish a relation between this spatial-complexity index and the comoving Lyapunov exponent. [S1063-651X(97)05810-8]

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The main property of chaos is the sensitive dependence of the evolution on the initial conditions, i.e., small perturbations on the initial state grow exponentially in time [1]. This is usually assumed as the characterizing property of chaos, and it is quantified by a positive value of the maximal Lyapunov exponent λ_1.

Beside the chaotic systems, highly nontrivial behaviors can appear also in systems which are not chaotic (i.e., $\lambda_1<0$). Let us mention the systems with asymptotically stable fixed points, but with fractal boundaries of the attraction basins [2], and the chaotic scattering phenomenon [3], where the "chaos" is just transient.

An interesting situation can occur in high dimensional systems, like the following chain of maps with unidirectional coupling:

$$x_n(t+1)=(1-c)f_a(x_n(t))+cf_a(x_{n-1}(t)), \qquad (1)$$

where t is the discrete time, $n=1,2,3,\ldots$ is a spatial index, and $x_0(t)$ is a given boundary condition; a typical choice for the local map is $f_a(x)=ax(1-x)$. This type of models are quite natural candidates for the description of systems with a privileged direction, e.g., boundary layer, thermal convection, and wind-induced water waves [4].

After the seminal papers of Deissler and Kaneko [5] it is now well known that nontrivial phenomena can take place in systems with asymmetric couplings, even in the absence of chaos ($\lambda_1\leqslant 0$). In particular, if the system is convectively unstable the spatial structure can be very complex and the external noise can have an important role in the formation and amplification of this structure [5,6]. Some authors, e.g., Pikovsky [7] and Kozlov *et al.* [8], stressed the fact that the "irregularity" of these systems seems to increase with n. An analysis of x_n as a function of t (by means of some standard methods for the characterization of dynamical systems, such as, for instance, the power spectrum, the return map, the Grassberger-Procaccia correlation dimension [9]) typically shows that x_1 is more irregular than x_0, x_2 more irregular than x_1, and so on.

In spite of the clear evidence of a spatial "complexity" in these nonchaotic systems, up to now, as far as we know, there is not a simple and systematic quantitative characterization of this phenomenon and its possible quantitative relation with the comoving Lyapunov exponents. The definition of the comoving Lyapunov exponent $\lambda(v)$, for these extended systems, may be given as follows [5]. If $\delta x_0(0)$ is a perturbation on the boundary at the time $t=0$, in a frame of reference that moves along the system with velocity $v>0$, at large t, this perturbation is $O(\exp[\lambda(v)t])$; when there exists a range of velocity for which $\lambda(v)$ is positive, then the system is said to be convectively unstable. The interesting situation arises when the usual Lyapunov exponent, $\lambda_1=\lambda(v=0)$, is negative.

In many papers the boundary condition is kept fixed, i.e., $x_0(t)=x^*$, where often x^* is an unstable fixed point of the single map $x(n+1)=f(x(n))$ [10]. Here, following Deissler [5] and Pikovsky [7], we adopt a more general time dependent boundary condition: $x_0(t)=f(t)$ with $f(t)$ a known function, that may be periodic, quasiperiodic, or obtained by a chaotic system. It is natural to wonder how an uncertainty $\delta x_0(t)=O(\epsilon)$, with $\epsilon\ll 1$, on the knowledge of the boundary conditions will affect the system. In this letter we consider only the case of infinitesimal perturbations, so that we may safely assume that δx_n evolves according to the tangent vector equations of the system (1).

For the moment we do not consider, for the sake of simplicity, intermittency effects, that is, we neglect finite time fluctuations of the comoving Lyapunov exponent. The uncertainty $\delta x_n(t)$, on the determination of the variable at the site n, is given by the superposition of the evolved $\delta x_0(t-\tau)$ with $\tau=n/v$:

$$\delta x_n(t)\sim\int \delta x_0(t-\tau)e^{\lambda(v)\tau}dv=\epsilon\int e^{[\lambda(v)/v]n}dv. \qquad (2)$$

Since we are interested in the asymptotic, in space, behavior, i.e., large n, we can write

$$\delta x_n(t)\sim\epsilon e^{\gamma n}, \qquad (3)$$

where, in the particular case of a nonintermittent system,

$$\gamma=\max_v\frac{\lambda(v)}{v}. \qquad (4)$$

Equation (4) gives a link between the comoving Lyapunov exponent and the "spatial" Lyapunov exponent γ, a more precise and operative definition of which is given by

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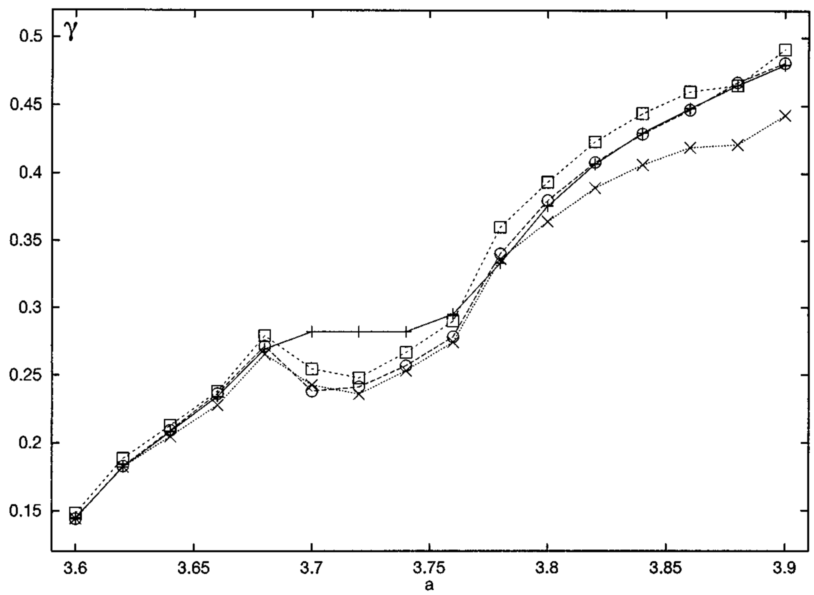

FIG. 1. γ (+), γ_1 (×), γ_p (□), and γ^* (○) versus a at fixed $c=0.7$ for the logistic map $f_a(x)=ax(1-x)$; the boundary condition is quasiperiodic: $x_0(t)=0.5+0.4\sin(\omega t)$, with $\omega=2\pi(5^{1/2}-1)/2$.

$$\gamma=\lim_{n\to\infty}\frac{1}{n}\left\langle \ln\frac{|\delta x_n|}{\epsilon}\right\rangle, \tag{5}$$

where the brackets mean a time average. So Eq. (4) establishes a relation between the convective instability of a system and its sensitivity to the boundary conditions, which can be considered a sort of spatial complexity.

Let us remark again that Eq. (4) holds exactly only in the absence of intermittency; in this case it can be shown from Eq. (4) that our spatial index can be written in a simple way in terms of the ''spatial Lyapunov exponents'' $\mu(\Lambda)$ introduced in Ref. [11]:

$$\gamma=\max_v\frac{\lambda(v)}{v}=\mu(\Lambda=0). \tag{6}$$

In the general case the relation is rather more complicated. If we define the effective comoving Lyapunov exponent $\widetilde{\lambda}_t(v)$, instead of Eq. (2) we have

$$\delta x_n(t)\sim\epsilon\int e^{[\widetilde{\lambda}_t(v)/v]n}dv, \tag{7}$$

and therefore

$$\gamma=\lim_{n\to\infty}\frac{1}{n}\left\langle \ln\frac{|\delta x_n|}{\epsilon}\right\rangle=\lim_{n\to\infty}\frac{1}{n}\ln\frac{|\delta x_n^{\text{typical}}|}{\epsilon}=\left\langle \max_v\frac{\widetilde{\lambda}_t(v)}{v}\right\rangle. \tag{8}$$

In a generic case, because of the fluctuations, it is not possible to write γ in terms of $\lambda(v)$. Nevertheless it is possible to state a lower bound:

$$\gamma\geqslant\max_v\frac{\langle\widetilde{\lambda}_t(v)\rangle}{v}=\max_v\frac{\lambda(v)}{v}\equiv\gamma^*. \tag{9}$$

The evaluation of the function $\lambda(v)$ needs a heavy computational effort, however, one can find good approximations of the quantity γ^*. A first simple approximation for it, actually a lower bound, is given by

$$\gamma_1=\frac{\lambda(v^*)}{v^*}, \tag{10}$$

where v^* is the velocity at which λ attains its maximum value. The analysis of the long time behavior of many impulsive perturbations makes it possible to obtain v^* and $\lambda(v^*)$ without the knowledge of $\lambda(v)$ as a function of v. An improvement of this approximation can be performed in the following way. Beside $\lambda(v^*)$, one computes the usual Lyapunov exponent $\lambda_1=\lambda(0)$, then one estimates the function $\lambda(v)$, by assuming it is the parabola $\lambda_p(v)$ passing through the point $(0,\lambda_1)$ with maximum $\lambda(v^*)$ for $v=v^*$ and, finally, one determines $\gamma_p=\max_v[\lambda_p(v)/v]$. Typically γ_p is very close (within a few percent) to γ^*.

In Fig. 1 we show γ, γ^*, γ_1, and γ_p versus a at a fixed value of c ($c=0.7$) for the logistic map, $f_a(x)=ax(1-x)$, and using a quasiperiodic boundary condition, $x_0(t)=0.5+0.4\sin(\omega t)$, with $\omega=2\pi(\sqrt{5}-1)/2$. There is a large range of values of the parameter a for which γ is rather far from γ^*; for instance, at $a=3.74$ we have $\gamma=0.34$ and $\gamma^*=0.26$. The difference is an effect of the intermittency; this may be pointed out by looking at what happens with the map $f_a(x)=ax$ mod 1: in this case we find that, all over the explored range of variation of a, γ and γ^*, from a numerical point of view, are indistinguishable (their relative difference is smaller than 10^{-6}).

We may obtain a further indication of the fact that the non-negligible fluctuations of the comoving Lyapunov exponents are at the origin of the marked difference of γ from its lower bound, by introducing, following Ref. [12], the generalized spatial Lyapunov exponents $L_s(q)$. These quantities allow us to characterize the fluctuations in the growth of the perturbations along the chain:

$$L_s(q)=\lim_{n\to\infty}\frac{1}{n}\ln\left\langle\left|\frac{\delta x_n}{\epsilon}\right|^q\right\rangle. \tag{11}$$

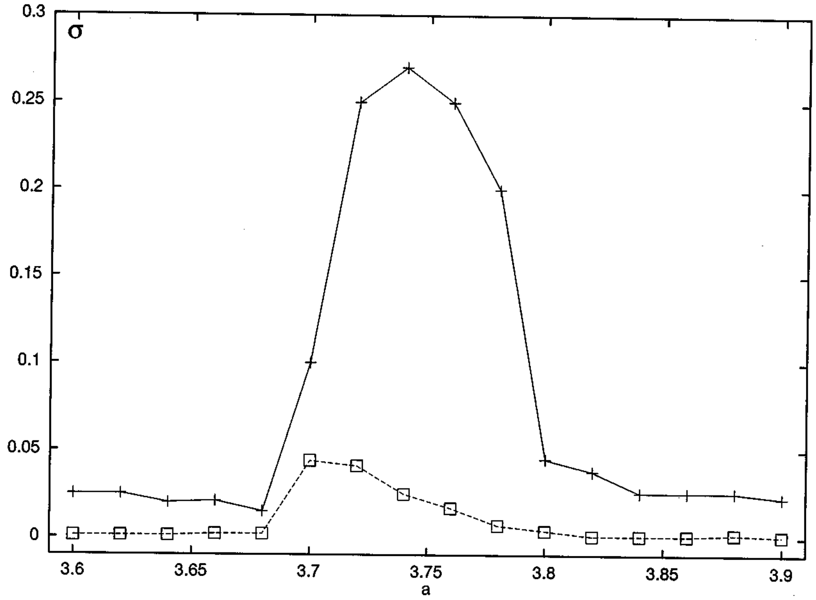

FIG. 2. σ^2 (+) and $\gamma-\gamma^*$ (□) versus a at $c=0.7$; the boundary condition is the same as Fig. 1.

By means of standard arguments of probability theory, one has that (a) $L_s(q)/q$ is a monotonic nondecreasing function of q; (b) $dL_s(q)/dq|_{q=0}=\gamma$; and (c) $L_s(q)=\gamma q+\frac{1}{2}\sigma^2 q^2$, for small q, where $\sigma^2=\lim_{n\to\infty}\langle(\ln|\delta x_n/\epsilon|-\gamma n)^2\rangle/n$. The shape of $L_s(q)/q$ depends on the details of the dynamics, however, $L_s(q)$ is roughly determined by the two parameters γ and σ^2. The reason for having introduced this function is that one expects some relation between the fluctuations of the spatial-complexity index γ, and the fluctuations of the effective comoving Lyapunov exponent, and it is much easier to compute the former than the latter. Figure 2 shows that, in the case of the logistic map, as we expected, the parameter σ^2 (that is related to the variance of the spatial fluctuations) is small (large) in the region where γ^* is a good (bad) approximation of γ.

We stress that all the results above do not depend too much on the details of the boundary conditions $x_0(t)$ used. Indeed we found that if $x_0(t)$ has a chaotic behavior, like that of the y variable of the Hénon map: $y(t+1)=-\alpha y(t)^2+\beta y(t-1)+1$ (with typical values of the parameters $\alpha=1.4$ and $\beta=0.3$), γ and σ^2 as functions of a are not very different from the case with $x_0(t)$ a quasiperiodic function.

In conclusion we have shown how in a nonchaotic, but convectively unstable flow, where the convective instability induces a spatial sensitivity to the boundary conditions, it is possible to introduce an index (a sort of ''spatial'' Lyapunov exponent) for the quantitative characterization of this ''spatial complexity.'' Moreover, there exists a relation (a bound) between this spatial complexity and the comoving Lyapunov exponents.

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