Mechanism of chaos synchronization and on-off intermittency

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We report a mechanism of synchronization that occurs when the system consisting of the variable differences of two identical chaotic systems generates an infinite laminar phase period that is connected with on-off intermittency. For the synchronization a signal of the master system is fed back to a corresponding slave system variable. The phenomenon is analyzed theoretically in logistic maps and demonstrated experimentally in electronic circuits based on the forced double-well Duffing equation. [S1063-651X(97)11808-6]

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In chaotic systems, when initial points of two identical systems are slightly different, they have different trajectories as time evolves, even though each map has the same attractor in phase space. This phenomenon is due to the sensitivity of the initial conditions [1]. Recently, Pecora and Carroll have suggested a method of synchronization by linking two identical systems with a common signal (or signals) so that the distance between the two corresponding trajectories converges to zero although they have different initial points [2]. After that, some modified or new methods have been developed and experimentally verified [3], since synchronization in chaotic systems has high potentiality of practical applications in secure communication [4], optics [5], and nonlinear dynamics model identification [6].

In estimating synchronization, sub-Lyapunov exponents are generally an important factor, for when synchronization occurs they are always negative [2]. This is a necessary, but not a sufficient, condition. When the parameters of the two chaotic systems are mismatched, synchronization is degraded even though the sub-Lyapunov exponents remain negative. And when the sub-Lyapunov exponents are positive, desynchronization events occur intermittently. It has been found by several authors that this phenomenon is related to on-off intermittency, but not fully analyzed [7].

So in this paper we analyze the mechanism of synchronization by using the system consisting of the variable differences of two identical chaotic systems, which we refer to as *error dynamics* (ED). The ED exhibits that desynchronization events are related to on-off intermittency. The synchronization occurs when the ED generates an infinite period of laminar phase (IPLP), which is connected with on-off intermittency [8]. In this synchronization the structure of the bifurcation diagram of the ED has an important role since the synchronization occurs when the parameters of the ED are forced through a bifurcation point chaotically by the master system signals. For this, we introduce a method of feeding back a signal (or signals) of the master system to the corresponding variable (variables) of the slave system. We analyze the phenomenon theoretically in the logistic maps in detail, and demonstrate it experimentally in electronic circuits based on the forced double-well Duffing equation.

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To show the point of our investigation, we briefly consider two identical chaotic systems:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, y), \quad \dot{y} = G(\mathbf{x}, y) \quad \text{(master system)},$$
$$\dot{\mathbf{x}}' = \mathbf{F}(\mathbf{x}', y'), \quad \dot{y}' = G(\mathbf{x}', y') \quad \text{(slave system)}. \tag{1}$$

In these systems, a signal of the master system $y(t)$ is added to y' with a scaling factor α, so the slave system is

$$\dot{\mathbf{x}}' = \mathbf{F}(\mathbf{x}', y' + \alpha[y(t) - y']),$$
$$\dot{y}' = G(\mathbf{x}', y' + \alpha[y(t) - y']). \tag{2}$$

If we let $\mathbf{x} - \mathbf{x}' = \mathbf{X}$ and $y - y' = Y$, the ED becomes

$$\dot{\mathbf{X}} = \mathbf{F}'(\alpha, \mathbf{x}(t), y(t); \mathbf{X}, Y),$$
$$\dot{Y} = G'(\alpha, \mathbf{x}(t), y(t); \mathbf{X}, Y). \tag{3}$$

Then α, $\mathbf{x}(t)$ and $y(t)$ become parameters of the ED. Here if the parameters of the ED are forced through a bifurcation point by $\mathbf{x}(t)$ and $y(t)$, which are chaotic, the system generates on-off intermittency or IPLP according to α [8]. In the region of α where the system generates IPLP, $\mathbf{X} \to 0$ and $Y \to 0$ as time evolves. Under this condition, the slave system is synchronized with the master system. We here note that if $\alpha = 0$ two identical systems are independent of each other, and that if $\alpha = 1$, the Pecora-Carroll synchronization occurs [3].

Now to analyze our synchronization theoretically we consider, for example, two identical logistic maps with a feedback signal x_n: the master and the slave,

$$x_{n+1} = \lambda x_n(1 - x_n),$$
$$x'_{n+1} = \lambda[x'_n + \alpha(x_n - x'_n)]\{1 - [x'_n + \alpha(x_n - x'_n)]\}, \tag{4}$$

respectively. If we let $x_n - x'_n = y_n$, the ED becomes

$$y_{n+1} = \lambda(1 - \alpha)y_n[(1 - 2x_n) + (1 - \alpha)y_n]. \tag{5}$$

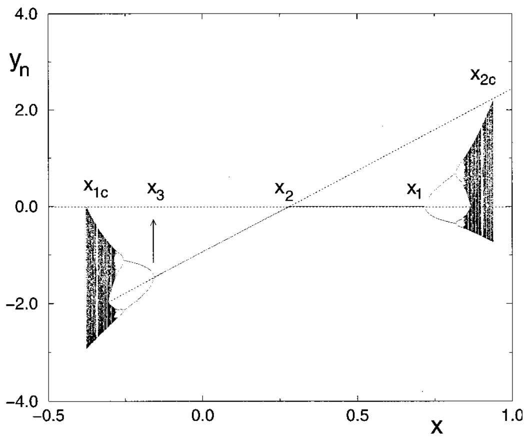

FIG. 1. Bifurcation diagram of the ED of the logistic maps depending on x. It shows a stable orbit, which has $y=0$ in the region from $\alpha=0.2807,\ldots$ to $\alpha=0.7192,\ldots$ and period doubling bifucations as the parameter decreases or increases.

From the ED it is obvious that the parameter of the system is forced chaotically by x_n as in the system generating on-off intermittency [8]. If the ED generates IPLP, the two logistic maps become synchronized.

To obtain the condition of α for synchronization, the bifurcation diagram of the ED according to x is obtained as given in Fig. 1 when $\lambda=3.8$ and $\alpha=0.4$. The figure shows a stable period 1-T orbit between $\frac{1}{2}\pm 1/2\lambda(1-\alpha)$ where $y=0$ and between $\frac{1}{2}-1/2\lambda(1-\alpha)$ and $\frac{1}{2}-3/\lambda(1-\alpha)$ where $y=[1-\lambda(1-\alpha)(1-2x)]/\lambda(1-\alpha)^2$. The points x_1, x_2 are determined by equations $F'(y)=\pm 1$ and $y=F(y)$ and the point x_3 by equation $y=F\circ F(y)$ so that the values of the points are $x_1=1/2+1/4.56$, $x_2=1/2-1/4.56$, and $x_3=1/2-3/4.56$, respectively. The unstable period 1-T orbit, the dotted line, is also determined by equation $y=F(x;y)$. As the parameter increases the stable period 1-T orbit, $y=0$, bifurcates into a period 2-T orbit at x_1 and develops into chaos through the period doubling bifurcation. And as the parameter decreases it bifurcates into another period 1-T orbit at x_2 and this orbit bifurcates again into a period 2-T orbit at x_3 and develops into chaos through period doubling bifurcation. At each end of the diagram, $x_{1c}=1/2-4/[2\lambda(1-\alpha)]$ and $x_{2c}=1/2+1/[\lambda(1-\alpha)]$, the chaotic bands abruptly terminate because of boundary crisis. Each crisis point is obtained by equations $F\circ F(x;y_M)=0$ and $F\circ F(x;y_M)=[1-\lambda(1-\alpha)(1-2x)]/\lambda(1-\alpha)^2$, respectively, where y_M are the values satisfying $dF(x;y_M)/dy=0$. Thus the critical points obtained are $-0.377\,19,\ldots$ and $0.938\,596,\ldots$.

If we let $\lambda(1-\alpha)(1-2x_n)=z_n$ the system has the form of $y_{n+1}=z_n y_n+O(y_n^2)$ similar to the equation in Ref. [7], where $z_n=-1$, 1, 0 at $x_n=x_2$, x_1, 1/2, respectively. Then the bifurcation points are at $z=\pm 1$, and z_n is in a chaotic process. So the ED generates chaos, on-off intermittency, or IPLP according to α. From the equation we can obtain the regions for IPLP since they are adjoined to the onset of on-off intermittency. The condition for the onset of on-off intermittency is $\langle \ln z\rangle=0$ when z_n is uniform, since the system has approximately the form of $y_{n+1}\approx e^{n\langle \ln z\rangle}y_1=(a/e)^n y_1$, where $z_n\in(0,a)$ [8].

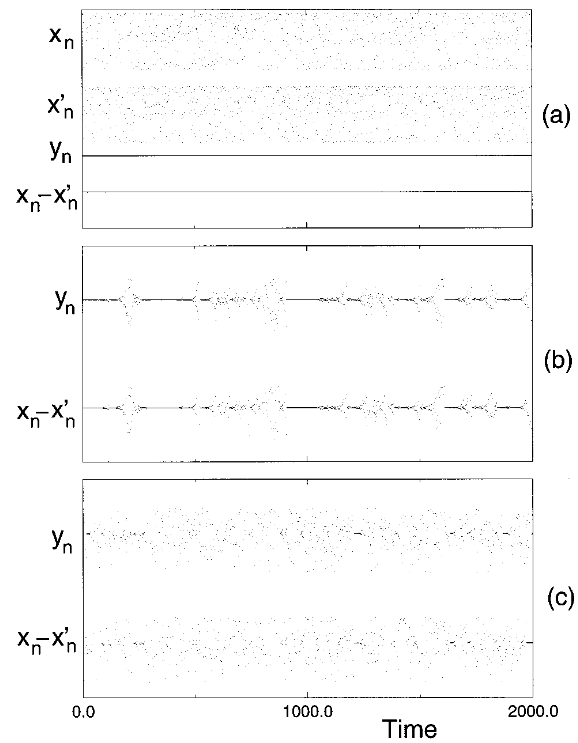

FIG. 2. Temporal behaviors of synchronization in the logistic maps when the maps are (a) synchronized ($\alpha=0.4$), (b) intermittently synchronized ($\alpha=0.33$), and (c) desynchronized ($\alpha=0.25$).

In our study, the condition for the onset of on-off intermittency is obtained by dividing the integration region of z_n into $z_U>z_n>0$ and $z_L<z_n<0$ with density function $P(z)$ as follows:

$$\int_0^{-z_L} P(z)\ln z\,dz+\int_0^{z_U} P(z)\ln z\,dz=0, \tag{6}$$

where z_L and z_U are the lower and upper limits of z. If we assume $P(z)$ is uniform, $P(z)=1/(z_U-z_L)$, and Eq. (6) yields $\ln[\lambda(1-\alpha)]=1-[(2x_U-1)\ln(2x_U-1)+(1-2x_L)\ln(1-2x_L)]/2(x_U-x_L)$ since $z_U=1-2x_L$, $z_L=1-2x_U$, $x_U=\lambda/4$, and $x_L=(\lambda^2/4)(1-\lambda/4)$. The critical value of α for the onset of on-off intermittency is $0.28466,\ldots$ for $\lambda=3.8$. The critical value obtained numerically is $\alpha_c=0.37524,\ldots$. Here when $\alpha>\alpha_c$ the system generates IPLP. The difference between the analytical and numerical results is caused by the following: the density function is not uniform; when y_n is above the unstable orbits (the dotted lines in Fig. 1) the trajectory diverges although $x_2<x_n<x_1$; and the upper limit of x_n exceeds x_{2c}. We note that $x_U=0.95>x_{2c}$ and that $x_L=0.1805>x_{1c}$. These effects are not considered in the analytic calculation, but the analytic result approaches the numerical one. So, to sum up, while when $\alpha<\alpha_c$ and α is close to α_c, on-off intermittency occurs, and when α is far below the critical point, only a chaotic burst appears, when $\alpha>\alpha_c$ the ED generates IPLP, which means the two logistic maps are synchronized.

TABLE I. The region of synchronization in chaotic systems.

System	Variable	Synchronizing region
Lorenz		
$\sigma=10$	x	$0.83\cdots<\alpha\leqslant1.0$
$r=8/3$	y	$0.85\cdots<\alpha\leqslant1.0$
$b=23$	z	$\alpha<-5.0,\ldots$
Duffing		
$k=0.2,$	x	$0.18\cdots<\alpha<0.26,\ldots,$ $0.91\cdots<\alpha\leqslant1.0$
$20.3\cos(1.4t)$	y	$-2.0\cdots<\alpha<-1.3,\ldots,$ $0.19\cdots<\alpha<0.25,\ldots,$ $0.95\cdots<\alpha\leqslant1.0$
Brusselator		
$A=0.2,\ B=0.24$	x	$0.19\cdots<\alpha\leqslant1.0$
$0.3\cos(20t)$	y	$\alpha=1.0,\ \alpha<-0.20,\ldots$

Now to observe the synchronizing phenomenon according to α, the temporal behaviors of y_n and x_n-x_n' are obtained when the two logistic maps are (a) synchronized, (b) intermittently synchronized, and (c) desynchronized for $\alpha=0.4$, 0.33, and 0.25, respectively, as given in Fig. 2. Figure 2(a) presents the signals of x_n, x_n', y_n, and x_n-x_n'. There y_n and x_n-x_n' converge to 0, that is, the ED generates IPLP. Then the two logistic maps are synchronized. Figure 2(b) presents the signals of y_n and x_n-x_n', which show on-off intermittency. This means the two logistic maps are intermittently synchronized. Figure 2(c) also presents the signals of y_n and x_n-x_n', which are chaotic. Then the two logistic maps are not synchronized. In the figures the temporal behavior of y_n is approximately the same as that of x_n-x_n'. This indicates that the ED well describes the signal differences of the master and the slave systems. It also verifies that synchronization occurs when the ED generates IPLP.

Now to demonstrate our synchronization in electronic circuits based on the forced double-well Duffing equation, we first consider the following master and replica slave system equations with a feedback signal $x(t)$:

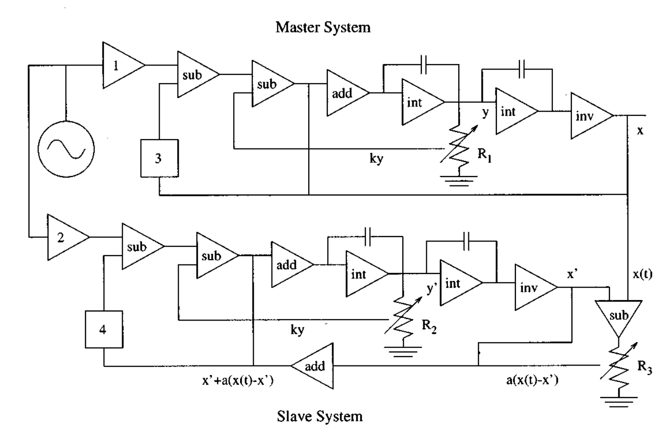

FIG. 3. Schematic diagram of the circuit based on the forced double-well Duffing equation to implement synchronization.

$$
\begin{aligned}
&\dot{x}=y,\\
&\dot{y}=-ky+x-x^3+B\cos(\omega t)\quad\text{(master system)},\\
&\dot{x}'=y',\\
&\dot{y}'=-ky'+\{x'+\alpha[x(t)-x']\}-\{x'+\alpha[x(t)-x']\}^3\\
&\qquad+B\cos(\omega t)\quad\text{(slave system)}.
\end{aligned}
\tag{7}
$$

If we let $x-x'=X$ and $y-y'=Y$, the equations are reduced to

$$
\begin{aligned}
&\dot{X}=Y,\\
&\dot{Y}=-kY-(3x(t)^2-1)(1-\alpha)X+3x(t)(1-\alpha)^2X^2\\
&\qquad-(1-\alpha)^3X^3.
\end{aligned}
\tag{8}
$$

Here it becomes obvious that the parameters are also modulated by $x(t)$. Since the parameters are forced through a bifurcation point, we obtain the regions of α for synchronization numerically, which are given in Table I at given parametric values.

The schematic diagram of the analog circuit of the Duffing equation for synchronization is given in Fig. 3, which implements Eqs. (7). In the figure, the signal from the external oscillator (Tektronix FG501A) is applied to the two chaotic systems in parallel through buffers (1 and 2) to synchronize the oscillator where the amplitude is about 3.4 V and the frequency is about 9.6 kHz. The operational amplifiers (OA's) and associated circuitry perform the operations of addition, subtraction, inversion, and integration. Analog multipliers (3 and 4) implement the nonlinear terms in the circuit equation. We guarantee that our circuit implementation of

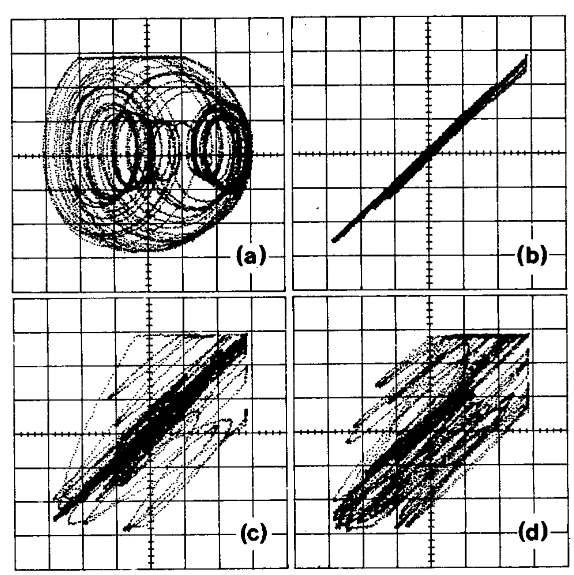

FIG. 4. Experimental outputs of the circuit. (a) Chaotic attractor of the forced double-well Duffing equation projected onto the xy plane, and the trajectories of the master system y vs slave system y' when the two systems are (b) synchronized ($\alpha=0.56$), (c) intermittently synchronized ($\alpha=0.59$), and (d) desynchronized ($\alpha=0.65$).

Eq. (7) is exact, and that the coefficient k can be independently varied by adjusting the variable resistors R_1 and R_2. The circuit time scale can be easily adjusted by changing the values of the capacitors for integration. For synchronization, the signal x from the master system is substracted by x'. The subtracted signal is reduced by a variable resistor R_3 (100 kΩ) to vary the scaling factor α and the reduced one is added to x'. The OA's and buffers used in this circuit are LF353, and multipliers (3 and 4) are MPY100. In this circuit, all the resistors connected to the OA's and buffers are 100 k Ω and the capacitors are 1 nF.

To illustrate the chaotic behavior of the master system, we obtain the phase diagram of x versus y as given in Fig. 4(a). The figure well shows the typical phase diagram of the Duffing oscillator. To observe synchronization, we tune the variable resistor R_3, and obtain the phase diagrams of y versus y' when the two chaotic systems are synchronized, intermittently synchronized, and desynchronized. When $\alpha=0.56$, the two systems are synchronized as the phase diagram shows in Fig. 4(b). When $\alpha=0.59$, the systems are synchronized intermittently as in Fig. 4(c). And when $\alpha=0.65$, the systems are not synchronized as in Fig. 4(d). In this circuit, the region of our synchronization is about $0.57>\alpha>0.49$. In addition we obtain our synchronization when $0.95<\alpha<1.0$, and the Pecora-Carroll synchronization when $\alpha=1.0$. The synchronization regions are similar to those obtained numerically as in Table I. (In experiment, we have obtained other various regions of synchronization as we vary the frequency or amplitude of the external force.)

Our technique of synchronization is applied in other nonlinear dynamical systems numerically such as the forced double-well Duffing, the Lorenz, the forced Brusselator systems. Then we observe our synchronization in various regions of α even if $\alpha\neq 1.0$ as given in Table I. In the regions where our synchronization occurs the ED's generate IPLP connected with on-off intermittency as analyzed in the above.

In conclusion we have observed a mechanism of synchronization, which occurs when the ED generates IPLP. We have developed for this synchronization a method of feeding back a signal of the master system to the corresponding variable of the slave system. The mechanism of synchronization is verified in the logistic maps, and demonstrated in electronic circuits based on the forced double-well Duffing equation. We expect this type of synchronization will be helpful in developing other methods of synchronization where, especially, on-off intermittency appears. It is also expected to be applicable to practical systems such as private communications.

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