Stochastic ratchets with colored thermal noise

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We study thermal ratchet systems, i.e., particles moving in asymmetric periodic potentials using a generalized Langevin equation. This scheme allows for a clear distinction of thermal noise, whether ''white'' or ''colored,'' and time-dependent external fields, deterministic or stochastic. It can then be verified that, as a consequence of the fluctuation-dissipation theorem, the ratchet does not drift if it is in interaction *only* with a thermal bath. That is, we show that a net current arises only if the forcing is done by an *external* source. Hence we find that the only necessary condition for rectifying an external field, producing a current, is the asymmetry of the potential. The use of the generalized Langevin equation gives access to a wider variation of the quantities involved; for instance, we find that an inverted current arises for external fields correlated in shorter time scales than the thermal noise. [S1063-651X(97)09110-1]

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Unexpected transport properties of the so-called stochastic ratchets, namely, dissipative systems of particles moving in asymmetric periodic potentials in the presence of thermal and external noise, have been the focus of attention of recent studies [1–8]. In addition to their potential application in the understanding of biological or molecular machines [5,6], one of the most intriguing aspects of stochastic ratchets is their ability to ''rectify'' symmetric *correlated* noise and thus being able to produce a net current. This result was pointed out by Magnasco [1] and further verified by others [2,3], using a one-dimensional ratchet system in the presence of external *colored* noise, in the overdamped (Smoluchowski) limit. That study was inspired by the classical work by Feynman [9], where the ratchet and pawl system is used to exemplify the second law of thermodynamics: Feynman shows that a system executing Brownian motion in contact with a thermal reservoir cannot produce work.

In this paper we verify that, as a consequence of the fluctuation-dissipation theorem [10], a ratchet in the presence of thermal noise *only* does not show any current, whether the fluctuations of the bath are time correlated or uncorrelated. Thus the net current or drift of the particle must be produced by the presence of an *external* source. In particular, we shall show that *any* external forcing may be used to produce the drift. That is, we shall argue that the onset of a net current is only due to the asymmetry of the potential and that it is not necessary that the external forcing be correlated in time.

In previous studies [1–7], the description of the dynamics of the system was done under the assumption of a large separation of time scales, those of the ratchet (e.g., a motor protein), and of the thermal bath. The latter approximation, however, allows for considering ''white'' thermal noise only and therefore any time correlation of the forcing must come from an external source. Here, by describing the ratchet system in the Brownian regime by means of a generalized Langevin equation, one is able to consider thermal noise with finite correlation times and thus it can be shown that the production of the current is due to a time-dependent external force.

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The *generalized* Langevin equation of a particle of mass m moving in an asymmetric periodic potential $V(x)$ (see Fig. 1) is

$$m\frac{d^2x}{dt^2}=-\int_{-\infty}^{t}d\tau\,\Gamma(t-\tau)\frac{dx(\tau)}{d\tau}-\frac{d}{dx}V(x)+f(t)+F_{ext}(t), \tag{1}$$

where $\Gamma(t)$ is the *memory* friction kernel and $f(t)$ is the stochastic fluctuating thermal force exerted by the bath. $F_{ext}(t)$ is a time-dependent *external* force. It is important to stress that the properties of $F_{ext}(t)$, whether stochastic or deterministic, correspond to a *given* external process and they are completely independent of the *internal* degrees of freedom of the system-bath composite. The thermal force $f(t)$ has the usual stochastic properties of being gaussian with zero mean $\overline{f(t)}=0$ and its second moment is related to the memory friction kernel by the fluctuation-dissipation expression [10]

$$\overline{f(t)f(t')}=2kT\Gamma(t-t') \quad \text{for} \quad t>t', \tag{2}$$

where k is Boltzmann constant and T the temperature of the bath. This theorem ensures proper equilibration of the overall system-bath composite when $F_{ext}(t)=0$. As is well known,

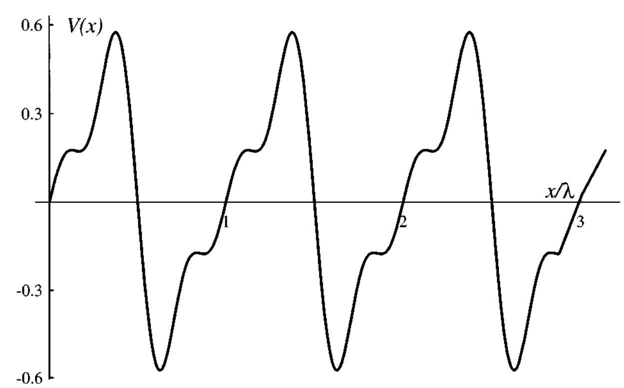

FIG. 1. Ratchet potential. See Eq. (6).

one recovers the usual Langevin equation with white noise when the memory kernel becomes δ correlated, that is, when

$$\Gamma(t)=\gamma\delta(t), \tag{3}$$

with γ the friction coefficient. It is important to recall that the use of a dissipation kernel $\Gamma(t)$ with or without memory depends on the time scales of evolution of what one considers to be the "system" and the "bath." In reality, of course, a thermal bath is always colored. However, if the time scales of evolution are such that one is able to approximate the bath as being white, in comparison with the time scale of the system, the description becomes Markovian. This property allows for a more complete mathematical description of the dynamics of the system than if the finite time correlations of the bath are kept. But certainly, if the time scales of the system and the bath are comparable, then one must face the non-Markovian character of the corresponding dynamical equations. One of the most serious difficulties due to the lack of Markovian character is that the whole hierarchy of multiple-time probability distributions cannot be constructed from the knowledge of the two-time conditional probability distribution [11]. Moreover, in general, one cannot write down a (generalized) Fokker-Planck equation for this distribution [12]. The latter result has consequences in the present study.

As we shall show below, by a numerical solution of Eq. (1), such an equation does not show a net current in the abscence of external forces, i.e., if $F_{ext}(t)=0$ in Eq. (1). Nevertheless, and this was truly unexpected [1], if one adds a *time-symmetric* external force, the system then shows a net current, in general. A different result here is the fact that the correlation time of the external force can even be shorter than that of the thermal bath in order to produce a current.

For comparison purposes we shall use as external forces a systematic deterministic oscillatory force, such as

$$F_{ext}(t)=F_0\cos(\omega_0 t), \tag{4}$$

with F_0 and ω_0 arbitrary and constant. In the other extreme, we shall consider $F_{ext}(t)$ to be a stochastic force and time symmetric as well.

As already mentioned, in the descriptions given in, e.g., Refs. [1–3], it is assumed that the motion takes place in the overdamped (Smoluchowski) regime and the memory of the bath cannot be taken into account. The dynamical equation then reads

$$\gamma\frac{dx}{dt}=-\frac{d}{dx}V(x)+\tilde{f}(t), \tag{5}$$

where we have written $\tilde{f}(t)$ to make the following points. As such, this equation represents the interaction of a particle with a thermal bath only if $\tilde{f}(t)$ is δ correlated. In other words, only a δ-correlated $\tilde{f}(t)$ obeys the fluctuation-dissipation theorem; see Eqs. (2) and (3). If $\tilde{f}(t)$ shows time correlations with correlation time different from zero, then these are produced necessarily by an external source. It has been argued (see, e.g., Ref. [3]) that these correlations are actually *nonequilibrium fluctuations*. This may indeed be the case, but nonetheless those fluctuations are still of an external origin. That is, an isolated system such as the overall system-bath composite, must reach a thermal equilibrium state in the absence of external forcing (such as a time-dependent field or a temperature gradient [8]) and there can be no (stationary) currents. This is in fact the situation for the stochastic process $x(t)$ defined by Eq. (5), with symmetric *correlated* force $\tilde{f}(t)$, and it is precisely in such a case that it has been found [1–4] that the system reaches a stationary state with a net current.

We now proceed to exemplify the above results via numerical solutions of the generalized Langevin equation (1). In dimensionless units ($m=1$ and $\lambda=1$, the mass of the particle and the period of the potential), the ratchet potential is a simple one [3] (see Fig. 1)

$$V(x)=\frac{V_0}{2\pi}\left(\sin(2\pi x)-\frac{1}{2}\sin(4\pi x)+\frac{1}{3}\sin(6\pi x)\right). \tag{6}$$

The thermal stochastic force is taken to be Gaussian, with zero mean and with the correlation function

$$\overline{f(t)f(t')}=A^2\frac{1}{2\tau_0}e^{-|t-t'|/\tau_0}, \tag{7}$$

where τ_0 is the bath correlation time. One recovers the δ-correlated situation in the limit $\tau_0\to 0$. The memory kernel $\Gamma(t)$ is defined through the fluctuation-dissipation relation (2). The solution to the generalized Langevin equation is dependent on several parameters: V_0, the amplitude of the ratchet potential; A, the amplitude of the stochastic force; τ_0, the correlation time, or memory time, of the bath; and T, the temperature that enters in the fluctuation-dissipation relation given by Eq. (2). If in addition we consider an external field $F_{ext}(t)$, as in Eq. (4), we have two more free parameters F_0 and ω_0; if the external field is stochastic, we must specify at least its strength and its correlation time. We have kept $V_0=2.5$ throughout and vary the rest of the parameters.

The goal here is to find out if the system shows a stationary current different from zero in a given situation defined by the parameters described above. It turns out, however, that in the generalized non-Markovian description it is not clear which is the function that measures the current, such that it reduces to the probability density current in the Markovian limit [13],

$$j(x,t)=-\frac{kT}{\gamma}\partial_x\rho(x,t)+\frac{1}{\gamma}[F_{ext}(t)-\partial_x V(x)]\rho(x,t), \tag{8}$$

with $F_{ext}(t)$ slowly varying. The latter identification follows from the conservation law for the probability density $\rho(x,t)$. This law can be deduced from the corresponding Fokker-Planck (or Smoluchowski) equation [13]. Now, since a "generalized" Fokker-Planck equation is not known [12] (due to the non-Markovian character introduced by the memory term) one cannot write down the conservation law for the generalized probability distribution $\rho(x,t)$. Thus, in order to find out whether or not a current exists, we shall mainly look for the behavior of the average position of the particle as a function of time $\overline{x(t)}$. This average is calculated over many realizations of the stochastic thermal force, and of

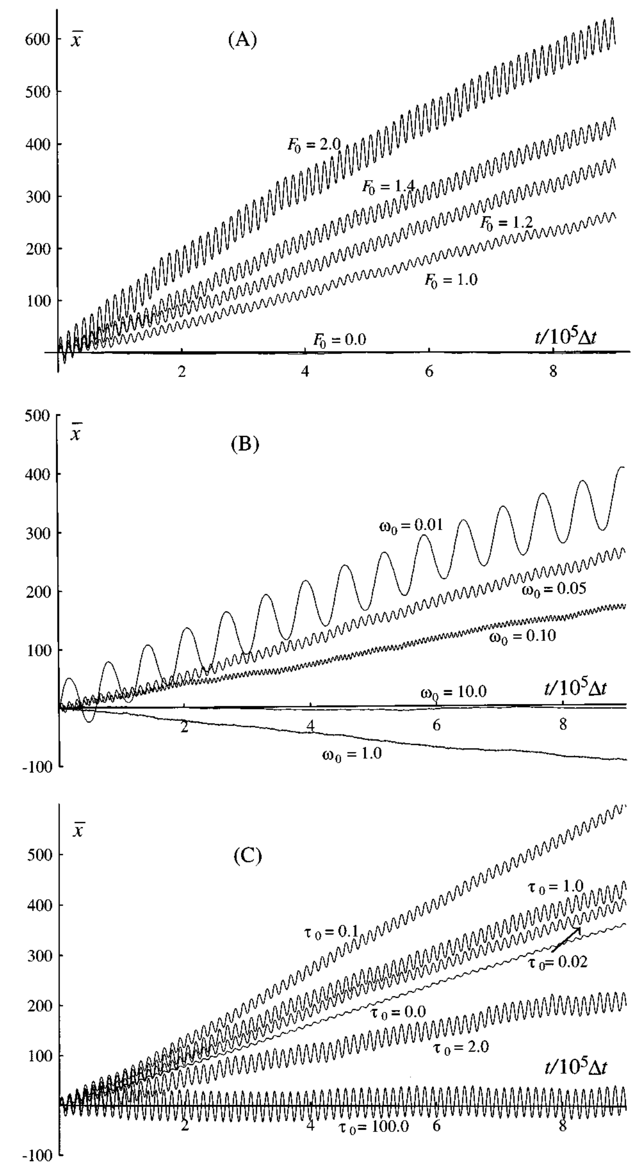

FIG. 2. Average position $\overline{x(t)}$ vs time (a) as a function of the strength F_0 of the external force for $\tau_0 = 1.0$ and $\omega_0 = 0.05$, Eq. (4); (b) as a function of the frequency ω_0 of the external force for $\tau_0 = 1.0$ and $F_0 = 1.0$, Eq. (4); and (c) as a function of the correlation time τ_0 of the fluctuations of the bath for $\omega_0 = 0.05$ and $F_0 = 1.4$, Eq. (7).

the external noise if the latter is also stochastic. If $\overline{x(t)} = 0$, there is no current. If $\overline{x(t)} \neq 0$, the particle is drifting. Every situation described below was obtained with 100 realizations of the stochatic force $f(t)$ and for runs of 900 000 time steps ($\Delta t = 0.01$) [14].

In Figs. 2(a)–2(c), the average position $\overline{x(t)}$ of a Brownian particle in a ratchet potential in the presence of a given periodic sinusoidal force is shown; cf. Eq. (4). The different curves correspond to different strengths of the external force F_0 [Fig. 2(a)], to different frequencies ω_0 of the external force [Fig. 2(b)], and to different values of the bath correlation time, or memory, τ_0 [Fig. 2(c)]. We can see that, in general, there is a net drift for almost all cases.

We want to highlight the following points. First, in Fig. 2(a), the case of *no external force* is also shown, i.e., $F_0 = 0.0$, and one finds that there is no net current. We stress that the fluctuations of the bath are *time correlated*, i.e., $\tau_0 \neq 0$. In accordance with the second law, one cannot extract a current from a thermal bath, whether white or colored.

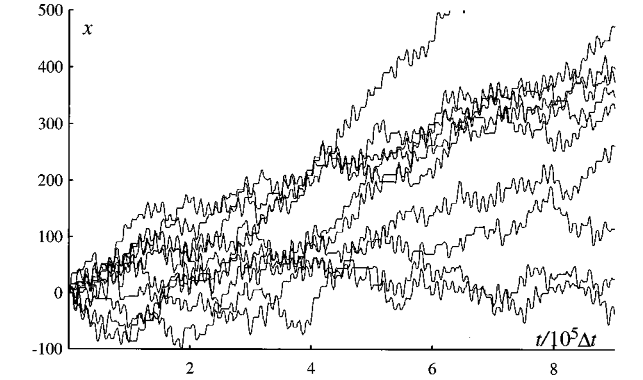

FIG. 3. Position $x(t)$ vs time, for ten different realizations of the stochastic thermal force $f(t)$, in the presence of an external force $F_{ext}(t)$, Eq. (9). $\tau_0 = 1.0$, $F_0 = 1.0$, and $\omega_0 = 0.05$.

Next, we direct the reader's attention [see Fig. 2(b)] to the striking oscillacions of the average position $\overline{x(t)}$. The frequency of this oscillation is, within statistical errors, the frequency ω_0 of the external force. For comparison, in Fig. 3, we show the actual ''walker'' $x(t)$ of ten runs, corresponding to different realizations of the thermal noise for given τ_0, F_0, and ω_0. One can hardly expect from Fig. 3, given the fact that we are facing a highly nonlinear system, that the *average* $\overline{x(t)}$ will show such a clean filtering of the forcing frequency. Note also the inversion of the current in Fig. 2(b) for a large driving frequency.

Finally, in Fig. 2(c), it is important to note the nonmonotonic behavior of the current as a function of the memory time τ_0 of the bath, for fixed external force. The case $\tau_0 = 0.0$ corresponds to a white bath. Note that when the bath has a very long correlation time ($\tau_0 = 100.0$) it becomes very inefficient in driving the particle.

We now discuss the case in which the time-dependent external force $F_{ext}(t)$ is stochastic. We take it to be Gaussian, with zero mean $\overline{F_{ext}(t)} = 0$ and the correlation function

$$\overline{F_{ext}(t)F_{ext}(t')} = F_0^2 \frac{1}{2\tau_{ext}} e^{-|t-t'|/\tau_{ext}}. \tag{9}$$

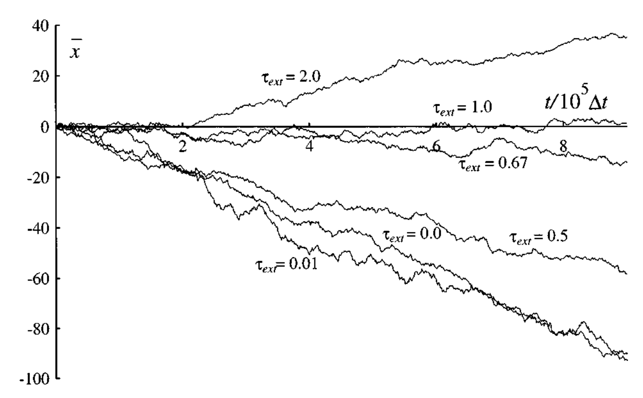

FIG. 4. Average position $\overline{x(t)}$ vs time, as a function of the correlation time τ_{ext} of a stochastic external force $F_{ext}(t)$, Eq. (9). $\tau_0 = 1.0$ and $F_0 = 10.0$.

The results are shown in Fig. 4. The different curves correspond to different correlation times τ_{ext} of the external force. Three main conclusions can be drawn from here. First, if the correlation time of the external force is equal to the correlation time of the bath $\tau_{ext}=\tau_0$, there is no drift (as it should because the ratchet comes into contact with a single colored noise). That is, the external force plus the thermal force is equivalent to increasing the intensity of the thermal noise, as can be seen from Eqs. (2), (7), and (9) (completely analogous to the case of a white bath with a δ-correlated external force, as shown in [1,3]). Second, for external correlation times longer than the thermal correlation time $\tau_{ext}>\tau_0$, the situation should approach that of a white bath in the presence of a colored external source: There must be a drift (''positive'' in this case), as it has already been demonstrated [1–3]. Third, for external correlation times *shorter* than the thermal correlation time $\tau_{ext}<\tau_0$, there is also current. It is important to point out that the current in this case is ''negative,'' or ''inverted.'' The origin of this inversion, not elucidated here, is clearly different from the current inversion found in previous studies [2–4,7].

An interesting case is the extreme one of a *colored* thermal bath, $\tau_0\neq 0$, with a *white*, or δ-correlated, external force $\tau_{ext}\rightarrow 0$. We found, as shown in Fig. 4, that the ratchet moves. That is, it is capable of rectifying even a symmetric δ-correlated external noise. This result indicates that the only necessary condition for a thermal system to show current is that the potential $V(x)$ be asymmetric. From the present work we can further conclude that the phenomenon is very robust: Perturb in (almost) any way an otherwise ratchet in thermal equilibrium and it will generate a current. We stress once more that the study of a system with a colored thermal bath, memory, and with time-dependent external forces (even δ correlated) can only be studied with a *generalized* Langevin equation, such as Eq. (1).

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