Analysis of synchronization of chaotic systems by noise: An experimental study

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The behavior of uncoupled chaotic systems under the influence of external noise has been the subject of recent work. Some of these studies claim that chaotic systems driven by the same noise do synchronize, while other studies contradict this conclusion. In this work we have undertaken an experimental study of the effect of noise on identically driven analog circuits. The main conclusion is that synchronization is induced by a nonzero mean of the signal and not by its stochastic character. [S1063-651X(97)09310-0]

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Recently, many papers have been devoted to stochastic resonance, i.e., the enhanced response of a system to an external signal induced by noise [1], a phenomenon in which noise has a *creative* role. Thus the claim by Maritan and Banavar [2] that *a pair of chaotic systems subjected to the same noise may undergo a transition at large enough noise amplitude and follow almost identical trajectories with complete insensitivity to the initial conditions* or in other words, that these systems may become synchronized simply by being subject to the *same* noise may sound completely reasonable at first. Several authors [3–8] have commented on these conclusions, which are in contradiction to the *common sense* understanding of the effect of noise on nonlinear dynamical systems [9], i.e., that noise acts as a disordering field that increases the degree of chaos of the system.

One of the arguments against the results of Ref. [2] is that, based on the theory of the statistics of trajectory separation in noisy dynamical systems [10,11], synchronization will occur in ensembles provided that the highest Lyapunov exponent is negative, while if this exponent is positive the systems will remain unsynchronized. In particular, Pikovsky [3] showed that the Lyapunov exponent of a noisy logistic equation is positive and thus one should not expect synchronization behavior to occur. Notice that this type of synchronization in random dynamical systems differs from the usual chaotic synchronization [12,13] in that in the present case the Lyapunov exponents of all the driven systems are identical (there are no conditioned, transverse, or the like, Lyapunov exponents), implying that when the different systems synchronize their behavior is no longer chaotic.

Pikovsky [3] also showed that the results in Ref. [2] depended on the numerical precision used in the calculations: *Synchronization appears only if the precision of the calculations is not very high*. This aspect has been considered also in Ref. [8], where the authors show that no synchronization occurs if iterations are done with infinite precision, thus confirming the conclusions in Ref. [3].

However, the point can be considered most clearly by studying carefully the statistical properties of the noise added to the system (this point has been treated in some detail by Herzel and Freund [5]). Maritan and Banavar [2] considered two examples of chaotic systems driven by the same noise: a discrete time system, the logistic equation, and a continuous time system, the Lorenz model. In the case of the Lorenz system the noise that is applied is uniformly distributed in an interval $[0,W]$. Thus this noise does not have zero mean. Here it is important to point out that these authors also remarked that the use of symmetric (zero mean) noise does not lead to synchronization [2]. The effect of this asymmetric noise can be understood because its mean acts as a constant bias to the system [5]. Thus one is altering the behavior of the system that is no longer chaotic (the highest Lyapunov exponent becomes negative). In this sense, the effect of this type of noise is to suppress the chaotic behavior of the system [14].

The analysis of the effect of the applied noise in the case of the logistic equation is subtler [5], as in this case Maritan and Banavar [2] apply a zero mean noise in the interval $[-W,W]$. The key remark is that the logistic equation works in the interval $[0,1]$ and sometimes the inclusion of external noise makes the system leave this interval and thus those realizations of the stochastic variable leading to a violation of the interval are excluded. This implies that the noise each system experiences becomes asymmetric, i.e., biased, and the corresponding Lyapunov exponents may be negative for some threshold noise W_c [5].

Some of the above-mentioned studies indicate that some nonphysical aspects inducing synchronization may be due to numerical artifacts in the calculations [3,8], pointing out that the observed synchronization depends on the numerical precision of the calculations, and for this reason we resort to experiments. In particular, electronic circuits are very useful devices for studying the behavior of nonlinear dynamical systems, both purely deterministic or subject to stochastic terms [15]. In both cases these circuits offer an analog simulation of systems that may not be trivial to solve by analytical calculations, especially in the case of systems subject to noise.

The experiments performed for the present work are based on the use of Chua's circuit, a paradigm of nonlinear

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analog circuits exhibiting chaotic behavior [16], which consists of three energy storage elements, one inductor and two capacitors (coupled through a resistance), and in parallel with a piecewise linear negative resistance implemented through the use of operational amps. It is defined by the evolution equations

$$C_1\frac{dV_1}{dt}=\frac{1}{R}(V_2-V_1)-h(V_1),$$

$$C_2\frac{dV_2}{dt}=\frac{1}{R}(V_1-V_2)+i_L, \qquad (1)$$

$$L\frac{di_L}{dt}=-V_2-r_0 i_L,$$

where V_1, V_2, and i_L, the voltages across C_1 and C_2 and the current through L, respectively, are the three variables that describe the dynamical system, resulting from straightforward application of Kirchhoff's law. The parameters have the following meaning: C_1 and C_2 are the two capacitances, L is the inductance, R is the resistance that couples the two capacitors, and r_0 is the inner resistance of the inductor. The three-segment piecewise-linear characteristic of the nonlinear resistor, a current, is defined by

$$h(V_1)=G_bV_1+\frac{1}{2}(G_a-G_b)[|V_1+B_p|-|V_1-B_p|], \quad (2)$$

where G_a and G_b are the slopes of the inner and outer regions of $h(V_1)$, respectively, and $B_p=1$ V defines the location of the breaking points of three-slope nonlinear characteristic $h(V_1)$.

An experimental setup of two equally driven Chua circuits whose components are defined by $(C_1,C_2,L,r_0,R)=(10\text{ nF},100\text{ nF},10\text{ mH},\ 20\ \Omega,1\text{ k}\Omega)$ has been built. The tolerances of the components employed here are 10% for inductors, 5% for capacitors, and 1% for resistors. The experiment has been designed in such a way that one can connect the individual circuits in a variety of ways. The circuits were sampled with a digital oscilloscope (Hewlett-Packard 54601B) with a maximum sample rate of 20×10^6 samples per second, eight bit analog to digital resolution, and a record length of 4000 points. The external noise has been generated by using a function generator (Hewlett-Packard 33120A) and its characteristics, Gaussian distribution and zero mean in the absence of an offset have been adequately checked. The slopes of the nonlinear characteristic $h(V)$ (2) are defined by $G_a=-8/7000S$ and $G_b=-5/7000S$.

The external noise has been introduced in two different ways: additively and multiplicatively. In the first case [see Fig. 1(a)] the stochastic voltage produced by the noise generator $\xi(t)$ has been converted in a current through a voltage controlled current source (VCCS), this contribution being added to that of the nonlinear element, yielding, through straightforward application of Kirchhoff's laws, the evolution equation for the voltage across capacitor C_1,

$$C_1\dot{V}_1=\frac{V_2-V_1}{R}-h(V_1)-f(\xi), \qquad (3)$$

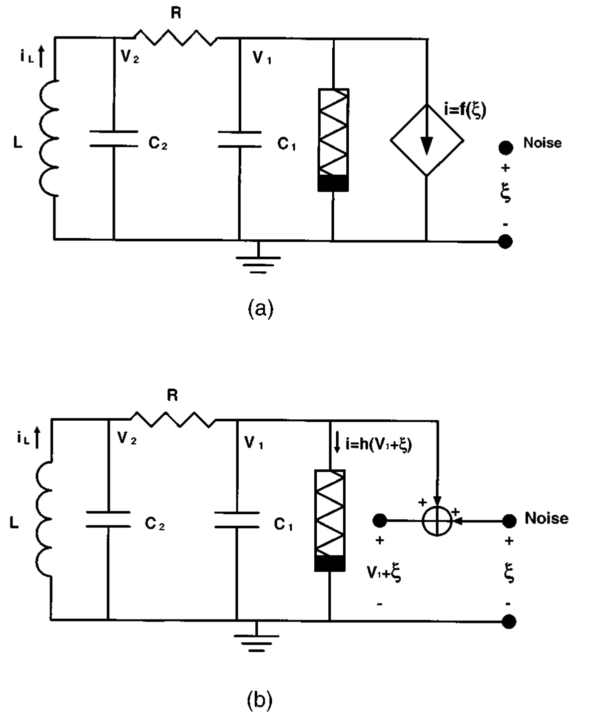

FIG. 1. Schematic diagram of the experimental setup corresponding to a single Chua circuit subject to noise in two different ways: (a) noise enters additively through a stochastic voltage that is converted into an intensity through a VCCS and added to the intensity produced by the nonlinear element of the circuit [see Eq. (3)] and (b) noise enters multiplicatively by adding a stochastic voltage to the voltage V_1 taken across the extremes of capacitor C_1, which is then fed into the nonlinear element. In both setups two identical Chua circuits have been subject to the same noise.

where $f(\xi)=\xi/R'$ with $R'=12.5\text{ k}\Omega$.

The second method to introduce the noise [see Fig. 1(b)] uses a recently introduced [17] circuit that enables one to drive the nonlinear element by using the voltage from an external source. The nonlinear element is driven, in general, by the voltage coming from an external source, not necessarily the voltage coming from capacitor C_1, as would happen in the case of a standard Chua circuit. Thus it is a VCCS with a characteristic defined by Eq. (2). We have added the stochastic signal to voltage V_1 by using an analog adder and the result has been used to drive the nonlinear element. This yields the evolution equation for the voltage across capacitor C_1,

$$C_1\dot{V}_1=\frac{V_2-V_1}{R}-h(V_1+\xi), \qquad (4)$$

where it is easy to see that the noise term now yields a multiplicative contribution.

The basic result of the present work is that the important aspect in the effect of noise on an ensemble of identically driven chaotic systems is whether its mean is zero (symmetrically distributed noise) or nonzero (asymmetrical noise). In the present work the noise that has been used is white, with a Gaussian distribution. If the noise has zero mean, the identically driven systems do not become synchronized to each other independently of the amplitude and vari-

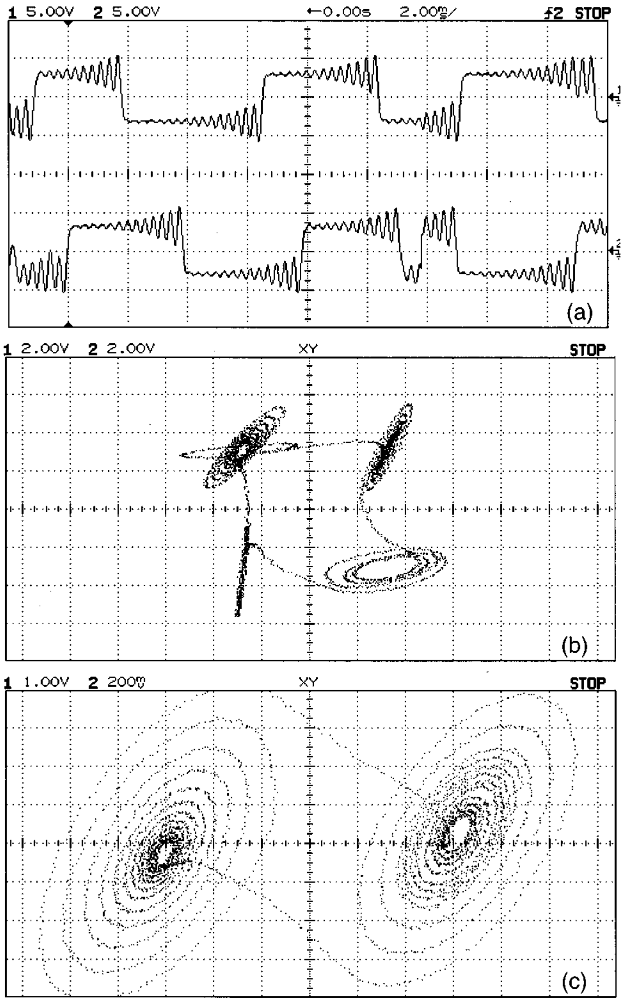

FIG. 2. Effect of symmetrically distributed (zero mean) noise on two Chua circuits identically driven accordingly to Eq. (3); see also Fig. 1(a): (a) V_1 taken in both circuits versus time, (b) the two V_1 voltages represented versus each other, and (c) phase portrait in one of the two circuits (V_1 vs V_2). It is important to remark that the oscilloscope is stopped, and if points were stored in memory, the phase portrait would fill a squared region in the phase portrait.

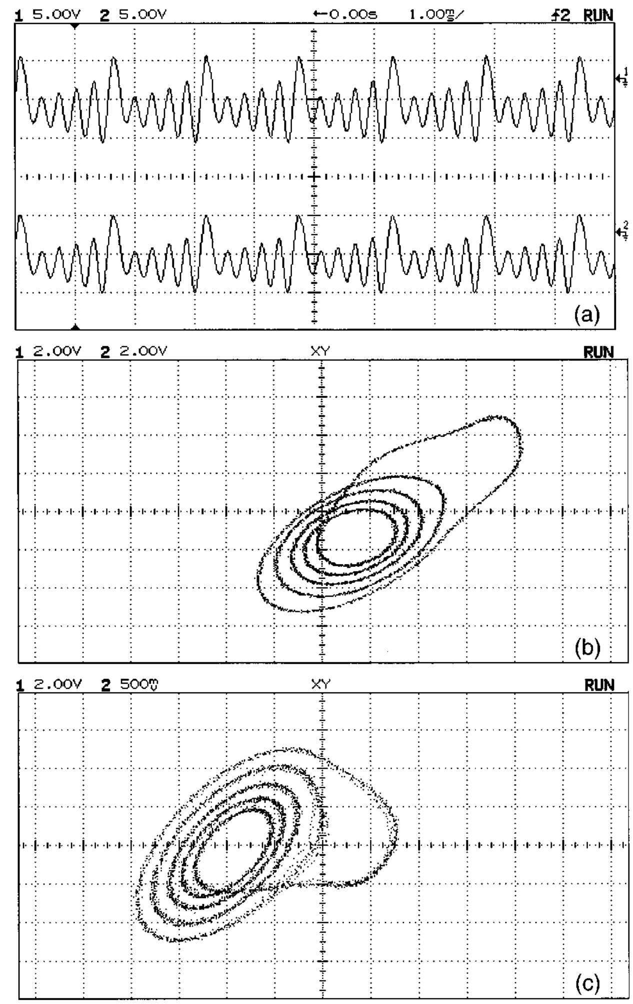

FIG. 3. Effect of asymmetrically distributed (nonzero mean or biased) noise on two Chua circuits identically driven as shown in Fig. 1(a): (a) V_1 taken in both circuits versus time, exhibiting a phase shift between the two signals, (b) the two V_1 voltages represented versus each other, and (c) phase portrait in one of the two circuits (V_1 vs V_2). Notice that panel (b) shows that there exists a generalized synchronization relationship between the two signals, although the signals are no longer chaotic [as seen better from panel (c)]. Notice that the stopped image presented in panel (b) is a genuine effect and does not stem from a technical feature of the oscilloscope, i.e., it exists on a scale longer than several minutes (to be compared to a sampling time of a few microseconds).

ance of the noise. The observations are in line with Ref. [9]: Noise produces a loss of the fine fractal details of the strange attractor, which becomes smeared out, while no evidence of synchronization behavior is observed. This can be seen in Fig. 2 for the case of additive noise (3), while the results are completely analogous to the case of multiplicative noise (4).

When the noise that is introduced in the system has instead nonzero mean, the stochastic signal induces qualitative changes in the system. This is shown in Fig. 3 for the case of additive noise (3), which proves that the relevant effect of the external stochastic signal is indeed closely linked to the fact that the mean of the stochastic signal is different from zero. Notice the similarity between Figs. 3(b) and 3(c). This confirms the validity in our case of the theory of statistics for trajectory separation in random dynamical systems [10,11]. Synchronization occurs, in a generalized sense, when the highest Lyapunov exponent becomes negative, although this implies that the system is no longer chaotic. The results obtained by introducing the external noise multiplicatively (4) are completely analogous.

To summarize, we have considered the effect of noise on the synchronization of identically driven chaotic systems. In agreement with the interpretation carried out by other authors [3,6–8], but especially as discussed in Ref. [5], the relevant aspect is not the stochastic nature of the signal used to drive the chaotic systems but instead its mean value. If it is zero, which, for example, is the situation that corresponds to the kind of thermal noise generated by a system in thermal equilibrium, the effect of the signal should be just to smear out the fractal structure of the strange attractor, but without yielding any observable effect on the behavior of an ensemble of identically driven systems. If the stochastic signal has instead nonzero mean, the net effect will be a biased signal that will induce a regularization in the system. This effects is completely analogous to that of some chaos suppression methods that achieve this result through perturbations in the system variables [18]. According to the theory of synchronization in dynamical systems driven by noise, synchronization will occur whenever the highest Lyapunov exponent of the system is negative [10,11]; this implies that the dynamics becomes regular (although somehow noisy). This is precisely what one observes in the cases that we have considered in the present work.

At this moment it is interesting to recall the results of Ref. [19]. Rajasekar and Lakshmanan concluded from numerical experiments that the addition of noise to a chaotic system has the effect of suppressing chaos, i.e., of making the highest Lyapunov exponent become negative. From the conclusions in the present work and also from Refs. [3,5] it is clear that these results emerge from the fact that Rajasekar and Lakshmanan generated a set of random numbers with a nonzero mean, that is, introduced a bias in the system. A well-known effect of noise in nonlinear systems is to allow the system to perform a noise-induced transition from the corresponding attractor to a second attractor (that has its own attraction basin) [20], but this is not the explanation of the behavior reported in Ref. [19]. Rajasekar and Lakshmanan reported also on the effect of the numerical integration step in the transition from chaos to order. One could think of representing the corresponding dynamical system by an analog circuit. In this case the conclusion is that unbiased noise cannot suppress chaos, although it may induce a transition to an existing (periodic) attractor. This representation is continuous and it is clear that the approximation of the underlying system by a numerical method should be such that its behavior cannot depend on the intregation time step, which is not a physical parameter of the system.

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