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Modulational instability of a circularly polarized wave in a magnetized electron-positron plasma with relativistic thermal energies

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A circularly polarized electromagnetic wave, propagating in the direction of an external magnetic field in an electron-positron plasma, is known to be unstable for frequencies less than half the plasma frequency. This has been shown by taking into account weakly relativistic effects on particle motion in the wave field, and nonrelativistic temperatures. Here, we include fully relativistic effects on the thermal motion of the particles, and show that in the ultrarelativistic limit the system is unstable for all frequencies satisfying $\omega \ll \eta\omega_c$ and $\omega_p \ll (\eta)^{1/2}\omega_c$, where ω_c is the gyrofrequency, ω_p is the plasma frequency, and η is the ratio between the rest energy density and the enthalpy of the system. In the limit of nonrelativistic thermal motions, the results obtained previously are recovered. [S1063-651X(97)03609-X]

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I. INTRODUCTION

It seems to be well established that electromagnetic radiation originating in pulsar magnetospheres experiences amplitude modulation [1–3]. Chian and Kennel [4,5] proposed a self-modulation mechanism to account for the observations. They derived a nonlinear Schrödinger equation that, unfortunately, was shown to be incorrect [6,7] because they ignored two sources of nonlinearity. On one hand, they did not include harmonic generation and, on the other hand, they did not consider the ponderomotive force.

Kates and Kaup [8], by observing that ponderomotive forces, relativistic corrections, and harmonic generation all contribute cubic terms in the amplitude, were able to derive a cubic nonlinear Schrödinger equation. Assuming weakly relativistic effects, they showed that in an unmagnetized cold electron-positron plasma there is no amplitude modulation, in contrast with Chian and Kennel [4]. The result was then generalized to include finite thermal effects. In this case, the plasma was shown to be modulationally unstable in a narrow range just above the plasma frequency.

Later on, Kates and Kaup [9] studied the propagation of an electromagnetic wave in the direction of an external magnetic field. They showed that for frequencies $\omega < \omega_p/2$, where $\omega_p = (4\pi n_0 e^2/m)^{1/2}$ is the plasma frequency, the system is modulationally unstable.

More recently, Gratton *et al.* [10], by extending the theory in order to include ultrarelativistic effects (relativistic temperatures) and phonon damping in an unmagnetized electron-positron plasma, showed that relativistic thermal energies change the stability results found by Kates and Kaup [8]. Three cases were analyzed in [10]. First, when the damping is $O(\epsilon^{(0)})$ and $O(\epsilon^{(1)})$ (ϵ is the perturbation parameter) a modulational instability is possible for all frequencies and temperatures. When the phonon damping is very small, $O(\epsilon^{(2)})$, the modulational instability occurs in a finite band near the reduced plasma frequency, for ultrarelativistic temperatures.

Here we extend the work of [10] to include an external magnetic field. We neglect phonon damping. In the nonrelativistic temperature limit, we recover the results of [9]—which also reduces to [8] for zero magnetic field—except for an overall factor of two, which is clearly a misprint and does not alter their conclusions. We show that in the ultrarelativistic thermal limit, the plasma is modulationally unstable for all frequencies satisfying $\omega \ll \eta\omega_c$, and for $\omega_p \ll (\eta)^{1/2}\omega_c$.

This paper is organized as follows. In Sec. II, the basic equations are discussed. In Sec. III, by using a multiscale space-time perturbation approach (see, e.g., [11]), we derive all required quantities necessary to obtain the nonlinear Schrödinger equation. In Sec. IV, we obtain the nonlinear Schrödinger equation. In Sec. V, the nonlinear Schrödinger equation is analyzed. In Sec. VI, the results are discussed.

II. BASIC EQUATIONS

We shall study an electromagnetic wave in an electron-positron plasma propagating along an external magnetic field. We will include weakly relativistic effects on the particle motion in the field of the electromagnetic wave, but fully relativistic effects in the particle thermal motions.

The basic equations are (see [10])

$$\frac{h}{c^2}\frac{d}{dt}(\gamma_l \vec{v}_i) = -\frac{1}{\gamma_l}\nabla p - \frac{\vec{v}_i}{c^2}\gamma_l\frac{dp}{dt} + n_i q_i\left(\vec{E} + \frac{1}{c}\vec{v}_i \times \vec{B}\right), \tag{1}$$

$$\frac{\partial \gamma_l n_i}{\partial t} + \frac{\partial \gamma_l n v_{z,i}}{\partial z} = 0, \quad l = e,p, \tag{2}$$

where

$$\gamma_l = \left(1 - \frac{\vec{v}_l^2}{c^2}\right)^{-1/2}, \quad l = e,p \tag{3}$$

and Eq. (2) is the continuity equation.

Equation (1) is obtained from the space components of

$$\partial_\nu T^{\mu\nu}=\frac{1}{c}\, j_\nu F^{\mu\nu}, \tag{4}$$

upon using the time component

$$h\,\frac{d\gamma}{dt}=\frac{1}{\gamma}\,\frac{\partial p}{\partial t}-\gamma\,\frac{dp}{dt}+qn\vec{v}\cdot\vec{E}. \tag{5}$$

In Eq. (4),

$$T^{\mu\nu}=\frac{h}{c^2}\,u^\mu u^\nu-pg^{\mu\nu}, \tag{6}$$

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu, \tag{7}$$

$$j^\mu=qnu^\mu. \tag{8}$$

The electromagnetic potentials, $A^\mu=(\phi,A^l)$, are related to the electric and magnetic fields through

$$\vec{E}=-\frac{1}{c}\,\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\phi, \tag{9}$$

$$\vec{B}=\vec{\nabla}\times\vec{A}. \tag{10}$$

The metric tensor $g_{\mu\nu}$ is defined by $ds^2=g_{\mu\nu}dx^\mu dx^\nu$, where $g_{\mu\nu}=(1,-1,-1,-1)$, $(\mu,\nu)=0,1,2,3$, and $dx^\mu=(cdt,dx^l)$ with $l=1,2,3$, and $u^\mu=\gamma(c,v^l)$.

In the energy momentum tensor, $T^{\mu\nu}$, $h=nmc^2+\overline{\epsilon+}p$ is the enthalpy, m is the proper mass of the particles, $\overline{\epsilon}$ is the internal energy, and p is the pressure.

We now assume that we have a finite amplitude circularly polarized wave propagating along the z direction—the direction of the external magnetic field—so that

$$\vec{A}=(A_x(z,t),A_y(z,t),0). \tag{11}$$

From $u_\mu\partial_\nu T^{\mu\nu}=0$ we obtain the equation for adiabatic motion,

$$\frac{d\varepsilon}{dt}=\frac{h}{n}\,\frac{dn}{dt}, \tag{12}$$

which is equivalent to

$$\frac{1}{n}\,\frac{dp}{dt}=\frac{d}{dt}\left(\frac{h}{n}\right). \tag{13}$$

Using this equation, along with Eqs. (9) and (10), the components of Eq. (1) become

$$\frac{d}{dt}\left(\frac{h}{nc}\,\gamma\,\frac{v_x}{c}+\frac{q}{c}\,A_x\right)=\frac{q}{c}\,v_yB_{0z}, \tag{14}$$

$$\frac{d}{dt}\left(\frac{h}{nc}\,\gamma\,\frac{v_y}{c}+\frac{q}{c}\,A_y\right)=-\frac{q}{c}\,v_xB_{0z}, \tag{15}$$

$$\frac{d}{dt}\left(\frac{h}{nc}\,\gamma\,\frac{v_z}{c}\right)=-q\,\frac{\partial\phi}{\partial z}-\frac{1}{\gamma n}\,\frac{\partial p}{\partial z}+\frac{q}{c}\left(v_x\,\frac{\partial A_x}{\partial z}+v_y\,\frac{\partial A_y}{\partial z}\right). \tag{16}$$

On the other hand, the potentials satisfy

$$\left(\frac{\partial^2}{\partial t^2}-c^2\,\frac{\partial^2}{\partial t^2}\right)\hat{A}_x=\frac{4\pi q^2}{m}\left(n_{L,p}\,\frac{v_{x,p}}{c}-n_{L,e}\,\frac{v_{x,e}}{c}\right), \tag{17}$$

$$c\,\frac{\partial^2\hat{\phi}}{\partial t\partial z}=\frac{4\pi q^2}{m}\left(n_{L,p}\,\frac{v_{z,p}}{c}-n_{L,e}\,\frac{v_{z,e}}{c}\right), \tag{18}$$

where $n_L=\gamma n$, and $\hat{A}^\mu=(q/mc^2)A^\mu$.

Finally, the equation of motion can be written in the following form:

$$\frac{d}{dt}\left(\frac{h}{nmc^2}\,\gamma\,\frac{v_x}{c}+\sigma_L\hat{A}_x\right)=\frac{\sigma_L}{c}\,v_yB_{0z}, \tag{19}$$

$$\frac{d}{dt}\left(\frac{h}{nmc^2}\,\gamma\,\frac{v_y}{c}+\sigma_L\hat{A}_y\right)=-\frac{\sigma_L}{c}\,v_xB_{0z}, \tag{20}$$

$$\frac{d}{dt}\left(\frac{h}{nmc^2}\,\gamma\,\frac{v_z}{c}\right)=\frac{c}{\gamma nmc^2}\,\frac{\partial p}{\partial z}-\sigma_Lc\,\frac{\partial\hat{\phi}}{\partial z}+\sigma_L\left(v_x\,\frac{\partial\hat{A}_x}{\partial z}+v_y\,\frac{\partial\hat{A}_y}{\partial z}\right), \tag{21}$$

where

$$\sigma_L=\frac{q}{|q|}. \tag{22}$$

It is convenient to introduce the constants δ and η as

$$\eta=\frac{n_0mc^2}{h_0}, \tag{23}$$

$$\delta=\frac{\Gamma p_0}{h_0}, \tag{24}$$

where $4/3\leqslant\Gamma\leqslant<5/3$ is a polytropic index such that $p/p_0=(n/n_0)^\Gamma$, $h=\Gamma p/(\Gamma-1)+nmc^2$, and $\overline{\epsilon=}p/(\gamma-1)$, with the understanding that as Γ approaches the value 5/3 we must set $p\ll nmc^2$ (see [10]).

For low thermal energies,

$$\eta=1-\tfrac{3}{2}\,\delta, \tag{25}$$

$$\delta=\frac{v_s^2}{c^2+(3/2)v_s^2}\ll 1. \tag{26}$$

On the other hand, in the ultrarelativistic limit,

$$\eta\ll 1, \tag{27}$$

$$\delta=\tfrac{1}{3}. \tag{28}$$

III. THE MULTISCALE PERTURBATION METHOD

We assume that all quantities can be written in the form

$$G=G_0+\epsilon G^{(1)}+\epsilon^2G^{(2)}+\epsilon^3G^{(3)}+\cdots. \tag{29}$$

Thus, to zeroth order in ϵ, the electromagnetic potential, the density, and the enthalpy, are given by

$$A_0=(-\tfrac{1}{2}yB_0,\tfrac{1}{2}B_0,0), \tag{30}$$

$$n_0=\text{const}, \tag{31}$$

$$h_0=n_0mc^2+p_0+\overline{\epsilon_0}, \tag{32}$$

where B_0 is the external magnetic field.

The first order electromagnetic potential $A^{(1)}$ is given by a typical Fourier component,

$$A^{(1)}=a(\hat{x}+i\hat{y})e^{(ikz-i\omega t)}. \tag{33}$$

We shall now solve the problem to order ϵ^3 by using the multiscale perturbation approach, which assumes that the amplitude depends weakly on space and time [11],

$$\frac{\partial a}{\partial t}=\epsilon T_1(a,a^*)+\epsilon^2 T_2(a,a^*)+\epsilon^3 T_3(a,a^*)+\cdots , \tag{34}$$

$$\frac{\partial a}{\partial z}=\epsilon Z_1(a,a^*)+\epsilon^2 Z_2(a,a^*)+\epsilon^3 Z_3(a,a^*)+\cdots \tag{35}$$

and the corresponding complex conjugate quantities.

Thus,

$$\partial_t=-\omega\partial_\theta+\epsilon(Z_1\partial_a+Z_1^*\partial_{a*})+\cdots, \tag{36}$$

$$\partial_z=k\partial_\theta+\epsilon(T_1\partial_a+T_1^*\partial_{a*})+\cdots. \tag{37}$$

To first order in ϵ, Eqs. (19) and (20) yield

$$\left(\frac{1}{\eta}\frac{d}{dt}\pm i\sigma_L\omega_c\right)\frac{v_x^{(1)}\pm iv_y^{(1)}}{c}=-\sigma_L T_1\frac{\partial}{\partial a}(\hat{A}_x^{(1)}\pm i\hat{A}_y^{(1)}). \tag{38}$$

From the last equation it follows that

$$\frac{\vec{v}_{p,e}^{(1)}}{c}=-\sigma_L\frac{q}{mc^2}\frac{\vec{a}(\hat{x}+i\hat{y})e^{i\theta}}{\eta_\pm}+\text{c.c.} \tag{39}$$

where

$$\hat{A}^{(1)}=\hat{a}(\hat{x}+i\hat{y})e^{i\theta}+\text{c.c.}, \tag{40}$$

$$\hat{a}=\frac{q}{mc^2}a, \tag{41}$$

$$\eta_\pm=\frac{1}{\eta}\pm\frac{\omega_c}{\omega}, \tag{42}$$

and $\omega_c=|q|B_0/mc$ is the gyrofrequency.

From Eqs. (17) and (39), to order ϵ, we obtain the dispersion relation,

$$\omega^2=c^2k^2+\omega_p^2\left(\frac{1}{\eta_+}+\frac{1}{\eta_-}\right), \tag{43}$$

$$\omega_p^2=\frac{4\pi q^2 n_0}{m}, \tag{44}$$

and all other quantities are zero to order one.

We shall now calculate the second order velocities.

From Eqs. (36) and (37), to order ϵ, we obtain

$$\frac{\partial^2}{\partial t^2}=\left(-\omega\frac{\partial}{\partial\theta}+\epsilon T_1\frac{\partial}{\partial a}\right)^2=-2\epsilon\omega T_1\frac{\partial^2}{\partial a\partial\theta}, \tag{45}$$

$$\frac{\partial^2}{\partial z^2}=\left(k\frac{\partial}{\partial\theta}+\epsilon Z_1\frac{\partial}{\partial a^2}\right)^2=2k\epsilon Z_1\frac{\partial^2}{\partial a\partial\theta}, \tag{46}$$

where

$$T_1=\frac{\partial a}{\partial t_1}, \tag{47}$$

$$Z_1=\frac{\partial a}{\partial z_1}, \tag{48}$$

and $t_1=\epsilon t$, and $z_1=\epsilon z$.

To order ϵ^2, Eqs. (19) and (20) yield

$$\frac{d}{dt}\left[\frac{h_0}{n_0mc^2}\frac{v_x^{(2)}\pm iv_y^{(2)}}{c}+\frac{1}{\eta}\frac{v_x^{(1)}\pm iv_y^{(1)}}{c}+\sigma_L(\hat{A}_x^{(1)}\pm i\hat{A}_y^{(1)})\right]=\mp\sigma_L\omega_c i\frac{v_x^{(2)}\pm iv_y^{(2)}}{c}. \tag{49}$$

From Eq. (49) it follows that

$$\frac{\vec{v}_{p,e}^{(2)}}{c}=-i\frac{\omega_c}{\omega^2\eta_{p,e}^2}\frac{\partial\vec{\hat{a}}}{\partial t}e^{i\theta}+\text{c.c.} \tag{50}$$

From Eq. (17), to order ϵ^2, we obtain

$$\left(\omega^2\frac{\partial^2}{\partial\theta^2}-c^2k^2\frac{\partial}{\partial\theta^2}\right)\hat{A}_x^{(2)}-2i\omega\left(T_1+\frac{c^2k}{\omega}Z_1\right)\frac{\partial}{\partial\hat{a}}\hat{A}_x^{(1)}=-\frac{i\omega_p^2\omega_c}{\omega^2}\left(\frac{1}{\eta_+^2}-\frac{1}{\eta_-^2}\right). \tag{51}$$

Since $A_x^{(2)}$ can be taken to be zero (see [8,9]), it follows that

$$T_1+Z_1\frac{c^2k}{\omega}\left[1+\frac{\omega_p^2\omega_c}{2\omega^3}\left(\frac{1}{\eta_-^2}-\frac{1}{\eta_+^2}\right)\right]^{-1}=0. \tag{52}$$

On the other hand, from the dispersion relation, Eq. (43), we obtain

$$v_g=\frac{c^2k}{\omega}\left[1+\frac{\omega_p^2\omega_c}{2\omega^3}\left(\frac{1}{\eta_-^2}-\frac{1}{\eta_+^2}\right)\right]^{-1}. \tag{53}$$

From Eqs. (52) and (53) it follows that

$$T_1+v_gZ_1=0, \tag{54}$$

which, in terms of the amplitude, is given by

$$\frac{\partial a}{\partial t_1}+v_g\frac{\partial a}{\partial z_1}=0. \tag{55}$$

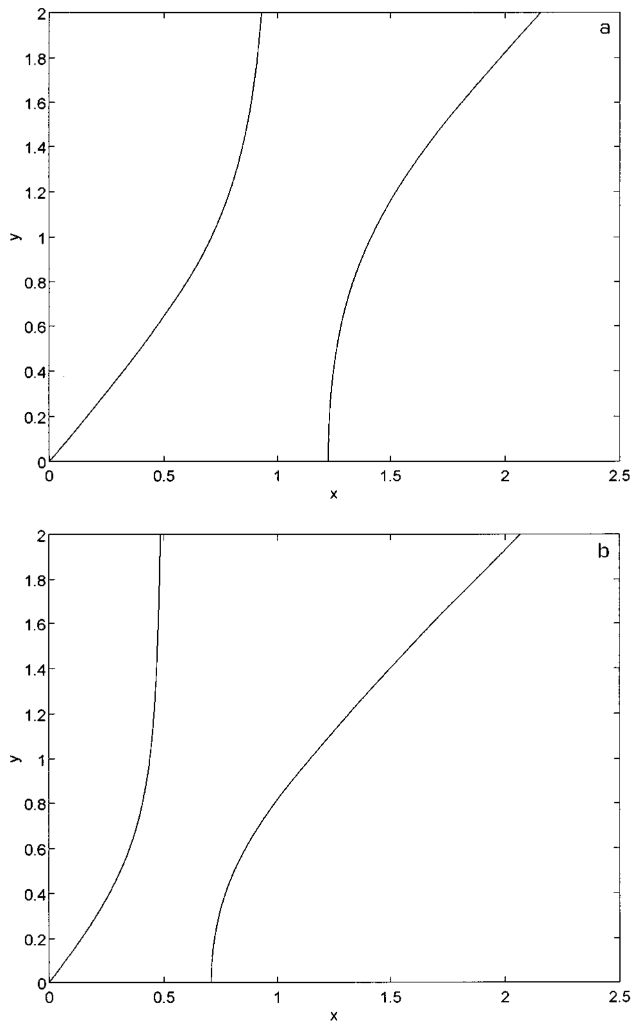

FIG. 1. Dispersion relation, Eq. (43). Normalized wave number, $y=kc/\overline{\omega_c}$, vs normalized frequency, $x=\omega/\overline{\omega_c}$, for $\overline{\omega_p}/\overline{\omega_c}=1/2$, and (a) $\eta=1$, (b) $\eta=0.5$.

This equation means that the amplitude is constant in a frame moving with group velocity of the wave.

Note that the effective plasma frequency is now given by $\overline{\omega_p^2}=\eta\omega_p^2$ (see [10]), and the resonances occur at $\overline{\omega_c}=\eta\omega_c$. This is illustrated In Fig. 1, for (a) $\eta=1$ and (b) $\eta=0.5$.

To third order in ϵ, Eq. (21) yields

$$\frac{\partial}{\partial t}\,\frac{v_z^{(3)}+v_z^{(2)}}{c}=-\delta c\,\frac{\partial}{\partial z}\left(\frac{n^{(3)}+n_L^{(2)}}{n_{(0)}}-\frac{2\overline{aa^*}}{\eta_\pm^2}\right)-\eta\sigma_L c\times\frac{\partial}{\partial z}\,\hat{\phi}^{(3)}+\hat{\phi}^{(2)}+\eta\sigma_L c\left[\left(\frac{v_x^{(2)}+v_x^{(1)}}{c}\right)\frac{\partial\hat{A}_x^{(1)}}{\partial z}+\left(\frac{v_y^{(2)}+v_y^{(1)}}{c}\right)\frac{\partial\hat{A}_y^{(1)}}{\partial z}\right]. \tag{56}$$

To third order in ϵ, the continuity equation yields

$$\frac{\partial}{\partial t}\,\frac{n_L^{(3)}+n_L^{(2)}}{n_0}=-\frac{\partial}{\partial z}\,v_z^{(3)}+v_z^{(2)}. \tag{57}$$

Upon elimination of secularities, one finds that neither $v_z^{(2)}$ nor $n_L^{(2)}$ depends on theta. Therefore, adding up the continuity equation for electrons and positrons yields

$$v_{z,p}^{(2)}+v_{z,e}^{(2)}=v_g\,\frac{n_{L,p}^{(2)}+n_{L,e}^{(2)}}{n_0}. \tag{58}$$

Using

$$A_x^{(1)}=ae^{i\theta}+\text{c.c.}, \tag{59}$$

$$A_y^{(1)}=iae^{i\theta}-ia^*e^{-i\theta}, \tag{60}$$

$$v_{x,p,e}^{(1)}=-\sigma_L\,\frac{a}{\eta_{p,e}}\,e^{i\theta}+\text{c.c.}, \tag{61}$$

$$v_{y,p,e}^{(1)}=-\sigma_L\left(\frac{a}{\eta_{p,e}}\,e^{i\theta}-\frac{a^*}{\eta_{p,e}}\,e^{-i\theta}\right), \tag{62}$$

$$v_{x,p,e}^{(2)}=-i\,\frac{\omega_c}{\omega^2\eta_\pm^2}\left(\frac{\partial a}{\partial t_1}\,e^{i\theta}-\frac{\partial a^*}{\partial t_1}\,e^{-i\theta}\right), \tag{63}$$

$$v_{y,p,e}^{(2)}=\frac{\omega_c}{\omega^2\eta_\pm^2}\,T_1e^{i\theta}+\text{c.c.} \tag{64}$$

and adding up Eq. (56) for electrons and positrons, upon elimination of secularities, we obtain

$$\frac{n_{L,p}^{(2)}+n_{L,e}^{(2)}}{n_0}=\frac{2\overline{n^{(2)}}}{n_0}-\frac{\delta c^2 2\hat{a}\hat{a}^*}{F}\left(\frac{1}{\eta_+^2}+\frac{1}{\eta_-^2}\right), \tag{65}$$

where

$$\frac{\overline{n^{(2)}}}{n_0}=\eta\sigma 2\hat{a}\hat{a}^*, \tag{66}$$

$$\sigma=\frac{c^2}{\omega_p^2F}\,(\omega^2+c^2k^2-2kv_g\omega), \tag{67}$$

$$F=v_g^2-\delta c^2. \tag{68}$$

On the other hand, to second order in ϵ, from Eq. (18) it follows that

$$v_{z,p}^{(2)}=v_{z,e}^{(2)}. \tag{69}$$

Subtracting the continuity equation for electrons and positrons,

$$\frac{\partial}{\partial t}\left(\frac{n_{L,p}-n_{L,e}}{n_0}\right)=\frac{\partial}{\partial z}\,(v_{z,p}-v_{z,e}), \tag{70}$$

and using Eq. (69), yields

$$n_{L,p}^{(2)}=n_{L,e}^{(2)}. \tag{71}$$

Note that

$$\frac{n_L^{(2)}}{n_0}=\frac{n^{(2)}}{n_0}+\frac{1}{2}\,\frac{{v_x^{(1)}}^2+{v_y^{(1)}}^2}{c^2}. \tag{72}$$

Therefore, from Eqs. (65) and (71), it follows that

$$\frac{n^{(2)}_{L,p,e}}{n_0}=\frac{\overline{n}}{n^{(2)}_0}-\frac{\delta c^2\hat{a}\hat{a}^*}{F}\left(\frac{1}{\eta^2_+}+\frac{1}{\eta^2_-}\right). \quad (73)$$

Let us now calculate v^3_x. From Eqs. (19), to order ϵ^3, it follows that

$$\frac{d}{dt}\left[\frac{1}{\eta}\frac{v_x^{(3)}+v_x^{(2)}+v_x^{(1)}}{c}+\frac{1}{\eta}\left(\frac{1}{2}\frac{v_x^{(1)^2}+v_y^{(1)^2}}{c}+\frac{\delta n^{(2)}}{n_0}\right)\frac{v_x^{(1)}}{c}+\sigma_L\right]\hat{A}=\sigma_L\omega_c\frac{v_y^{(3)}}{c}. \quad (74)$$

When this equation is combined with a similar one for v_y, Eq. (20), we obtain

$$\frac{\vec{v}^{(3)}_{e,p}}{c}=S_{e,p}(\hat{x}+i\hat{y})e^{i\theta}+\text{c.c.}, \quad (75)$$

where

$$S_{e,p}=-\frac{\omega_c}{\eta\omega^3}\frac{1}{\eta_+}\frac{\partial^2\hat{a}}{\partial t_1^2}+\sigma_L\frac{i}{\omega\eta^2_\mp}\left(\frac{1}{\eta}-\eta_\mp\right)\frac{\partial\hat{a}}{\partial\tau}-\sigma_L\frac{\eta k v_g\sigma\hat{a}^2\hat{a}^*}{\omega\eta^2_\mp}\left(\frac{1}{\eta}-\eta_\mp\right)+\sigma_L\frac{2\hat{a}^2\hat{a}^*}{\eta\eta^4_\mp}\left(1-\frac{v_g^2\delta}{F}\right)+\sigma_L\frac{\delta\sigma\hat{a}^2\hat{a}^*}{\eta^2_\mp}-\sigma_L\frac{k v_g\delta c^2 2\hat{a}^2\hat{a}^*}{\omega F\eta^3_\mp}, \quad (76)$$

and

$$t_1=\epsilon t, \quad (77)$$

$$\tau=\epsilon^2 t. \quad (78)$$

It is important to point out that $A^{(2)}$ and $A^{(3)}$ can be taken equal to zero (see [8,9]). We can also calculate other second and third order quantities, but we already have the necessary information to calculate the nonlinear Schrödinger equation. This is done in the next section.

IV. THE NONLINEAR SCHRÖDINGER EQUATION

Equation (17), to third order in ϵ, yields

$$\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial z^2}\right)\hat{A}^{(1)}_x=\epsilon^3\omega_p^2\left[\frac{v^{(3)}_{x,p}-v^{(3)}_{x,e}}{c}+\left(\frac{n^{(2)}_{L,p}}{n_0}\frac{v^{(1)}_{x,p}}{c}-\frac{n^{(2)}_{L,e}}{n_0}\frac{v^{(1)}_{x,e}}{c}\right)\right]. \quad (79)$$

Using Eqs. (39), (73), and (75), we obtain

$$2i\omega\frac{\partial\hat{a}}{\partial\tau}+\omega\frac{\partial v_g}{\partial k}\frac{\partial^2\hat{a}}{\partial\xi^2}+\hat{a}^2\hat{a}^*[C_p+2C_R]=0, \quad (80)$$

where

$$v_g=\frac{c^2k}{\omega}\left[1+\frac{\omega_p^2\omega_c}{2\omega^3}\left(\frac{1}{\eta^2_-}-\frac{1}{\eta^2_+}\right)\right]^{-1}, \quad (81)$$

$$\omega\frac{c^2k}{v_g\omega}\frac{\partial v_g}{\partial k}=\left[c^2-v_g^2-\frac{v_g^2\omega_c\omega_p^2}{\eta\omega^3}\left(\frac{1}{\eta^3_+}-\frac{1}{\eta^3_-}\right)\right], \quad (82)$$

$$C_p=-\frac{\omega v_g}{c^2k}\frac{c^2\eta}{F\omega_p^2}(2k\omega v_g-\omega^2-c^2k^2)^2+\omega_p^2\frac{\sigma\delta v_g\omega}{c^2k}\left(\frac{1}{\eta^2_+}+\frac{1}{\eta^2_-}\right), \quad (83)$$

$$C_R=\omega_p^2\frac{\omega v_g}{c^2k\eta}\left(\frac{1}{\eta^4_+}+\frac{1}{\eta^4_-}\right)(1-\delta)-\omega_p^2\frac{\omega v_g\delta^2}{2k\eta F}\left(\frac{1}{\eta^2_+}+\frac{1}{\eta^2_-}\right)^2-\frac{\omega_p^2}{2}\left(\frac{v_g^2\delta}{F}-\frac{\delta\omega v_g}{kF}\right)\times\left(\frac{1}{\eta^2_+}+\frac{1}{\eta^2_-}\right)\left(\frac{1}{\eta_+}+\frac{1}{\eta_-}\right). \quad (84)$$

From Eq. (42),

$$\eta_\pm=\frac{1}{\eta}\left(1\pm\eta\frac{\omega_c}{\omega}\right)=\frac{1}{\eta}\left(1\pm\overline{\frac{\omega_c}{\omega}}\right)=\frac{1}{\eta}\,\overline{\eta_\pm}. \quad (85)$$

Therefore, the coefficients C_P and C_R can be rewritten in the form

$$C_P=-\frac{\omega v_g}{c^2k}\frac{c^2\eta}{F\omega_p^2}(2k\omega v_g-\omega^2-c^2k^2)^2+\omega_p^2\frac{\sigma\delta v_g\omega\eta^2}{c^2k}\left(\frac{1}{\overline{\eta^2_+}}+\frac{1}{\overline{\eta^2_-}}\right), \quad (86)$$

and

$$C_R=\omega_p^2\frac{\omega v_g\eta^3}{c^2k}\left(\frac{1}{\overline{\eta^4_+}}+\frac{1}{\overline{\eta^4_-}}\right)(1-\delta)-\omega_p^2\frac{\omega v_g\delta^2\eta^3}{2kF}\left(\frac{1}{\overline{\eta^2_+}}+\frac{1}{\overline{\eta^2_-}}\right)^2-\omega_p^2\eta^3\left(\frac{\delta v_g^2}{F}-\frac{\delta\omega v_g}{kF}\right)\left(\frac{1}{\overline{\eta^2_+}}+\frac{1}{\overline{\eta^2_-}}\right)\left(\frac{1}{\overline{\eta_+}}+\frac{1}{\overline{\eta_-}}\right). \quad (87)$$

V. ANALYSIS OF THE NONLINEAR SCHRÖDINGER EQUATION

The first term in C_P is

$$C_{P1}=-\frac{4\omega_p^2\omega^7v_g^3\eta^3}{c^4k^3F}\frac{1}{(\omega^2-\overline{\omega_c^2})^2}\times\left[1+\frac{2\overline{\omega_c^2}}{\omega^2-\overline{\omega_c^2}}-\frac{2\overline{\omega_p^2}\overline{\omega_c^2}}{(\omega^2-\overline{\omega_c^2})^2}\right]^2, \quad (88)$$

and the second term is

$$C_{P2}=\frac{4\omega_p^2\delta v_g^2\omega^6\eta^3}{c^2k^2F(\omega^2-\overline{\omega_c^2})^2}\left[1+\frac{2\overline{\omega_c^2}}{\omega^2-\overline{\omega_c^2}}-2\frac{\overline{\omega_p^2}\,\overline{\omega_c^2}}{(\omega^2-\overline{\omega_c^2})^2}\right]\left[1+\frac{2\overline{\omega_c^2}}{\omega^2-\overline{\omega_c^2}}\right]. \tag{89}$$

On the other hand, $C_R=C_{R1}+C_{R2}+C_{R3}$, where

$$C_{R1}=\frac{2\omega_p^2\omega^5\eta^3 v_g}{c^2k(\omega^2-\overline{\omega_c^2})^2}\left[1+\frac{8\omega^2\overline{\omega_c^2}}{(\omega^2-\overline{\omega_c^2})^2}\right], \tag{90}$$

$$C_{R2}=\frac{4\overline{\omega_p^2}\omega^5 v_g\delta^2\eta^2}{2kF(\omega^2-\overline{\omega_c^2})^2}\left(1+\frac{2\omega^2}{\omega^2-\overline{\omega_c^2}}\right)^2, \tag{91}$$

$$C_{R3}=\frac{4\overline{\omega_p^2}\delta\eta^2 v_g\omega^4}{F(\omega^2-\overline{\omega_c^2})^2}\left(v_g-\frac{\omega}{k}\right)\left(1+\frac{2\omega^2}{\omega^2-\overline{\omega_c^2}}\right). \tag{92}$$

Assuming $\overline{\omega_p}/\overline{\omega_c}\ll 1$, and $\omega\ll\overline{\omega_c}$, the coefficients can be written in the following form:

$$C_{P1}\simeq-\frac{4\overline{\omega_p^2}\eta^2\omega^5 v_g^2}{kc^2F\overline{\omega_c^4}}\left(1+\frac{6\omega^2}{\overline{\omega_c^2}}+\frac{2\overline{\omega_p^2}}{\overline{\omega_c^2}}\right), \tag{93}$$

$$C_{P2}\simeq-\frac{4\overline{\omega_p^2}\delta v_g^2\omega^6\eta^2}{c^2k^2F\overline{\omega_c^4}}\left(1+\frac{4\omega}{\overline{\omega_c^2}}\right), \tag{94}$$

and

$$C_{R1}\simeq\frac{2\overline{\omega_p^2}\omega^5\eta^2 v_g}{c^2k\overline{\omega_c^4}}(1-\delta)\left(1+\frac{10\omega^2}{\overline{\omega_c^2}}\right), \tag{95}$$

$$C_{R2}\simeq\frac{2\overline{\omega_p^2}\omega^5 v_g\delta^2\eta^2}{kF\overline{\omega_c^4}}\left(1+\frac{6\omega^2}{\overline{\omega_c^2}}\right), \tag{96}$$

$$C_{R3}\simeq-\frac{4\overline{\omega_p^4}\delta\eta^2\omega^8 v_g^2}{k^2c^2F\overline{\omega_c^8}}\left(1+\frac{8\omega^2}{\overline{\omega_c^2}}\right). \tag{97}$$

Using Eqs. (93)–(97), we obtain

$$C_P+2C_R\simeq\frac{4\overline{\omega_p^2}\eta^2\omega^3 v_g}{kc^2\overline{\omega_c^4}}\left[-\frac{1}{1-\frac{\delta c^2}{v_g^2}}\left(1+\frac{6\omega^2}{\overline{\omega_c^2}}+\frac{2\overline{\omega_p^2}}{\overline{\omega_c^2}}\right)-\frac{\omega\delta}{v_gk(1-\delta c^2/v_g^2)}\left(1+\frac{4\omega}{\overline{\omega_c^2}}\right)+\left(1+\frac{10\omega^2}{\overline{\omega_c^2}}\right)(1-\delta)-\frac{c^2\delta^2}{v_g^2\left(1-\frac{\delta^2c^2}{v_g^2}\right)}\times\left(1+\frac{6\omega^2}{\overline{\omega_c^2}}\right)+\frac{2\overline{\omega_p^2}\omega^3\delta}{kv_g(1-\delta c^2/v_g^2)\overline{\omega_c^4}}\times\left(1+\frac{8\omega^2}{\overline{\omega_c^2}}\right)\right]. \tag{98}$$

From Eq. (81), it follows that

$$\frac{c^2k}{v_g\omega}=1+\frac{2\overline{\omega_p^2}\,\overline{\omega_c^2}}{(\omega^2-\overline{\omega_c^2})^2}\simeq 1+\frac{2\overline{\omega_p^2}}{\overline{\omega_c^2}}, \tag{99}$$

and, from Eq. (43), we obtain

$$\frac{\omega^2}{c^2k^2}=\frac{1}{1-2\overline{\omega_p^2}/(\omega^2-\overline{\omega_c^2})}\simeq 1-\frac{2\overline{\omega_p^2}}{\overline{\omega_c^2}}. \tag{100}$$

Using Eqs. (99) and (100) in Eq. (98), yields

$$C_P+2C_R\simeq\frac{4\overline{\omega_p^2}\eta^2\omega^5 v_g}{kc^2\overline{\omega_c^4}}\frac{1}{1-\frac{\delta c^2}{v_g^2}}\left[-\left(1+\frac{6\omega^2}{\overline{\omega_c^2}}+\frac{2\overline{\omega_p^2}}{\overline{\omega_c^2}}\right)-\delta\left(1+\frac{\omega^2}{\overline{\omega_c^2}}+\frac{4\overline{\omega_p^2}}{\overline{\omega_c^2}}\right)+\left(1-\delta-\frac{2\delta\overline{\omega_p^2}}{\overline{\omega_c^2}}\right)\times\left(1+\frac{10\omega^2}{\overline{\omega_c^2}}\right)(1-\delta)-\delta^2\left(1+\frac{6\omega^2}{\overline{\omega_c^2}}+\frac{2\overline{\omega_p^2}}{\overline{\omega_c^2}}\right)+\frac{\overline{\omega_p^2}\omega^2}{\overline{\omega_c^4}}\left(1+\frac{8\omega^2}{\overline{\omega_c^2}}\right)\right]. \tag{101}$$

VI. DISCUSSION

The last term in Eq. (101) is negligible with respect to the others.

In the ultrarelativistic limit, $\delta=1/3$ and $\eta\ll 1$, so that

$$C_P+2C_R<0. \tag{102}$$

The general condition for instability of the nonlinear Schrödinger equation, Eq. (80), is

$$\omega\frac{\partial v_g}{\partial k}(C_p+2C_R)>0. \tag{103}$$

From the dispersion relation, Eq. (43), it follows that, for $\omega<\overline{\omega_c}$, the second derivative of ω with respect to k is always negative (see Fig. 1). Therefore, the system is modulationally unstable for all frequencies satisfying $\omega\ll\overline{\omega_c}$.

For nonrelativistic thermal energies, $\delta \ll 1$ and $\eta = 1$ [see Eqs. (27) and (28)], so that Eq. (101) reduces to

$$C_P + 2C_R \simeq \frac{4\omega_p \omega^5 v_g}{kc^2 \omega_c^4 (1 - c_s^2/v_g^2)} \times \left[-\left(1 + \frac{6\omega^2}{\omega_c^2} + \frac{2\omega_p^2}{\omega_c^2} \right) + 1 + \frac{10\omega^2}{\omega_c^2} \right]. \quad (104)$$

This is the result of Kates and Kaup [9], except for an overall factor of 2. Note also that the plasma frequency of the system is $2\omega_p$.

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