

## Perturbative and nonperturbative analysis of the third-order zero modes in the Kraichnan model for turbulent advection

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The anomalous scaling behavior of the  $n$ th-order correlation functions  $\mathcal{F}_n$  of the Kraichnan model of turbulent passive scalar advection is believed to be dominated by the homogeneous solutions (zero modes) of the Kraichnan equation  $\hat{\mathcal{B}}_n \mathcal{F}_n = 0$ . In this paper we present an extensive analysis of the simplest (nontrivial) case of  $n=3$  in the isotropic sector. The main parameter of the model, denoted as  $\zeta_h$ , characterizes the eddy diffusivity and can take values in the interval  $0 \leq \zeta_h \leq 2$ . After choosing appropriate variables we can present nonperturbative numerical calculations of the zero modes in a projective two dimensional circle. In this presentation it is also very easy to perform perturbative calculations of the scaling exponent  $\zeta_3$  of the zero modes in the limit  $\zeta_h \rightarrow 0$ , and we display quantitative agreement with the nonperturbative calculations in this limit. Another interesting limit is  $\zeta_h \rightarrow 2$ . This second limit is singular, and calls for a study of a boundary layer using techniques of singular perturbation theory. Our analysis of this limit shows that the scaling exponent  $\zeta_3$  vanishes as  $\sqrt{\zeta_2 / |\ln \zeta_2|}$ , where  $\zeta_2$  is the scaling exponent of the second-order correlation function. In this limit as well, perturbative calculations are consistent with the nonperturbative calculations. [S1063-651X(97)06306-X]

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### I. INTRODUCTION

The Kraichnan model of turbulent passive scalar advection [1] pertains to a field  $T(\mathbf{r}, t)$  which satisfies the equation of motion

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} + \mathbf{u}(\mathbf{r}, t) \cdot \nabla T(\mathbf{r}, t) = \kappa \nabla^2 T(\mathbf{r}, t) + \xi(\mathbf{r}, t). \quad (1)$$

Here  $\xi(\mathbf{r}, t)$  is a Gaussian white random force,  $\kappa$  is the molecular diffusivity, and the driving field  $\mathbf{u}(\mathbf{r}, t)$  is chosen to have Gaussian statistics, and to be ‘‘rapidly varying’’ in the sense that its time correlation function is proportional to  $\delta(t)$ . The statistical quantities that one is interested in are the many-point correlation functions

$$\mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2n}) \equiv \langle \langle T(\mathbf{r}_1, t) T(\mathbf{r}_2, t) \cdots T(\mathbf{r}_{2n}, t) \rangle \rangle, \quad (2)$$

where double angle brackets denote an ensemble average with respect to the (stationary) statistics of the forcing and the statistics of the velocity field. Assuming that these correlation functions are scale invariant one is interested in the scaling (or homogeneity) exponent  $\zeta_{2n}$  of  $\mathcal{F}_{2n}$  which is defined by

$$\mathcal{F}_{2n}(\lambda \mathbf{r}_1, \lambda \mathbf{r}_2, \dots, \lambda \mathbf{r}_{2n}) = \lambda^{\zeta_{2n}} \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2n}). \quad (3)$$

One expects such a scale invariant solution to exist in the inertial range, i.e., when all the separations  $r_{ij}$  satisfy  $\eta \ll r_{ij} \ll L$  where  $\eta$  and  $L$  are the inner and outer scales, respectively. It is known [1] that for  $\mathcal{F}_2$  such a solution exists with  $\zeta_2 = 2 - \zeta_h$ , where  $\zeta_h$  is the exponent of the eddy diffusivity, see Eq. (6).

The Kraichnan model is unique in the field of turbulence in that it allows the derivation [2] of an exact differential equation for this correlation function,

$$\left( -\kappa \sum_{\alpha} \nabla_{\alpha}^2 + \hat{\mathcal{B}}_{2n} \right) \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2n}) = \mathbf{R}_{2n}. \quad (4)$$

The operator  $\hat{\mathcal{B}}_{2n} \equiv \sum_{\alpha > \beta}^{2n} \hat{\mathcal{B}}_{\alpha\beta}$ , and  $\hat{\mathcal{B}}_{\alpha\beta}$  are defined by

$$\hat{\mathcal{B}}_{\alpha\beta} \equiv \hat{\mathcal{B}}(\mathbf{r}_{\alpha}, \mathbf{r}_{\beta}) = h_{ij}(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}) \partial^2 / \partial r_{\alpha,i} \partial r_{\beta,j}, \quad (5)$$

where the ‘‘eddy-diffusivity’’ tensor  $h_{ij}(\mathbf{R})$  is given by

$$h_{ij}(\mathbf{R}) = h(R) [(\zeta_h + d - 1) \delta_{ij} - \zeta_h R_i R_j / R^2], \quad (6)$$

and  $h(R) = H(R/\mathcal{L})^{\zeta_h}$ . Here  $\mathcal{L}$  is some characteristic outer scale of the driving velocity field. The parameter that can be varied in this model is the scaling exponent  $\zeta_h$ ; it characterizes the  $R$  dependence of  $h_{ij}(\mathbf{R})$  and it can take values in the interval  $[0, 2]$ . Finally, the right-hand side in Eq. (4) is known explicitly, but is not needed here. The reason is that it was argued [3] that the solutions of this equation for  $n > 1$  are dominated by the homogeneous solutions (‘‘zero modes’’), in the sense that deep in the inertial interval the inhomogeneous solutions are negligible compared to the homogeneous one. Also, it was claimed [3] that in the inertial interval one can neglect the Laplacian operators in Eq. (4), and remain with the simpler homogeneous equations  $\hat{\mathcal{B}}_{2n} \mathcal{F}_{2n} = 0$ .

Exact solutions of these homogeneous equations are not easy; even in the simplest case of  $n=2$  the function  $\mathcal{F}_4$  depends on six independent variables (for dimensions  $d > 2$ ), and one faces a formidable analytic difficulty for exact solutions. Accordingly, several groups considered perturbative solutions in some small parameter, like  $\zeta_h$  [3] or the inverse dimensionality  $1/d$  [4]. The rationale for this approach is that

at  $\zeta_h=0$  and  $d\rightarrow\infty$  one expects ‘‘simple scaling’’ with  $\zeta_{2n}=n\zeta_2$ . The exponent  $\zeta_4$ , and later also the set  $\zeta_{2n}$ , were computed as a function of  $\zeta_h$  near these simple scaling limits. The other limit of  $\zeta_h\rightarrow 2$  invites perturbation analysis as well, since one expects that at  $\zeta_h=2$  all the scaling exponents  $\zeta_{2n}$  would vanish. Such a perturbation theory turned out to be elusive.

Recently we reported [5] that it is possible to solve exactly the homogeneous equation satisfied by the third-order correlation function  $\mathcal{F}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ . Note that in Kraichnan’s model all the odd-order correlation functions  $\mathcal{F}_{2n+1}$  are zero because of symmetry under the transformation  $T\rightarrow -T$ . This symmetry disappears, for example [6], if the random force  $\xi(\mathbf{r}, t)$  is not Gaussian (but  $\delta$  correlated in time), and in particular if it has a nonzero third-order correlation

$$\mathcal{D}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \equiv \int dt_1 dt_2 \langle \xi(\mathbf{r}_1, t_1) \xi(\mathbf{r}_2, t_2) \xi(\mathbf{r}_3, 0) \rangle. \quad (7)$$

With such a forcing the third-order correlator is nonzero, and it satisfies the equation

$$\hat{\mathcal{B}}_3 \mathcal{F}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \mathcal{D}_3, \quad \hat{\mathcal{B}}_3 \equiv \hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{23}. \quad (8)$$

This equation pertains to the inertial interval and accordingly we neglected the Laplacian operators. We also denoted by  $\mathcal{D}_3$  the value of  $\mathcal{D}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  when all the separations  $|\mathbf{r}_i - \mathbf{r}_j|$  tend to zero. The solution of this equation is a sum of inhomogeneous and homogeneous contributions, and below we study the latter. We will focus on scale invariant homogeneous solutions which satisfy  $\mathcal{F}_3(\lambda \mathbf{r}_1, \lambda \mathbf{r}_2, \lambda \mathbf{r}_3) = \lambda^{\zeta_3} \mathcal{F}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ . We refer to these as the ‘‘zero modes in the scale invariant sector.’’ We note that the scaling exponent of the *inhomogeneous* scale invariant contribution can be read directly from power counting in Eq. (8) (leading to  $\zeta_3 = \zeta_2$ ). Any different scaling exponent can arise only from homogeneous solutions that do not need to balance the constant right-hand side. The scale invariant solutions of Eq. (8) live in a projective space whose dimension is lowered by unity compared to the most general form. These solutions do not depend on three separations but rather on two dimensionless variables that are identified below. It will be demonstrated how boundary conditions arise in this space for which the operator  $\hat{\mathcal{B}}_3$  is neither positive nor self-adjoint.

In Sec. II we present the transformation of variables that leads to a precise identification of the projective space. In this space we present the differential equation that needs to be studied, and derive the boundary conditions in the projective space. In Sec. III we discuss the perturbation theory that leads to the solution of the scaling exponents of the zero modes in the limit  $\zeta_h \rightarrow 0$ . It is shown that the choice of coordinates of Sec. II leads to a particularly transparent theory in this limit. In Sec. IV we present the perturbation theory in the limit  $\zeta_h \rightarrow 2$ . It turns out that this is a singular perturbation theory, and we discuss the analytic matchings across boundary layers and near the ‘‘fusion singularity’’ which are required to understand this limit, leading to a nonanalytic dependence of  $\zeta_3$  on  $\zeta_2$ . In Sec. V we deal with the nonperturbative calculation, culminating with solutions of  $\zeta_3$  as a function of  $\zeta_h$  throughout the range  $0 \leq \zeta_h \leq 2$ . It is demonstrated that the nonperturbative solutions are in agree-

ment with the perturbative calculations at the two ends of this interval. Section VI is devoted to a summary and a discussion.

## II. TRANSFORMATION OF VARIABLES

In this section we describe the transformation of variables in the operator  $\mathcal{B}_3$  to new variables that are denoted below as  $s, \rho, \phi$ . We first note that Eq. (8) is invariant under space translation, under the action of the  $d$  dimensional rotation group  $\text{SO}(d)$ , and under permutations of the three coordinates. Accordingly, we may seek solutions in the scalar representation of  $\text{SO}(d)$ , where the solution depends on the three separations  $r_{12}$ ,  $r_{23}$  and  $r_{31}$  only. In the first stage we transform coordinates to the variables  $x_1 = |\mathbf{r}_2 - \mathbf{r}_3|^2$ ,  $x_2 = |\mathbf{r}_3 - \mathbf{r}_1|^2$ ,  $x_3 = |\mathbf{r}_1 - \mathbf{r}_2|^2$ , defining

$$F_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = f_3(x_1, x_2, x_3). \quad (9)$$

By the chain rule,

$$\partial_{1i} F_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = 2(r_{13i} \partial_2 f_3 + r_{12i} \partial_3) f_3(x_1, x_2, x_3), \quad (10)$$

where  $\partial_{1i} \equiv (\partial/\partial r_{1i})$ ,  $\partial_2 \equiv (\partial/\partial x_2)$ , and  $r_{12i} \equiv r_{1i} - r_{2i}$ , etc.

Another application of the chain rule gives

$$\begin{aligned} \partial_{2j} \partial_{1i} = & 4r_{13i} r_{23j} \partial_1 \partial_2 + 4r_{12i} r_{23j} \partial_1 \partial_3 - 4r_{13i} r_{12j} \partial_2 \partial_3 \\ & - 4r_{12i} r_{12j} \partial_3^2 - 2\delta_{ij} \partial_3. \end{aligned} \quad (11)$$

(For brevity we display only the differential operators explicitly.) Using  $2\mathbf{r}_{12} \cdot \mathbf{r}_{13} = -x_1 + x_2 + x_3$ , and similar identities, we can now obtain

$$\begin{aligned} \partial_{ij} \partial_{2j} \partial_{1i} = & 2(x_1 + x_2 - x_3) \partial_1 \partial_2 + 2(-x_1 + x_2 - x_3) \partial_1 \partial_3 \\ & + 2(x_1 - x_2 - x_3) \partial_2 \partial_3 - 4x_3 \partial_1 \partial_2 - 2d \partial_3, \end{aligned} \quad (12)$$

and

$$\begin{aligned} \frac{r_{12i} r_{12j}}{r_{12}^2} \partial_{2j} \partial_{1i} = & \frac{(x_1 - x_2 - x_3)(x_1 - x_2 + x_3)}{x_3} \partial_1 \partial_2 \\ & + 2(-x_1 + x_2 - x_3) \partial_1 \partial_3 \\ & + 2(x_1 - x_2 - x_3) \partial_2 \partial_3 - 4x_3 \partial_1 \partial_2 - 2d \partial_3. \end{aligned} \quad (13)$$

Further calculations, using Eqs. (12) and (13), give

$$\begin{aligned} \mathcal{B}_{12} \equiv & r_{12}^{\zeta_h} \left( (d + \zeta_h - 1) \delta_{ij} - \zeta_h \frac{r_{12i} r_{12j}}{r_{12}^2} \right) \partial_{2j} \partial_{1i} \\ = & x_3^{\zeta_h/2} \left( (d-1) o_1 + (d-1)(d + \zeta_h) o_2 + \frac{\zeta_h}{x_3} o_3 \right), \end{aligned} \quad (14)$$

where

$$\begin{aligned} o_1 = & 2(x_1 + x_2 - x_3) \partial_1 \partial_2 + 2(-x_1 + x_2 - x_3) \partial_1 \partial_3 \\ & + 2(x_1 - x_2 - x_3) \partial_2 \partial_3 - 4x_3 \partial_1 \partial_2, \end{aligned} \quad (15)$$

$$o_2 = -2\partial_3, \quad (16)$$

$$o_3 = [x_3(x_1 + x_2 - x_3) - (x_1 - x_2 - x_3)(x_1 - x_2 + x_3)]\partial_1\partial_2. \quad (17)$$

The reader should note that the  $o_i$  operators do not depend on the parameters of the problem. The operators  $\mathcal{B}_{23}$  and  $\mathcal{B}_{31}$  are obtained from  $\mathcal{B}_{12}$  by cyclic permutations of the indices, thus completing the transformation of  $\mathcal{B}_3$  to the  $x$  variables.

Note that not every point in the  $x_1, x_2, x_3$  space corresponds to a physical configuration. The triangle inequalities between the pairwise distances translate to the condition

$$2(x_1x_2 + x_2x_3 + x_3x_1) \geq x_1^2 + x_2^2 + x_3^2. \quad (18)$$

This inequality describes a circular cone in the  $x_1, x_2, x_3$  space whose axis is the line  $x_1 = x_2 = x_3$ , tangent to the planes  $x_1 = 0$ ,  $x_2 = 0$ , and  $x_3 = 0$ . The group of permutations between the  $x_i$  axes acts very simply on this cone, corresponding to a  $C_6$  operation.

The presence of symmetries motivates a new parametrization of the cone by three new coordinates  $s, \rho, \phi$ :

$$x_n = s[1 - \rho \cos(\phi + \frac{2}{3}n\pi)], \quad (19)$$

$$0 \leq s < \infty, \quad 0 \leq \rho \leq 1, \quad 0 \leq \phi \leq 2\pi.$$

The new space is a direct product of three intervals, and this fact will simplify the discussion of the boundary conditions. The  $s$  coordinate measures the overall scale of the triangle defined by the original  $r_i$  coordinates, and configurations of constant  $\rho$  and  $\phi$  correspond to similar triangles. The  $\rho$  coordinate describes the deviation of the triangle from the equilateral configuration ( $\rho = 0$ ) up to the physical limit of three collinear points attained when  $\rho = 1$ ;  $\phi$  does not have a simple geometric meaning. Finally we note that the variables  $s$ ,  $\rho$ , and  $\cos(3\phi)$  are symmetric in the  $x_i$  variables. Accordingly any function of these variables is automatically invariant under the permutation of the  $x_i$  variables. We will use this property below.

The final form of the equation is achieved by transforming  $o_i$  operators to the variables  $s, \rho, \phi$ . To this end we compute the Jacobian of the transformation (19) using MATHEMATICA,

$$J \equiv \frac{\partial(s, \rho, \phi)}{\partial(x_1, x_2, x_3)} = \frac{1}{\sqrt{3}s^2\rho} \begin{bmatrix} 1 & -\rho + \cos(\phi) + \sqrt{3}\sin(\phi) & \sqrt{3}\cos(\phi) - \sin(\phi) \\ 1 & -\rho + \cos(\phi) - \sqrt{3}\sin(\phi) & -\sqrt{3}\cos(\phi) + \sin(\phi) \\ 1 & -\rho - 2\cos(\phi) & 2\sin(\phi)/\sqrt{3} \end{bmatrix}. \quad (20)$$

The Jacobian matrix  $J$  is substituted in the chain rule to give the transformation of the derivatives of  $f_3$  with respect to the  $x$  variables in terms of the new variables  $s, \rho, \phi$ . It is convenient to perform the tedious calculations using MATHEMATICA ending up with expressions for  $o_1, o_2$ , and  $o_3$ .

We present the final long result as a table, in which each item is of the form

$$\{\bar{s}, \bar{\rho}, \bar{\phi}, n, m, c\}$$

representing a term of the form

$$c\rho^n \frac{\cos}{\sin}(m\phi) s^{\bar{s}} \partial_s^{\bar{s}} \partial_\rho^{\bar{\rho}} \partial_\phi^{\bar{\phi}}.$$

The trigonometric functions  $\cos$  or  $\sin$  appear if  $\bar{\phi}$  is even or odd, respectively. The  $o_i$  operators are sums of such terms. The tables listing the terms appearing in each of the operators are presented in the Appendix.

The upshot of the transformation of the linear operator  $\hat{\mathcal{B}}_3$  to the new coordinates is that we derive a second-order linear partial differential operator in the  $s, \rho, \phi$  variables. At this point we take advantage of the scale invariance of the differential equation which allows us to seek scale invariant solutions of the form  $s^{\zeta_3/2} f(\rho, \phi)$ . Acting on functions of this form, the operators  $s\partial_s$  and  $s^2\partial_s^2$  become scalar multiplications by  $\frac{1}{2}\zeta_3$  and  $\frac{1}{2}\zeta_3(\frac{1}{2}\zeta_3 - 1)$ , respectively. The action of the

operator  $\hat{\mathcal{B}}_3$  yields an equation for  $f(\rho, \phi)$ , which is the basic equation we study in this paper,

$$\begin{aligned} \hat{\mathcal{B}}_3(\zeta_3)f(\rho, \phi) = & [a(\rho, \phi)\partial_\rho^2 + b(\rho, \phi)\partial_\phi^2 + c(\rho, \phi)\partial_\rho\partial_\phi \\ & + u(\rho, \phi, \zeta_3)\partial_\rho + v(\rho, \phi, \zeta_3)\partial_\phi \\ & + w(\rho, \phi, \zeta_3)]f(\rho, \phi) = 0. \end{aligned} \quad (21)$$

We note that Eq. (21) can be written in a coordinate-free form as

$$[-\nabla \cdot \vec{P}(\rho, \phi) \cdot \nabla + \mathbf{q}(\rho, \phi, \zeta_3) \cdot \nabla + w(\rho, \phi, \zeta_3)]f(\rho, \phi) = 0, \quad (22)$$

where  $\nabla$  is the gradient operator in the  $\rho, \phi$  space. The identification of the tensor  $\vec{P}$  and the row vector  $\mathbf{q}$  is obtained by comparison with the explicit form (21). The new operator  $\hat{\mathcal{B}}_3$  depends on  $\zeta_3$  as a parameter and it acts on the unit circle described by the polar  $\rho, \phi$  coordinates. The circle represents the projective space of the physical cone described above. We will see that the availability of a compact domain (the projective space) will lead to the existence of a discrete spectrum of the zero modes.

The discrete permutation symmetry of the original Eq. (8) results in a symmetry of Eq. (21) with respect to the six element group generated by the transformation  $\phi \rightarrow \phi + 2\pi/3$  (cyclic permutation of the coordinates in the physical space) and  $\phi \rightarrow -\phi$  (exchange of coordinates). This

symmetry extends to a full  $U(1)$  symmetry in the two marginal cases of  $\zeta_h=0$  and  $\zeta_h=2$  (see [7,8] for a discussion of the latter limit) for which all the coefficients in Eq. (21) become  $\phi$  independent. The coefficients in Eq. (21) all have a similar structure, and, for example,  $a(\rho, \phi)$  reads

$$a(\rho, \phi) = \sum_n [1 - \rho \cos(\phi + \frac{2}{3}\pi n)]^{(\zeta_h-2)/2} \tilde{a}(\rho, \phi + \frac{2}{3}\pi n), \quad (23)$$

where  $\tilde{a}(\rho, \phi)$  is a low-order polynomial in  $\rho$ ,  $\cos\phi$ , and  $\sin\phi$  which vanishes at  $\rho=1, \phi=0$ . We see that the coefficients are analytic everywhere on the circle except at the three points  $\rho=1, \phi=2\pi n/3$  where  $n=0,1,2$ . These points correspond to the fusion of one pair of coordinates, and the coefficients exhibit a branch point singularity there. This singularity is a consequence of the nonanalyticity of the driving velocity field whose eddy diffusivity is therefore characterized by a noninteger exponent. This singularity leads to a nontrivial asymptotic behavior of the solutions which had been described before in terms of the fusion rules [9,10]. Since the coefficient  $\tilde{a}$  vanishes at the fusion point, the singular term in Eq. (23) behaves like a positive power of the distance to the fusion point. Thus it is always sub-leading with respect to other terms that come from nonfusing coordinates and which do not vanish at the fusion point. Indeed, in [9,10] it was explained that exposing the singularity calls for taking a derivative with respect to the fusing coordinates. Note that for  $\zeta_h=2$  the singularity disappears trivially. For  $\zeta_h=0$  there is also no singularity since  $\tilde{a}$  exactly compensates for the inverse power.

The boundary conditions follow naturally when one realizes that  $\hat{B}_3$  is elliptic for points strictly inside the physical circle. In the presentation of Eq. (22) this means that

$$\det(\vec{P}) > 0 \quad \text{for } \rho < 1. \quad (24)$$

This property is a consequence of the ellipticity of the original operator  $\hat{B}_3$ . On the other hand,  $\hat{B}_3$  becomes singular on the boundary  $\rho=1$ , where the coefficients  $a(\rho, \phi)$  and  $c(\rho, \phi)$  vanish. In other words,

$$\vec{P} \cdot \mathbf{n} \Big|_{\rho=1} = 0, \quad (25)$$

where  $\mathbf{n}$  is a unit vector normal to the boundary. This singularity reflects the fact that this is the boundary of the physical region. It follows that  $\hat{B}_3$  restricted to the boundary becomes a relation between the function  $f(\rho=1, \phi) \equiv g(\phi)$  and its normal derivative  $\partial_\rho f(\rho, \phi) \Big|_{\rho=1} \equiv h(\phi)$ . The relation is

$$bg''(1, \phi) + uh(1, \phi) + vg'(1, \phi) + wg(1, \phi) = 0. \quad (26)$$

Solutions of Eq. (21) which do not satisfy this boundary condition are singular, with infinite  $\rho$  derivatives at  $\rho=1$ . Such solutions are not physical since they involve infinite correlations between the dissipation (second derivative of the field) and the field itself when the geometry becomes collinear, but without fusion.

It is important to stress that the reason for the regularity of the solutions that do satisfy the boundary conditions is that  $\det \vec{P}$  vanishes as a simple zero near the boundary, i.e., as

$(1-\rho)$ . Consequently it is possible to find solutions that behave as  $(1-\rho)^0 \sim O(1)$  near the boundary. If  $\det \vec{P}$  had a stronger zero the condition (26) could not ensure regularity. The physical reason for this regularity is the finite value of three-point correlation functions of collinear configurations.

Finally, we are facing the problem of solving Eq. (21) together with the boundary condition (26). This homogeneous boundary value problem always has a trivial solution  $f=0$ . The condition for the existence of a nontrivial solution is that  $\hat{B}_3$  is not invertible. We expect that noninvertibility will not be a generic phenomenon for an arbitrary value of  $\zeta_3$ . Finding the set of  $\zeta_3$  for which  $\hat{B}_3$  is not invertible becomes a generalized eigenvalue problem with the exponents  $\zeta_3$  playing the role of eigenvalues. We first discuss this program in the limit of small  $\zeta_h$ .

### III. PERTURBATION THEORY NEAR $\zeta_h=0$

The shape of our compact domain invites a Fourier representation in  $\phi$  for the function  $f(\rho, \phi)$ . The permutation symmetry implies that only  $\cos$  will appear in this representation, and the index will be divisible by 3. The general equation (21) will then mix different Fourier modes. On the other hand, in the limit  $\zeta_h=0$  the situation simplifies considerably.

#### A. Zero modes at $\zeta_h=0$

The differential equation reduces in the case  $\zeta_h=0$  to the simple form

$$[\rho^2(1-\rho^2)\partial_\rho^2 + \rho(1-d\rho^2)\partial_\rho + \partial_\phi^2 + \lambda\rho^2]f(\rho, \phi) = 0, \quad (27)$$

where

$$\lambda \equiv \frac{1}{2}\zeta_3(\frac{1}{2}\zeta_3 + d - 1). \quad (28)$$

Since the coefficients of Eq. (27) are independent of  $\phi$  the Fourier modes are decoupled, and we may seek solutions of the form  $f_m(\rho)\cos m\phi$  with  $m$  divisible by 3. The functions  $f_m$  obey the ordinary differential equations (ODEs)

$$[\rho^2(1-\rho^2)\partial_\rho^2 + \rho(1-d\rho^2)\partial_\rho - m^2 + \lambda\rho^2]f_m(\rho) = 0. \quad (29)$$

Examining the resulting differential equation we note that the coefficient of the highest derivative in  $\rho$  vanishes as a double zero at  $\rho=0$  and as a single zero at  $\rho=1$ . Since the order of the zero is not more than the order of the derivative the Frobenius theory [11] of regular singularities is applicable to both boundaries. Namely, the complete family of solutions in the vicinity of a given singular point  $\rho_0$  is spanned by functions that can be represented as

$$f_m^{(i)}(\rho) = (\rho - \rho_0)^{z_i} \ln(\rho - \rho_0)^{k_i} \sum_{p=0}^{\infty} a_{i,p}(\rho - \rho_0)^p, \quad (30)$$

$$i = 1, 2, \quad k_i = 0 \text{ or } 1$$

where the sum is convergent in a neighborhood of  $\rho_0$ . One of the indices  $k_i$  is always zero. When the indices  $z_{1,2}$  are dif-

ferent, we choose arbitrarily  $z_1 > z_2$  and then  $k_1$  is zero. When  $|z_1 - z_2|$  is not an integer one also has  $k_2 = 0$ . In cases in which the indices  $z_i$  coincide we will choose  $k_1 = 0$ , and  $k_2 = 1$ . The numerical values of the indices are obtained by substituting a solution  $(\rho - \rho_0)^z$  in the differential equation, collecting the coefficients of the leading terms near the singularity, and equating to zero. We refer to the indices  $z_i$  as the ‘‘Frobenius exponents.’’ Equation (29) has regular singularities at both boundaries  $\rho = 0, 1$ , with Frobenius exponent sets of  $-m, m$  and  $0, (3-d)/2$ , respectively. The singularity at  $\rho = 1$  arises from the singularity of the original partial differential equation (PDE) (21) at this boundary, and the boundary condition picks the regular solution, i.e.,  $f_m(\rho) \sim (1-\rho)^0$  as  $\rho \rightarrow 1$ . On the other hand, the singularity at  $\rho = 0$  is an artifact of the transformation to polar coordinates; however, analyticity of the solution at  $\rho = 0$  requires that  $f_m(\rho) \sim \rho^m$  as  $\rho \rightarrow 0$ , specifying the second boundary condition for Eq. (29).

To proceed with the solution of Eq. (29) we make the transformations  $\tilde{f}_m(\tilde{\rho}) = \rho^m f_m(\rho)$  and  $\tilde{\rho} = \rho^2$ , and obtain the hypergeometric equation for  $\tilde{f}_m$ ,

$$\begin{aligned} \{(1-\tilde{\rho})\tilde{\rho}\partial_{\tilde{\rho}}^2 + [m+1 - (m+\frac{1}{2}(d+1))\tilde{\rho}]\partial_{\tilde{\rho}} \\ + \frac{1}{4}[\lambda - m(m-1+d)]\}\tilde{f}_m(\tilde{\rho}) = 0. \end{aligned} \quad (31)$$

This equation is standard, see [12]. The unique solution of Eq. (31) which satisfies the boundary condition at 0 (up to a multiplication by a constant) is

$$\tilde{f}_m(\tilde{\rho}) = {}_2F_1(a, b; m+1; \tilde{\rho}), \quad (32)$$

where

$$a = \frac{1}{2}(m - \frac{1}{2}\zeta_3), \quad b = \frac{1}{2}(m + d - 1 + \frac{1}{2}\zeta_3). \quad (33)$$

It follows from the theory of the hypergeometric functions that the function  $\tilde{f}_m$  defined in Eq. (32) is regular at  $\rho = 1$  only if either  $a$  or  $b$  equals  $-n$ , where  $n$  is a non-negative integer. In such a case  $\tilde{f}_m$  becomes a polynomial of degree  $n$ . The spectrum of  $\zeta_3$  now follows from Eq. (33) and consists of two sets,

$$\zeta_{3(m,n)}^+ = 2(m+2n), \quad \zeta_{3(m,n)}^- = -2(d-1+m+2n). \quad (34)$$

Since  $\lambda(\zeta_{3(m,n)}^+) = \lambda(\zeta_{3(m,n)}^-)$  [cf. Eq. (28)], the corresponding eigenfunctions depend only on  $n$  and  $m$ . Expressed in terms of the original variables the eigenfunctions are

$$\begin{aligned} f_{(m,n)}(\rho, \phi) \\ = \rho^m {}_2F_1(-n, n+m+(d-1)/2; m+1; \rho^2) \cos(m\phi). \end{aligned} \quad (35)$$

The zero modes of the first set, with positive values of  $\zeta_3$ , are the ones to be matched at the outer scale. Of these, the most relevant nontrivial zero mode (i.e., the zero mode with the smallest positive exponent) is  $f_{0,1}$  with  $\zeta_{3(0,1)}^+ = 4$ , still less relevant than the scaling of the forced solution. We

remind the reader that the neglect of the diffusive terms means that anomalous scaling must be obtained using the outer scale as the renormalization scale, leading to dominance of the *smallest* positive exponents.

### B. Zero modes for $0 < \zeta_h \ll 1$

Perturbation theory for  $\zeta_h \rightarrow 0$  will be carried out by expanding the solution for small  $\zeta_h$  in terms of the  $\zeta_h = 0$  eigenfunctions  $f_{(m,n)}$ , in a procedure very similar to time independent perturbation theory in quantum mechanics. We first cast Eq. (29) in Sturm-Liouville form

$$\begin{aligned} \left( \frac{1}{\rho(1-\rho^2)^{(d-3)/2}} \partial_\rho [\rho(1-\rho^2)^{(d-1)/2} \partial_\rho] - \frac{m^2}{\rho^2} \right) f_m(\rho) \\ = -\lambda f_m(\rho). \end{aligned} \quad (36)$$

It follows that the zero modes  $f_{(m,n)}$  form an orthogonal set with respect to the inner product

$$\langle f, g \rangle_0 \equiv \int_{\rho \leq 1} \frac{d\rho d\phi}{2\pi} \rho(1-\rho^2)^{(d-3)/2} f(\rho, \phi) g(\rho, \phi). \quad (37)$$

We now assume that we may expand the eigenfunction and eigenvalues as

$$f(\rho, \phi) = f_{(m,n)}(\rho, \phi) + \zeta_h f^{(1)}(\rho, \phi) + O(\zeta_h^2), \quad (38)$$

$$\zeta_3 = \zeta_{3(m,n)} + \zeta_3^{(1)} + O(\zeta_h^2). \quad (39)$$

Writing

$$\begin{aligned} B(\zeta_h, \zeta_3) = B(0, \zeta_{3(m,n)}) + \zeta_h [\partial_{\zeta_h} B(0, \zeta_{3(m,n)}) \\ + \zeta_3^{(1)} \partial_{\zeta_3} B(0, \zeta_{3(m,n)})] + O(\zeta_h^2), \end{aligned} \quad (40)$$

Eq. (21) becomes, to order  $\zeta_h$ ,

$$\begin{aligned} B(0, \zeta_{3(m,n)}) f^{(1)}(\rho, \phi) + [\partial_{\zeta_h} B(0, \zeta_{3(m,n)}) \\ + \zeta_3^{(1)} \partial_{\zeta_3} B(0, \zeta_{3(m,n)})] f_{(m,n)}(\rho, \phi) = 0. \end{aligned} \quad (41)$$

Taking the inner product  $\langle \rangle_0$  of Eq. (41) with  $f_{(m,n)}$ , and using the self-adjointness of  $B(0, \zeta_{3(m,n)})$  with respect to this inner product, there follows an expression for the first correction to  $\zeta_3$ ,

$$\zeta_3^{(1)} = - \frac{\langle f_{(m,n)}, \partial_{\zeta_h} B(0, \zeta_{3(m,n)}) f_{(m,n)} \rangle_0}{\langle f_{(m,n)}, \partial_{\zeta_3} B(0, \zeta_{3(m,n)}) f_{(m,n)} \rangle_0}. \quad (42)$$

The integrals in Eq. (42) were programmed using MATHEMATICA, and we used the program to generate expressions for the first few values for  $\zeta_3^{(1)}$  presented below:

$m$	$n$	$\zeta_{3(m,n)}^+$	$\zeta_3^{+(1)}$
0	0	0	0
0	1	4	$\frac{2(2-d)}{d-1}$
0	2	8	$\frac{2(159+91d-31d^2-19d^3-2d^4)}{-48+4d+32d^2+11d^3+d^4}$
0	3	12	$\frac{6(780+503d-89d^2-103d^3-19d^4-d^5)}{-480-8d+324d^2+142d^3+21d^4+d^5}$
3	0	6	$\frac{-3(15+8d+d^2)}{-2+d+d^2}$
3	1	10	$-\frac{(9+d)(-489-131d+136d^2+53d^3+5d^4)}{-144-36d+100d^2+65d^3+14d^4+d^5}$
6	0	12	$\frac{3(3+d)(11+d)^2(370+216d+37d^2+2d^3)}{3840+544d-2584d^2-1460d^3-310d^4-29d^5-d^6}$
$m$	$n$	$\zeta_{3(m,n)}^-$	$\zeta_3^{-(1)}$
0	0	$-2d+2$	$-1$
0	1	$-2d-2$	$\frac{3-d}{d-1}$
0	2	$-2d-6$	$\frac{-3(90+62d-10d^2-9d^3-d^4)}{-48+4d+32d^2+11d^3+d^4}$
0	3	$-2d-10$	$-\frac{4200+3010d-210d^2-476d^3-93d^4-5d^5}{-480-8d+324d^2+142d^3+21d^4}$
3	0	$-2d-4$	$-\frac{(5+d)(7+2d)}{2-d-d^2}$
3	1	$-2d-8$	$-\frac{(9+d)(441+135d-104d^2-42d^3-4d^4)}{-144-36d+100d^2+65d^3+14d^4+d^5}$
6	0	$-2d-10$	$\frac{5(11+d)(6558+6508d+2409d^2+412d^3+33d^4+d^5)}{-3840-544d+2584d^2+1460d^3+310d^4+29d^5+d^6}$

(43)

#### IV. PERTURBATION THEORY IN THE LIMIT $\zeta_h \rightarrow 2$

In this section we treat the case  $\zeta_2 = 2 - \zeta_h \ll 1$  perturbatively. The perturbation theory is singular, since the leading-order approximation, obtained by setting the small parameter  $\zeta_2 = 0$  is not valid throughout the entire domain. We employ boundary layer techniques and matching of asymptotics to obtain an asymptotic approximation for  $\zeta_3$  in this limit. As expected,  $\zeta_3$  goes to 0 with  $\zeta_2$ , but with a nontrivial dependence

$$\zeta_3 = O(\sqrt{\zeta_2 / |\ln \zeta_2|}). \quad (44)$$

##### A. Leading-order solution

As in the case  $\zeta_h = 0$  discussed above, substituting  $\zeta_h = 2$  into the zero-mode equation (21) yields an equation with coefficients which are independent of the variable  $\phi$ . This equation reads

$$[\rho^2(1-\rho^2)^2 \partial_\rho^2 + \rho(1-\rho^2)(1-2\rho^2 + \zeta_3 \rho^2) \partial_\rho + (1-\rho^2) \partial_\phi^2 + w(\rho)] f(\rho, \phi) = 0, \quad (45)$$

$$w(\rho) = \rho^2 \frac{\zeta_3}{2d} \left\{ (d+2)(d-1) + \left( \frac{\zeta_3}{2} - 1 \right) \times [(d-1)(\rho^2+1) + \rho^2 - 1] \right\}. \quad (46)$$

This significant simplification is a consequence of the higher symmetry of the passive scalar equation in this limit, as discussed in detail in Refs. [7,8].

Making use of the symmetry, we look for solutions of the form

$$f(\rho, \phi) = f_m(\rho) \cos m \phi, \quad m = 3n, \quad n = 0, 1, 2, \dots \quad (47)$$

The functions  $f_m$  satisfy the ODEs

$$[\rho^2(1-\rho^2)^2\partial_\rho^2+\rho(1-\rho^2)(1-2\rho^2+\zeta_3\rho^2)\partial_\rho -m^2(1-\rho^2)+w(\rho)]f_m(\rho)=0. \quad (48)$$

Equation (48) has regular singularities at the points 0 and 1. In close analogy with the case  $\zeta_h=0$  we find that the Frobenius exponents at 0 are  $\pm m$ , and choose the solution which behaves as  $\rho^m$  at 0, thus providing the boundary condition at 0.

The behavior near the boundary  $\rho=1$  is different. In contrast with the case  $0\leq\zeta_h<2$  where one of the Frobenius exponents is always 0, when  $\zeta_h=2$  the Frobenius exponents depend on the value of  $\zeta_3$ , and generally neither is 0. Thus the qualitative behavior of the zero modes for  $0<\zeta_2=2-\zeta_h\leq 1$  near  $\rho=1$  is different from that of the solutions of Eq. (46) where  $\zeta_h$  is set equal to 2. In other words, we expect the existence of a boundary layer near  $\rho=1$ ; Eq. (46) describes well only the behavior of the *outer* solution, namely, the leading-order approximation away from the boundary layer.

The outer solution may now be written explicitly. Using standard transformations [12] Eq. (48) may be reduced to a hypergeometric equation, whose solution which obeys the  $\rho=0$  boundary condition is

$$f_m(\rho)=\rho^m {}_2F_1(m/2,(m+1)/2;m+1;\rho^2). \quad (49)$$

Since we anticipate that  $\zeta_3\leq 1$ , we give in Eq. (49) the solution of Eq. (48) taking  $\zeta_3=0$ . This approximation is justified to leading order *a posteriori*, when we find that  $\zeta_3$  is indeed small. The outer solution is constructed from the  $f_m$ 's by

$$f^{\text{out}}(\rho,\phi)=\sum_{m=0,3,6,\dots} \nu_m f_m(\rho)\cos m\phi. \quad (50)$$

The coefficients  $\nu_m$  are unknown at this stage and they will be determined by matching with the inner solution.

The outer solution is a valid asymptotic approximation of the true solution only when  $\zeta_2\ll 1-\rho$ . We expect the solution to vary very rapidly within a boundary layer near  $\rho=1$ . The physical reason for this is that when  $\zeta_h=2$  the shearing action of the velocity field on any configuration of three points is area preserving. Thus any collinear configuration is ‘‘stuck’’ for infinitely long times as long as the molecular diffusivity is neglected. When  $\zeta_h\rightarrow 2$  this property affects strongly configurations that are almost collinear. Therefore the inner solution, valid within the boundary layer, should be expressed in terms of a ‘‘fast’’ variable  $\tau$  that changes on a scale inversely proportional to  $\zeta_2$ , namely,

$$\rho=1+\zeta_2\tau. \quad (51)$$

We therefore change the variable  $\rho$  to  $\tau$  in the differential equation (21) and keep the leading terms in  $\zeta_2$ . The resulting equation for the inner solution is

$$\left[\tau\left(\tau-\frac{d+1}{2d}p(\phi)\right)\partial_\tau^2+\left(\frac{(d+3)(d-1)}{4d}p(\phi)-\frac{1}{2}\tau\right)\times\partial_\tau+w(1)/4\right]f^{\text{in}}(\tau,\phi)=0, \quad (52)$$

where

$$p(\phi)=\sum_{\phi'=\phi,\phi\pm 2\pi/3} \ln(1-\cos\phi')(\cos\phi'-\cos 2\phi'), \quad (53)$$

and the function  $w$  is defined in Eq. (46). It is crucial for the matching between inner and outer solution to realize that Eq. (52) is not valid near the fusion points  $\phi=2\pi n/3$ . The significance of this fact is explained below.

Equation (52) has a regular singularity at  $\tau=0$  which corresponds to the  $\rho=1$  boundary, and as in the other cases demanding regularity gives a boundary condition at this point. The equation can be again transformed into the hypergeometric equations, so it has solutions of the form

$$f^{\text{in}}(\tau,\phi)=\mu(\phi) {}_2F_1\left(-\frac{(d-1)\zeta_3}{4},-\frac{1}{2};-\frac{(d+3)(d-1)}{2(d+1)};\frac{2d}{d+1}\frac{\tau}{p(\phi)}\right). \quad (54)$$

Here again we used the smallness of  $\zeta_3$  and neglected terms of higher order in it. The function  $f^{\text{in}}$  provides an asymptotic approximation to the actual solution when  $-\tau\ll(1/\zeta_2)$ . It depends on the function  $\mu(\phi)$  that will be determined by matching to the outer solution.

### B. Asymptotic matching

The next step involves matching of the two approximations  $f^{\text{in}}$  and  $f^{\text{out}}$  in their common region of validity

$$\zeta_2\ll 1-\rho\leq 1. \quad (55)$$

We perform the matching using standard boundary layer methods by examining the asymptotic behavior of  $f^{\text{in}}$  and  $f^{\text{out}}$  in the matching region (55) and balancing coefficients.

We make use of the asymptotic behavior of the hypergeometric functions [12]. For the outer solution we need the asymptotics of the hypergeometric function with argument close to 1, yielding

$$f^{\text{out}}(\rho,\phi)\sim\sum_{m=0}^{1-\rho\ll 1} 2^m \nu_m [1-m\sqrt{2(1-\rho)}]\cos(m\phi). \quad (56)$$

For the inner solution we use the behavior of the hypergeometric function for large negative values, and obtain, using the relation between  $\rho$  and  $\tau$ ,

$$f^{\text{in}}(\rho,\phi)\sim_{\zeta_2\ll 1-\rho} \mu(\phi)\left[1+c\zeta_3\sqrt{\frac{1-\rho}{\zeta_2 p(\phi)}}\right], \quad (57)$$

where

$$c = \frac{\sqrt{\pi}\Gamma\left(\frac{(d+3)(d-1)}{2(d+1)}\right)}{\Gamma\left(\frac{(d+3)(d-1)}{2(d+1)} + \frac{1}{2}\right)} \frac{d-1}{4} \sqrt{\frac{d+1}{2d}}. \quad (58)$$

To proceed with the matching of Eqs. (56) and (57), we first match the  $O(1)$  terms [for small  $(1-\rho)$ ], giving

$$\mu(\phi) = \sum_m 2^m \nu_m \cos(m\phi), \quad (59)$$

i.e.,  $2^m \nu_m$  are just the coefficients of the Fourier series of  $\mu(\phi)$ . Next we have to match the  $O(\sqrt{1-\rho})$  terms giving

$$\frac{c\zeta_3}{\sqrt{\zeta_2}p(\phi)} \mu(\phi) = \sum_m -m\sqrt{2}2^m \nu_m \cos(m\phi). \quad (60)$$

We expand  $p(\phi)^{-1/2}$  in Fourier series,

$$p(\phi)^{-1/2} = \sum_m p_m \cos(m\phi) \quad (61)$$

and substitute in Eq. (60) to get, using Eq. (59),

$$\begin{aligned} \sum_{m,n} \frac{c\zeta_3}{2\sqrt{\zeta_2}} p_n \nu_m \{\cos[(n-m)\phi] + \cos[(n+m)\phi]\} \\ = \sum_m -m\sqrt{2} \nu_m \cos(m\phi). \end{aligned} \quad (62)$$

Equating Fourier coefficients of the same order yields finally

$$\sum_n \frac{c\zeta_3}{2\sqrt{\zeta_2}} p_n (\nu_{m+n} + \nu_{|m-n|}) = -m\nu_m. \quad (63)$$

The matching condition (63) is a generalized eigenproblem for the infinite vector  $\nu_m$ , with eigenvalues  $\zeta_3$ .

Since  $\zeta_3$  appears in Eq. (63) only through the combination  $\zeta_3/\sqrt{\zeta_2}$ , one would be led to conclude that  $\zeta_3 = O(\sqrt{\zeta_2})$  for all nontrivial solutions of Eq. (63) with different numerical coefficients. This consideration is modified, however, since the coefficients  $p_n$  themselves also depend on  $\zeta_2$  as will now be demonstrated.

It follows from the definition of  $p(\phi)$ , Eq. (53), that

$$p(\phi) = O[\phi^2 \ln(\phi)] \quad (64)$$

for  $\phi$  small. Note that the Fourier coefficients  $p_n$  are written as integrals

$$p_n = \frac{1}{\pi} \int_0^{2\pi} \frac{d\phi}{\sqrt{p(\phi)}} \quad \text{for } n > 1. \quad (65)$$

All these integrals diverge at  $\phi=0$ , which is precisely the fusion point where the approximation (54) ceases to be valid, requiring a more careful examination of the behavior in this region. In deriving Eq. (54) we made the approximation

$$\begin{aligned} (1-\rho \cos\phi)^{-\zeta_2/2} &\sim 1 - \frac{1}{2}\zeta_2 \ln(1-\rho \cos\phi) \\ &\sim 1 - \frac{1}{2}\zeta_2 \ln(1-\cos\phi), \end{aligned} \quad (66)$$

for  $1-\rho = O(\zeta_2)$ . When  $\phi^2 = O(\zeta_2)$  the second approximation is no longer valid, and one has instead

$$(1-\rho \cos\phi)^{-\zeta_2/2} \sim 1 - \frac{1}{2}\zeta_2 \ln[(1-\rho) + \frac{1}{2}\phi^2], \quad (67)$$

so that instead of Eq. (52) we get a similar equation in which  $p(\phi)$  is replaced by  $\ln(-\zeta_2\tau + \frac{1}{2}\phi^2)$ . The resulting equation is no longer integrable in terms of hypergeometric functions. However, we still expect the asymptotic approximation (57) to be valid, but for small  $\phi$  the function  $p(\phi)$  is no longer given by Eq. (53). We may estimate  $p(\phi)$  for small  $\phi$  by the following consideration. Examining Eq. (52) we see that  $p(\phi)$  is the value of  $\tau$  where the coefficient of the second derivative crosses over from quadratic behavior to linear behavior, and the coefficient of the first derivative crosses over from linear to constant behavior. When  $\phi^2 = O(\zeta_2)$  there is also a crossover, but to a linear or constant function times a logarithmic function of  $\tau$ . The logarithmic function changes very slowly, and may be approximated roughly by a constant. The crossover occurs when

$$|\tau| \sim \zeta_2 \ln(\zeta_2 |\tau| + \phi^2) \sim \zeta_2 \ln(\zeta_2). \quad (68)$$

We are thus led to make the approximation

$$p(\phi) \sim \zeta_2 \ln \zeta_2 \quad \text{for } \phi^2 = O(\zeta_2). \quad (69)$$

This estimate implies that when calculating the Fourier coefficients in Eq. (65) the integral should be cut off at  $\phi \sim \sqrt{\zeta_2}$ , giving the coefficients a  $\sqrt{|\ln \zeta_2|}$  dependence on  $\zeta_2$ . It is now possible to balance powers in the matching Eq. (63), obtaining the order of magnitude relation (44).

## V. SOLUTIONS FOR GENERAL VALUES OF $\zeta_h$

For general values of  $\zeta_h$  and  $d$  the differential equation (21), which has variable coefficients, is not accessible to analytic techniques. In this section we present numerical solutions of the scaling exponents  $\zeta_3$  for arbitrary values of  $\zeta_h$ , using a discretized version of the operator  $\hat{B}_3$ . Since the differential problem is a linear homogeneous equation with linear homogeneous boundary conditions, the discretized problem is also a homogeneous linear equation, implying that nontrivial solutions exist only when the determinant of the discretized operator vanishes. This determinant depends parametrically on  $\zeta_3$ . Since the differential operator is defined on a compact domain we expect the determinant to vanish only for discrete values of  $\zeta_3$  for any given values of  $\zeta_h$  and the dimensionality  $d$ . One solution is known to exist always, a constant zero mode associated with  $\zeta_3=0$ . Our aim is to find the lowest lying positive real solutions  $\zeta_3$  for which the determinant vanishes.

The discretization of the operator  $\hat{B}_3$  was carried out as follows: We defined a nine-point finite difference scheme for the second-order equation (21). The discretization of the boundary conditions at  $\rho=1$ , Eq. (26), is achieved using the same scheme, which requires in this case only three bound-



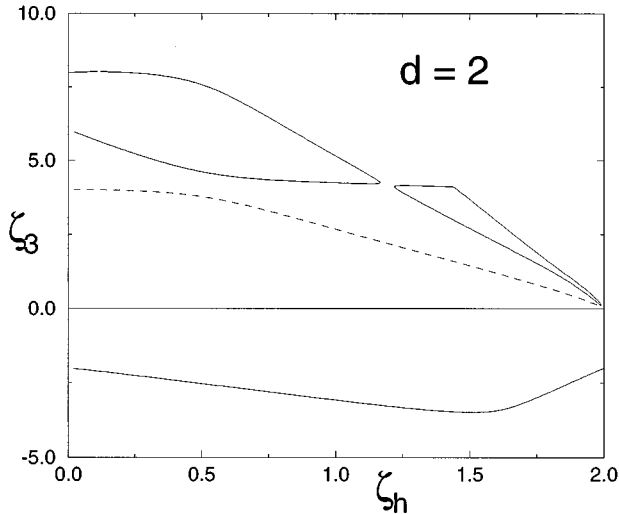


FIG. 1. The scaling exponent  $\zeta_3$  as a function of  $\zeta_h$  found as the loci of zeros of the determinant of the matrix, for  $d=2$ .

any points and one interior point, since on the boundary the radial derivative appears in first order. Using the symmetry of the problem we restricted the domain to one-sixth of the circle,  $0 < \phi < \pi/3$ . The symmetry implies that the original problem on the full circle is equivalent to the problem on the reduced domain with simple Neumann boundary conditions  $\partial_\phi f(\rho, \phi) = 0$ . On the new boundary lines  $\phi = 0, \pi/3$ . As explained above, the discretized problem is a matrix eigenvalue problem  $B_3 \Psi = 0$ , where  $B_3$  is a large sparse matrix, whose rank depends on the mesh of the discretization, and  $\Psi$  is the discretized version of the zero mode  $f$ . We used NAGs (NAG is a numerical analysis package) sparse Gaussian elimination routines to find the zeros of  $\det(B_3)$ , and determined the values of  $\zeta_3$  for these zeros as a function of  $\zeta_h$ . The results of this procedure for space dimensions  $d=2,3,4$  are presented in Figs. 1, 2, and 3. The zero modes that correspond to any given value of  $\zeta_3$  can be found straightforwardly by inverse iterations of the matrix  $B_3$  with an arbitrary initial vector.

### A. Results

The various branches shown in Figs. 1–3 can be organized on the basis of the perturbation theory near  $\zeta_h = 0$  which was presented in Sec. II. At  $\zeta_h = 0$  we identify the actual starting points of the branches with the analytic solutions for the lowest lying positive values of  $\zeta_3$ , which are 4, 6, 8, etc. In addition, for  $d=2$  we observe the highest negative value which is  $-2$ . Measuring the slopes of the various branches at  $\zeta_h = 0$  we find full agreement with the perturbative predictions.

We see that in all dimensions the branch which begins at  $\zeta_h = 0, \zeta_3 = 4$  continues in the nonperturbative region without crossing any other branch until it ends at  $\zeta_h = 2, \zeta_3 = 0$ . This branch is a continuation of the lowest lying positive branch predicted by the perturbation theory. The negative branch (shown only for  $d=2$ ) never rises above its perturbative limit and is not relevant for the scaling behavior at any value of  $\zeta_h$ . Note also that the point  $\zeta_h = 2, \zeta_3 = 0$  is an accumulation point of many branches, and we display only a

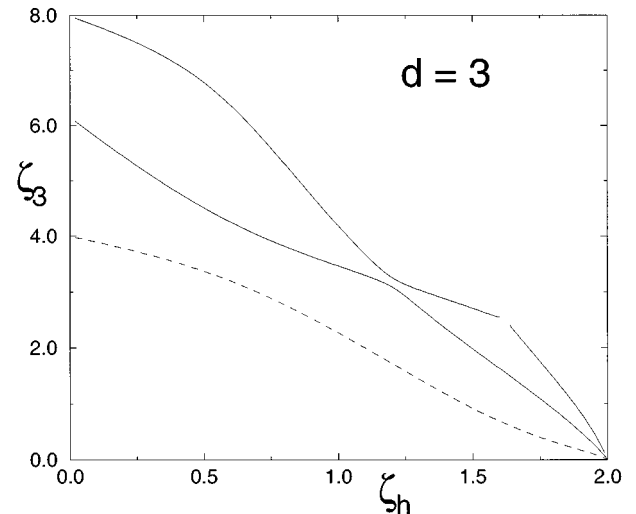


FIG. 2. Same as Fig. 1, but for  $d=3$ .

part of the actual spectrum near this point. These branches seem to approach the accumulation point with a slope that grows without limit. This finding is in agreement with the analytic result of the perturbation calculation presented in Sec. IV.

## VI. CONCLUSIONS

It is well known by now that there exists a disagreement between the scaling exponents  $\zeta_4$  and the higher-order exponents  $\zeta_n$  computed via perturbative approaches and the predictions of another approach based on the fully fused structure functions. The latter approach seems to be consistent with the results of numerical simulations in two spatial dimensions. The main conclusion of the present paper is that this disagreement cannot be ascribed to a formal failure of the perturbation theory. If we accept the statement that the scalar diffusivity is irrelevant, and compare the predictions of perturbation theory at both ends of the range of the allowed values of the parameter  $\zeta_h$ , we find that they are in excellent agreement with the nonperturbative calculation of

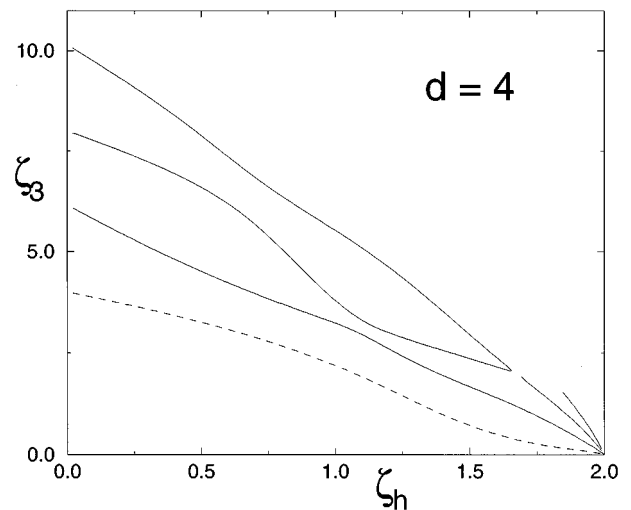


FIG. 3. Same as Fig. 1, but for  $d=4$ .

the scaling exponent, again subject to the assumption that the diffusivity is irrelevant. Therefore, if we want to understand the discrepancy between the two approaches mentioned above, there are a few possibilities that have to be sorted out by further research.

(i) The crucial assumption that goes to the fully fused approach, which is the linearity of the conditional average of the Laplacian of the scalar, is wrong.

(ii) The computation of the zero modes which is achieved by discarding the viscous terms in  $\hat{B}_n$  is irrelevant for the physical solution. It is not impossible that the limits  $\zeta_h \rightarrow 0$  and  $\kappa \rightarrow 0$  do not commute, giving rise to some wicked properties of the very small  $\zeta_h$  regime. That this is a possibility is underlined by recent calculations of a shell model of the Kraichnan model [13], in which it was shown that the addition of any minute diffusivity changes the nature of the zero modes qualitatively.

(iii) Lastly, and maybe most interestingly, it is possible that the physical solution is not strictly scale invariant through all the range of allowed distances [14]. In other words, it is possible that  $\mathcal{F}_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  is not a homogeneous function with a fixed homogeneity exponent  $\zeta_3$ , but rather

(for example), that  $\zeta_3$  depends on the ratios of the separations (or, in other words, the geometry of the triangle defined by the coordinates). If this were also the case for even correlation functions  $\mathcal{F}_{2n}$ , this would open an exciting route for further research to understand how non-scale-invariant correlation functions turn, upon fusion, to scale invariant structure functions.

In light of the numerical results of Ref. [15] and the experimental results displayed in [16,17] we tend to doubt option (i). If we were to guess at this point we would opt for possibility (ii). More work, however, is needed to clarify this important issue beyond doubt.

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#### APPENDIX: THE COEFFICIENTS

$$\begin{array}{l}
 \{0, 0, 1, -1, 1, 8\} \\
 \{0, 0, 2, -1, 1, -4\} \\
 \{0, 1, 0, 0, 1, -12\} \\
 \{0, 2, 0, 0, 0, -2\} \\
 o_1: \{0, 2, 0, 2, 0, 2\} \\
 \{1, 0, 1, -1, 1, -8\} \\
 \{1, 1, 0, 2, 1, -8\} \\
 \{2, 0, 0, 1, 1, 4\} \\
 \{0, 0, 1, -1, 1, 8\} \\
 \{0, 0, 2, -2, 0, -2\} \\
 \{0, 1, 0, -1, 0, -2\} \\
 \{0, 1, 0, 2, 1, 8\} \\
 \{0, 2, 0, 1, 1, -4\} \\
 \{0, 2, 0, 3, 1, 4\} \\
 \{1, 1, 0, 0, 1, 8\} \\
 \{2, 0, 0, 0, 0, -2\}
 \end{array} \tag{A1}$$

$$\begin{array}{l}
 \{0, 0, 1, -1, 1, -4\} \\
 o_2: \{0, 1, 0, 1, 0, 2\} \\
 \{0, 1, 0, 0, 1, 4\} \\
 \{1, 0, 0, 0, 0, -2\}
 \end{array} \tag{A2}$$

$$\begin{array}{l}
 o_3: \{0, 0, 1, 0, 2, -4\} \\
 \{0, 0, 2, 0, 0, 1\} \\
 \{0, 0, 2, 0, 2, 2\} \\
 \{0, 1, 0, -1, 0, -1\} \\
 \{0, 1, 0, 3, 0, -2\} \\
 \{0, 1, 0, 2, 1, 2\} \\
 \{0, 1, 0, 1, 2, 2\} \\
 \{0, 1, 1, 2, 1, -2\} \\
 \{0, 1, 1, 1, 2, 4\} \\
 \{0, 2, 0, 2, 0, 2\} \\
 \{0, 2, 0, 1, 1, -2\} \\
 \{0, 2, 0, 0, 2, 2\} \\
 \{1, 0, 1, -1, 1, -2\} \\
 \{1, 1, 0, 1, 0, -2\} \\
 \{1, 1, 0, 0, 1, 2\} \\
 \{2, 0, 0, 0, 0, 1\} \\
 \{0, 0, 1, -2, 2, 4\} \\
 \{0, 0, 2, -2, 0, -1\} \\
 \{0, 0, 2, -2, 2, -2\} \\
 \{0, 1, 0, 1, 0, 3\} \\
 \{0, 1, 0, 0, 1, -2\} \\
 \{0, 1, 0, -1, 2, -2\} \\
 \{0, 1, 1, 0, 1, 2\} \\
 \{0, 1, 1, -1, 2, -4\} \\
 \{0, 2, 0, 0, 0, -1\} \\
 \{0, 2, 0, 4, 0, -1\} \\
 \{0, 2, 0, 3, 1, 2\} \\
 \{0, 2, 0, 2, 2, -2\} \\
 \{1, 0, 1, 1, 1, 2\} \\
 \{1, 1, 0, 3, 0, 2\} \\
 \{1, 1, 0, 2, 1, -2\} \\
 \{2, 0, 0, 2, 0, -1\}.
 \end{array} \tag{A3}$$

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