Bifurcations and chaos for the quasiperiodic bouncing ball

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We investigate the influence of a second frequency on the classical periodic bouncing-ball problem, and call it the quasiperiodic bouncing ball. We indicate how to compute the Lyapunov exponent for implicit maps and confirm the presence of chaos for the periodic bouncing ball. We have numerically found a series of nontrivial bifurcations for the quasiperiodic bouncing ball. We have also found several cases of nonperiodic attractors with negative Lyapunov exponents. [S1063-651X(97)02710-4]

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The standard bouncing-ball problem consists of a particle, in a constant gravitational field, falling down vertically on a periodically oscillating table. This model is naturally connected to a series of physical and engineering problems; further details and references can be found in the book by Tufillaro *et al.* [1]. It seems that the periodic bouncing-ball problem (PBB) was first proposed by Zaslavsky [2] in relation to the Fermi-Ulam model. The main goal of this work is to investigate the influence of a second frequency on the bouncing-ball motion.

In many opportunities the quasiperiodicity has been responsible for rather exotic behaviors, mainly in quantum mechanics. Consider, for example, the appearance of singular continuous spectra for tight-binding Schrödinger operators [3], and N-level systems driven by quasiperiodic forces whose generalized Floquet operators have continuous spectra [4].

From the point of view of nonlinear dynamics the PBB is very interesting since it presents a cascade of period doubling bifurcations, strange attractors, under certain limiting conditions it is described by a classical kicked rotator-like model, and, finally, it is relatively easy to study experimentally. In fact, it became one of the standard models for experiments on nonlinear effects in dynamics [5]. In general, the control parameter is the amplitude of the oscillating table.

More recently, quantum calculations on the PBB have been reported [6] (in the case of elastic bouncing). In [6] the authors focused on the level statistics of the quasienergies; see also [7], for different quantum approaches of related models. From now on we only consider the classical bouncing ball.

Let $m(t)$ be the time dependence of the oscillating table and $m'(t)$ its velocity. If t_n is the instant of the nth impact and $v(t_n)$, $u(t_n)$ are, respectively, the departing and approaching ball velocities, we have [1] $v(t_n)-m'(t_n)=\alpha[m'(t_n)-u(t_n)]$, where $0<\alpha<1$ denotes the coefficient of restitution (we shall not consider the elastic bouncing case, i.e., $\alpha=1$). Between two consecutive impacts the ball moves under the action of the constant gravitational field g; it is also assumed that the impacts do not affect the motion of the table. By using Newton's law of motion and imposing as the condition for impact that the difference in position between the ball and the table vanishes, one gets the so-called impact map.

In the case of periodic oscillating table $m(t)=A[1+\sin(\omega t)]$, $A\geqslant 0$, the impact map takes the form [1]

$$v_{n+1}=(1+\alpha)\omega A\ \cos\theta_{n+1}-\alpha[v_n-g(t_{n+1}-t_n)];$$

$$A(\sin\theta_n-\sin\theta_{n+1})+v_n(t_{n+1}-t_n)-\frac{g}{2}\,(t_{n+1}-t_n)^2=0;$$

$$\theta_{n+1}=\omega t_{n+1}(\mathrm{mod}2\pi);$$

we have just introduced $\theta=\omega t(\mathrm{mod}2\pi)$, so that $\theta_n=\omega t_n(\mathrm{mod}2\pi)$. Notice that, by using the variables (v_n,θ_n), the motion due to the impact map, in this case, becomes restricted to a cylinder.

Although this impact map is exact, it is an implicit function and this puts some difficulties in its theoretical and numerical investigations. For instance, in theoretical studies one usually assumes that the maximum height the ball travels between impacts is much larger than the amplitude of oscillations of the table [8], and we are not aware of any calculations of the largest Lyapunov exponent for this model. One of the goals of this work is to use the implicit function theorem to calculate the Lyapunov exponent for the bouncing-ball model.

We fix $g=980$ and $\omega=120\pi$, in most calculations we set $\alpha=0.5$, and the control parameter is the amplitude A. Now we summarize the classical behavior of the PBB problem in a way that is convenient for later reference [1]. In the dissipative case there is an upper bound for the ball velocity, i.e., there is a trapping region in phase space (v,θ). There are sticking solutions which, we simply discard. For some values of the amplitude $0.01<A<0.0105$ there is a stable equilibrium solution, so that it is a periodic attractor of period 1. Increasing the value of the amplitude, a pitchfork bifurcation occurs and a stable orbit of period 2 appears (the equilibrium point becomes unstable); then a period doubling cascade—in which one sees only orbits of period 2^k—follows. For still larger values of A one sees nonperiodic motions represented by the well-known strange attractors.

According to the accepted classification of strange attractors as chaotic or nonchaotic [9], in order to characterize those bouncing ball strange attractors as chaotic we need to compute the largest Lyapunov exponent in each case, and check whether they are actually greater than zero. The largest

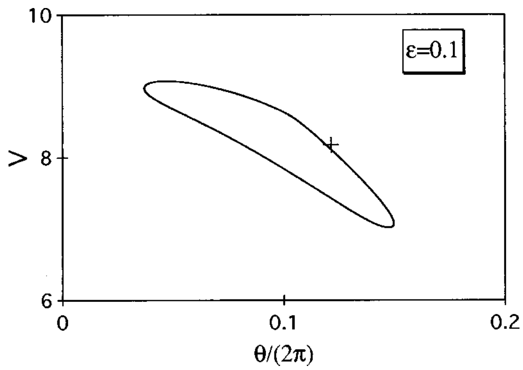

FIG. 1. The projection on Σ, i.e., the space (v,θ), of the attractor for the QBB with $A=0.01$; this value corresponds to a stable equilibrium point (+) in the PBB case, i.e., $\varepsilon=0$. The value of the perturbation parameter ε is indicated in the figure.

Lyapunov exponent measures the exponential rate of separation of close initial conditions as a function of time. Let us describe the procedure for a more general set of implicit equations

$$f(u_n,u_{n+1},\varphi_n,\varphi_{n+1})=0 \quad h(u_n,u_{n+1},\varphi_n,\varphi_{n+1})=0.$$

An effective way to compute the largest Lyapunov exponent λ is through

$$\lambda=\lim_{n\to\infty}\frac{1}{n}\ln\|J(n)J(n-1)\cdots J(1)V\|,$$

where V is a (almost arbitrary choice of) normalized vector and

$$J(n)=\begin{pmatrix}\dfrac{\partial u_{n+1}}{\partial u_n} & \dfrac{\partial u_{n+1}}{\partial\varphi_n}\\[2ex] \dfrac{\partial\varphi_{n+1}}{\partial u_n} & \dfrac{\partial\varphi_{n+1}}{\partial\varphi_n}\end{pmatrix}.$$

The goal now is to compute those partial derivatives. This can be accomplished by deriving both functions f and h with respect to u_n and φ_n, taking into account that $u_{n+1}=u_{n+1}(u_n,\varphi_n)$ and $\varphi_{n+1}=\varphi_{n+1}(u_n,\varphi_n)$, and then solving the linear system of equations that result (whose variables are exactly the partial derivatives we are interested in). If one first iterates the implicit equations f and h (we used the bisection method) one is able to obtain the required partial derivatives, and then the Jacobian $J(n)$. Notice this procedure can be seen as an immediate application of the implicit function theorem.

For the impact map, given an initial condition (v_0,θ_0), we iterated the equations and discarded the first 3000 iterates in order to reach the attractor (discarding eventual transients), and then used 10^5 iterates to compute the attractor and its largest Lyapunov exponent λ. For $A=0.0116$ the attractor of the impact map for the PBB is a stable periodic orbit of period 4, and we obtained $\lambda=-0.54$. We have also confirmed the presence of chaotic strange attractors for the PBB, for example, in the case $A=0.012$ we obtained $\lambda=0.34$. We present in Fig. 3(a) the geometric shape of part of this chaotic attractor.

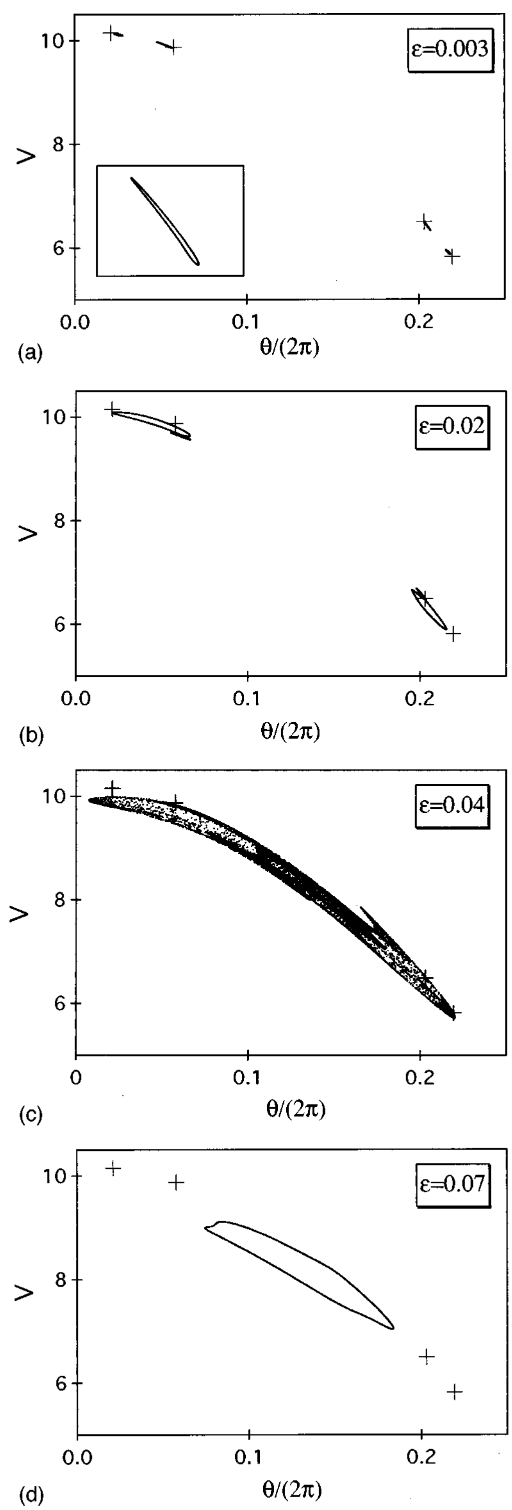

FIG. 2. The projections on Σ of some attractors for the QBB with $A=0.0116$; this value corresponds to a stable periodic orbit of period 4 (+) in the PBB case. The values of the perturbation parameter ε are indicated in each figure. The inset in (a) shows a magnification of the second small curve presented in (a).

Now we investigate the influence of a second frequency on the bouncing ball problem. In [10] the case of two frequencies was also considered, but from different approaches. For definiteness we take

$$m(t)=A[1+(1-\varepsilon)\sin(\omega t)+\varepsilon\ \sin(\tau\omega t)],$$

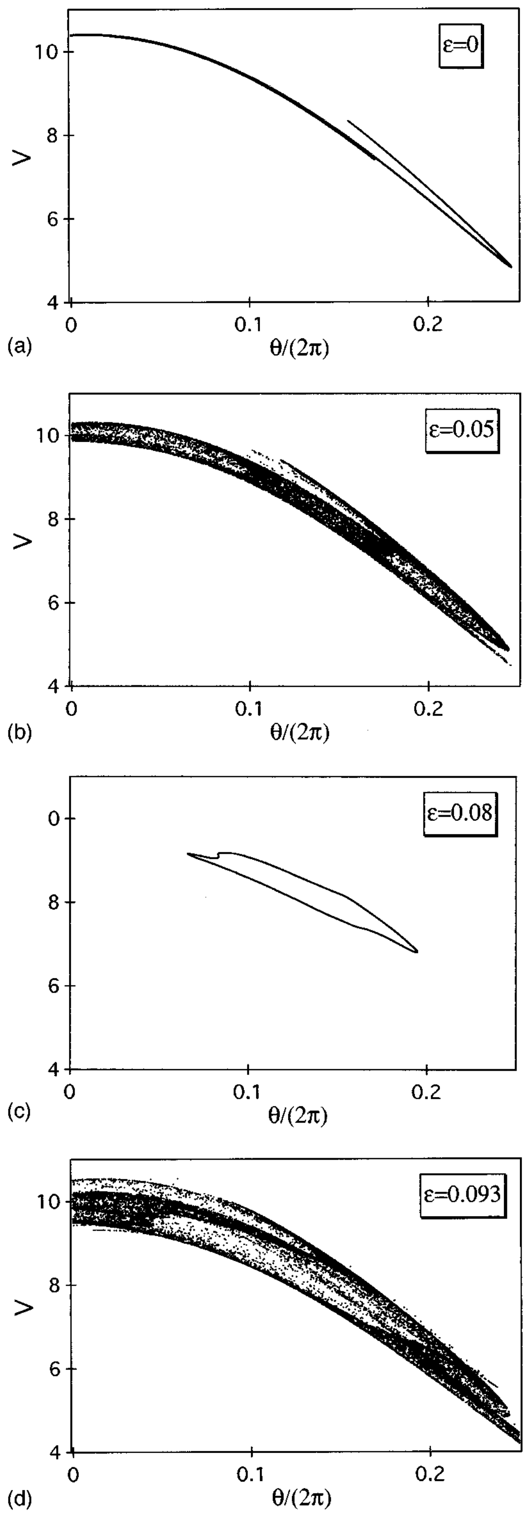

FIG. 3. The projections on Σ of the main branch of some attractors for the QBB with $A=0.012$; this value corresponds to a chaotic strange attractor in the PBB case. The values of the perturbation parameter ε are indicated in each figure. In fact, (a) refers to the PBB problem.

with $\tau=(\sqrt{5}-1)/2$. We restrict the values of the parameter ε to the range $0\leqslant\varepsilon\leqslant 0.1$ in order to keep the second frequency as a perturbation of the PBB motion. We call this model with nonzero ε the quasiperiodic bouncing ball (QBB).

As usual when quasiperiodic terms are present (there is a supplementary degree of freedom for each fundamental frequency), we enlarge the phase space by introducing a new phase variable $\xi=\tau\omega t(\mathrm{mod}2\pi)$, with $\xi_n=\tau\omega t_n(\mathrm{mod}2\pi)$. The phase space of the QBB is described by (v_n,θ_n,ξ_n), and the impact map is given by

$$v_{n+1}=(1+\alpha)\omega A[(1-\varepsilon)\cos\theta_{n+1}+\varepsilon\tau\cos\xi_{n+1}]-\alpha[v_n-g(t_{n+1}-t_n)];$$

$$A[(1-\varepsilon)(\sin\theta_n-\sin\theta_{n+1})+\varepsilon(\sin\xi_n-\sin\xi_{n+1})]+v_n(t_{n+1}-t_n)-\frac{g}{2}(t_{n+1}-t_n)^2=0;$$

$$\theta_{n+1}=\omega t_{n+1}(\mathrm{mod}2\pi);$$

$$\xi_{n+1}=\tau\omega t_{n+1}(\mathrm{mod}2\pi).$$

In Fig. 1 we show the projection of the attractor onto the space $\Sigma=(v,\theta)$ (it is just the projection, not the Poincaré section) for the case $A=0.01$ (stable equilibrium point for the PBB case) with $\varepsilon=0.1$. As soon as ε is taken different from zero the stable equilibrium point becomes unstable and the projection of the attractor onto Σ assumes the shape of closed simple curves (sometimes with "whirls"); since the projection of these attractors seems to lie on closed curves we have an indication that the motion on them are quasiperiodic. We call this kind of attractor a *quasiperiodic limit cycle* (QLC). We then applied the above described procedure to compute the largest Lyapunov exponent. Here we have our first example of nonperiodic attractor with negative Lyapunov exponents since we have found $\lambda=-0.30$ in this case (see also the discussion at the end of this work).

If we consider values of parameters such that the attractor in the PBB case is a periodic orbit and increase ε from zero, each point of the periodic orbit takes the shape of a closed curve; those curves grow with ε, intersect themselves, and eventually become a unique strange attractor. For larger values of ε the strange attractor disappears and we get a QLC. We indicate this process for $A=0.0116$ in Fig. 2, which corresponds to a stable periodic orbit of period 4 in the PBB case. We have found that this behavior is typical; but if we start with an orbit of period 1 (in the case $\varepsilon=0$) no strange attractor was found in the range $0<\varepsilon\leqslant 0.1$.

We remark that when each point of a periodic orbit projection on Σ becomes a closed curve (by turning the perturbation on), the orbits on the attractor keep the original ordering of the periodic orbit while jumping among these curves. Generally, in such cases we have also got negative values of λ; e.g., in the case of $A=0.0116$ we have $\lambda=-0.54$ for $\varepsilon=0$, $\lambda=-0.21$ for $\varepsilon=0.003$, and $\lambda=-0.23$ for $\varepsilon=0.07$, see Fig. 2.

We have also checked that, in some cases, the strange attractor that arises from this process is chaotic, e.g., for $A=0.0116$ and $\varepsilon=0.04$ we found $\lambda=0.06$ [see Fig. 2(c)]. Although we have shown only the projection of the attractors on Σ, it is worth mentioning that all strange attractors we have found on Σ appeared also as strange attractors in the three-dimensional phase space (v,θ,ξ).

In order to check possible artificial effects of particular parameter values, we have changed slightly the values of A and ε for the cases with chaotic attractors as well as QLC with negative values of λ. All reported cases have presented stable values of the Lyapunov exponent; indeed, it seems that λ is a continuous function of ε.

Our last point is the influence of the second frequency on the chaotic motion of the PBB. We concentrated on $A=0.012$. In Fig. 3 we indicate the typical behavior we found by showing the projection on Σ of the attractor for some values of ε with $A=0.012$. We found $\lambda=0.18$ for the case of Fig. 3(b), i.e., $\varepsilon=0.05$; then the attractor turns to a QLC [Fig. 3(c)] with negative Lyapunov exponents (for $\varepsilon=0.08$ we have found $\lambda=-0.22$). For still larger values of ε a chaotic strange attractor reappears [e.g., for $\varepsilon=0.093$ we found $\lambda=0.15$; see Fig. 3(d)]. We remark that in all cases we analyzed the absolute value of λ decreases as soon as the second frequency is taken into account.

Although we have presented results for $\tau=(\sqrt{5}-1)/2$ with the coefficient of restitution $\alpha=0.5$, similar results were found for other values of α and also for $\tau=1/\sqrt{2}$. Of course one can argue that such cases of QLC with negative Lyapunov exponents can be periodic orbits of a very long period, this is not the case in the space (v,θ,ξ) for τ being an irrational number the sequence (θ_n,ξ_n) cannot be periodic. Also, there are examples of nonanalytic maps whose attractor is an equilibrium point with positive Lyapunov exponents [11]. It is a theorem that $\lambda<0$ implies the attractor is a stable periodic orbit [11,12] for $C^{1+\mu}$ maps, but the QBB impact map has a nonautonomous character, so one can not discard—on basis of known results—the possibility of nonperiodic attractors with $\lambda<0$.

In summary, we have indicated how to compute the largest Lyapunov exponent λ for systems given by implicit maps and applied that procedure to the cases of PBB and QBB. We checked that, in fact, chaotic attractors occur for the PBB. We have found a series of bifurcations when a second frequency perturbs the PBB, with transitions from regular motion to chaotic attractors and vice versa. We have found robust cases of nonperiodic attractors with negative values of λ, and no case of nonchaotic strange attractor.

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