Coherence resonance at noisy precursors of bifurcations in nonlinear dynamical systems

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A general mechanism of coherence resonance that occurs in noisy dynamical systems close to the onset of bifurcation is demonstrated through examples of period-doubling and torus-birth bifurcations. Near the bifurcation of a periodic orbit, noise produces the characteristic peaks of ''noisy precursors'' in the power spectrum. The signal-to-noise ratio evaluated at these peaks is maximal for a certain optimal noise intensity in a manner that resembles a stochastic resonance. [S1063-651X(97)06307-1]

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Nonlinear systems perturbed by noise have the potential to display a wide range of complex responses including, somewhat paradoxically, an enhancement of net order and coherence as noise levels increase. A distinguished example of this phenomenon is stochastic resonance (SR) [1] which has attracted considerable attention over the last decade (see for references the reviews [2]). Conventional SR occurs in noisy dynamical systems when perturbed by a weak external periodic signal. For such systems, significant amplification of the weak periodic signal may occur solely by increasing the level of the noise intensity. The signal-to-noise ratio (SNR), and other appropriate measures of signal coherence, pass through a maximum at an optimal noise strength when the noise-controlled time scale of the system matches the period of the external signal.

A similar effect of noise-induced coherence may also be observed in systems which lack an external signal, but whose intrinsic dynamics are controlled by noise intensity. In earlier studies [3,4] the noise-induced enhancement of coherence in underdamped nonlinear oscillators has been found. The noise-induced peak at zero frequency appeared in the vicinity of a pitchfork bifurcation [3], whereas the decrease of the width of a fluctuating peak in the power spectrum is shown for an underdamped oscillator, whose eigenfrequency possesses an extreme in energy, in [4]. Recently, a noise-induced coherent motion has been observed for autonomous systems in [5], where the effect of noise on a nonuniform limit cycle has been studied, and in [6], where a coherence resonance in a noise-driven excitable system has been reported. This group of phenomena can be called coherence resonance or ''internal'' SR, which underlines the fact that one can observe SR-like phenomena without an external periodic signal.

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In the present paper we study the response of nonlinear dynamical systems to noise excitation near the onset of dynamical instabilities of periodic orbits. Our starting point is the key paper of Wiesenfeld [7], which carefully elaborates the way in which noise controls the qualitative structure of the power spectrum. In brief, Wiesenfeld demonstrates that the power spectrum of a system observed after a bifurcation point can, nevertheless, be visible even before the bifurcation actually occurs if there is noise present. We thus observe a *noisy precursor* of the bifurcation.

To follow this line of thought further, let us suppose that noise induces a peak of height H at the frequency ω_p in the power spectrum, so that the noisy precursor of an instability is observed. We then ask what might happen to the shape of the spectrum if the noise intensity is increased? Two tendencies can be suggested.

(i) With the increase of noise, the model's trajectory is kicked further away from the stable periodic orbit which leads to damped oscillations at the frequency ω_p. This boosts the height H of the peak in the power spectrum.

(ii) Because of the nonlinearity of the system, increasing noise will increase the peak's relative width $W=\Delta\omega/\omega_p$ (which is none other than the inverse of the familiar quality factor Q) [8]. This increase in W makes it difficult to resolve the peak from the noise background.

In order to measure the coherence of the system at the noise-induced peak we define the signal-to-noise ratio as $\mathrm{SNR}=H/W$, as in [5]. We aim to show that because H and W vary differently with noise intensity, the SNR will very often pass through a maximum, and in a manner that is typical for conventional SR.

Firstly, however, we note that it would be impossible to observe such behavior in a linear system perturbed by additive noise, since the height of a noise-induced peak is known to increase monotonically as a function of noise intensity, whereas the width of the peak is constant against noise. As a result, in linear systems the SNR increases monotonically with noise. On the other hand, noise excitation of a self-sustained oscillator which has a stable limit cycle far from a bifurcation leads to the well-known effect of washing the spectral line out [8] so that the SNR decreases monotonically with the increase of noise.

In order to test our prediction of a resonant behavior versus noise strength, as found in noisy precursors, we study the

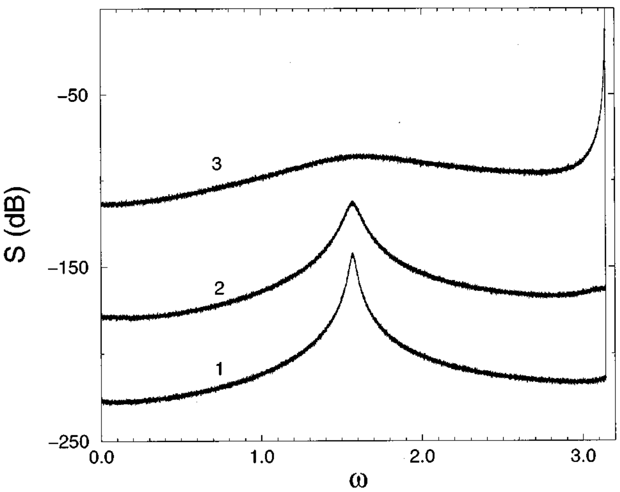

FIG. 1. The power spectrum of the noisy Feigenbaum map (1) at $a=1.24$ for different noise intensities: (1) $D=10^{-3}$, (2) $D=10^{-2}$, and (3) $D=5\times10^{-2}$.

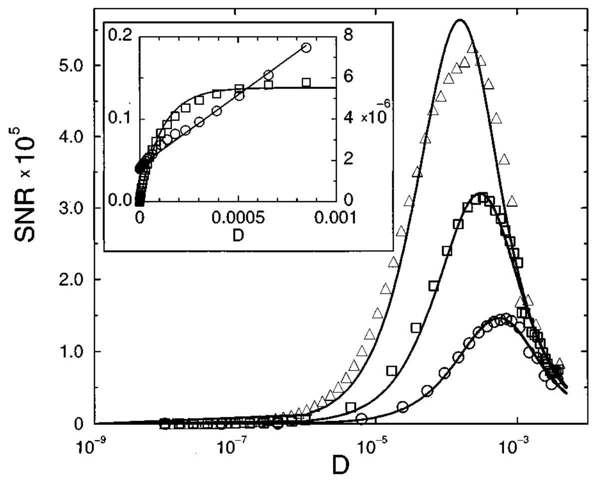

FIG. 2. SNR at $\omega_p=\pi/2$ vs D for the noisy Feigenbaum map near the second period-doubling bifurcation for different values of the parameter a: (○) $a=1.2$, (□) $a=1.22$, and (△) $a=1.23$. The solid lines represent the approximation given by Eq. (2). Inset: the dependence of H (□, right scale) and W (○, left scale) vs D for $a=1.23$.

effects of white noise forcing on: (a) period-doubling bifurcations in the celebrated discrete Feigenbaum map [9]; (b) torus-birth bifurcations found in two coupled discrete Feigenbaum maps; and (c) period-doubling bifurcations in the Rössler equations.

For the case of the *period-doubling bifurcations,* consider first the noisy Feigenbaum map (or logistic map) as defined by the following stochastic difference equation:

$$x_{n+1}=1-ax_n^2+\sqrt{D}\xi_n, \tag{1}$$

where a is the control parameter of the map and D measures the intensity of white noise ξ_n. The universal behavior of a noisy period-doubling sequence has been studied in detail in [10]. In the absence of noise, the bifurcation sequence of fixed points of period 2^k takes place for the parameter values a_k: $a_1=0.75$, $a_2=1.25$, $a_3=1.368\,099, \ldots$. Figure 1 displays the power spectrum of the noisy map (1) just before the second period doubling bifurcation ($a=1.24$) for different noise levels. Note that when there is no noise ($D=0$), the map has a stable fixed point of period 2 and a δ peak at the frequency $\omega_0=\pi$ in the power spectrum. With the noise switched on, the noisy precursor of a period-four cycle becomes visible as a peak at the subharmonic frequency $\omega_p=\pi/2$. The increase of noise makes this peak more pronounced. However for large noise levels, the width of the peak becomes so wide that it is difficult to distinguish the peak from the noise background.

To better quantify this behavior we present the results of numerical calculations of H, W, and SNR in Fig. 2 which clearly support our prediction for a resonant effect. The inset graph in Fig. 2 displays the dependence of H and W on the noise intensity D. The width of the noise-induced peak increases linearly with the increase of D, as is known for a classical self-sustained oscillator perturbed by noise [8]. On the other hand, the height of the peak increases linearly for a small noise intensity and then flattens out and saturates due to the nonlinearity of the system. Appropriate scalings for W and H were found to be $W\propto W_0+D$, $H\propto 1-\exp(-\alpha D)$, with constants W_0 and α. Competition between the growth in the height of the peak and its width therefore gives rise to a bell shaped curve for the SNR which can be fitted as

$$\mathrm{SNR}\propto\frac{1-\exp(-\alpha D)}{W_0+D}. \tag{2}$$

The optimal noise intensity D_{opt}, at which the SNR is maximum, corresponds to the situation in which the noise-induced peak is most pronounced (cf. Fig. 1).

The behavior of the precursors versus the control parameter a is of interest as well. We therefore introduce the critical parameter $\epsilon=a_k-a$, where a_k is the parameter value of the kth period-doubling bifurcation. For any fixed noise level D, we found that the SNR(D) scales with ϵ as SNR(D) $\propto\epsilon^{-3}$ which fits the theoretical predictions of Ref. [7]. Figure 2 also shows that as the control parameter a approaches the point of bifurcation $a_2=1.25$, the optimal noise intensity D_{opt} shifts towards smaller values and the SNR increases. Our simulations revealed that the optimal noise intensity and control parameter near a period-doubling bifurcation are connected linearly: $D_{opt}\propto\epsilon$. Beyond the bifurcation point $a\gtrsim a_2=1.25$, there is a period-four fixed point with δ peaks at the subharmonic $\omega_p=\pi/2$ in the power spectrum. However, as noise increases, these peaks are gradually ''washed out'' and the SNR monotonically decreases. Finally, we also analyzed Eq. (1) for parameter values close to other period-doubling bifurcations and obtained results in close agreement to all those reported above.

Let us now consider the *torus-birth bifurcation*. In the language of Poincaré maps this bifurcation refers to the case when a pair of complex conjugate characteristic multipliers cross the unit circle. This bifurcation is thus closely akin to the Hopf bifurcation in a flow system [11]. A system of two coupled Feigenbaum maps, for example, can easily generate

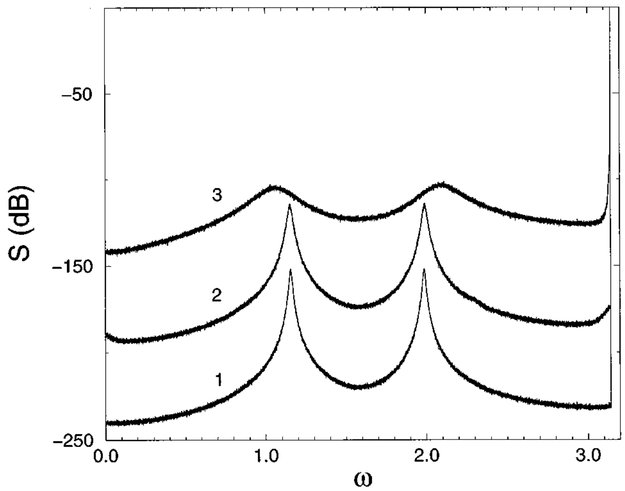

FIG. 3. The power spectrum of the coupled Feigenbaum map (3) at $a=0.405$, $\gamma=0.4$ for different noise intensities: (1) $D=10^{-3}$, (2) $D=10^{-2}$, and (3) $D=10^{-1}$.

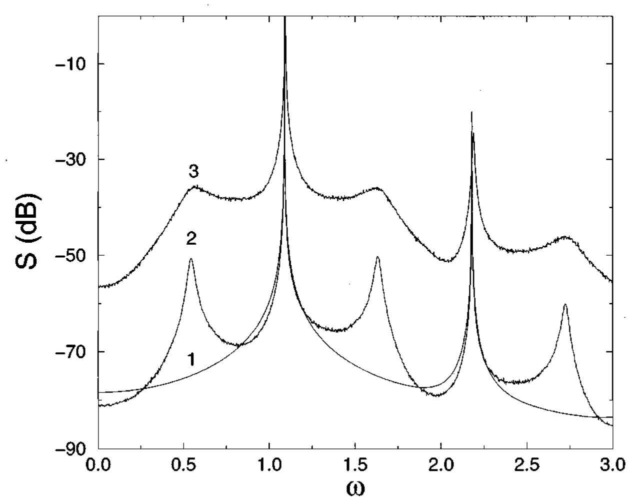

FIG. 5. The power spectrum of the Rössler system (4) at $a=b=0.2$, $c=2.7$ for several values of noise intensity: (1) $D=0$, (2) $D=10^{-4}$, and (3) $D=2\times10^{-3}$.

such a bifurcation [12,11]. With additive noise applied, the system is described by the two stochastic difference equations

$$x_{n+1}=1-ax_n^2+\gamma(y_n-x_n)+\sqrt{D}\xi_n,$$

$$y_{n+1}=1-ay_n^2+\gamma(x_n-y_n)+\sqrt{D}\eta_n, \qquad (3)$$

where γ is the coupling strength and ξ_n, η_n are statistically independent white noises. The bifurcations in the noiseless system (3) have been studied in detail (see, for example, [11]). The stable fixed point of period 1 is born as a increases beyond $a=-0.25$, with a bifurcation to a period-two fixed point when $a=(4\gamma^2-8\gamma+3)/4$. Increasing the parameter a further leads to the characteristic multipliers of the period-two fixed point to become a complex conjugate and when a is large enough, the multipliers cross the unit cycle. This Hopf bifurcation corresponds to the birth of an invariant curve or torus in the phase space of the system.

Consider now the case when $\gamma=0.4$, and for which a Hopf bifurcation is known to occur at $a=a_t\approx0.409\,98$. With $a=0.405<a_t$ the system is below the Hopf bifurcation and, in the absence of noise, the power spectrum contains only a single δ peak at the frequency π corresponding to the stable period-two fixed point. With noise applied to the system, the noisy precursors of the Hopf bifurcation become readily apparent (see Fig. 3) and two new peaks in the power spectrum appear. These peaks correspond in frequency to the case of torus dynamics that arises when there is no noise and $a>0.405$ (i.e., after the Hopf bifurcation). A close analysis

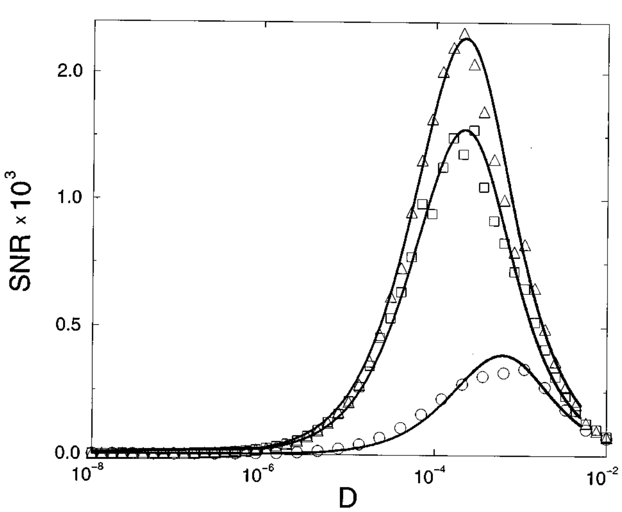

FIG. 4. SNR vs D for the coupled Feigenbaum map at $\gamma=0.4$ for several values of parameter a: (○) $a=0.4$, (□) $a=0.407$, and (△) $a=0.408$. The solid lines represent the approximation given by Eq. (2).

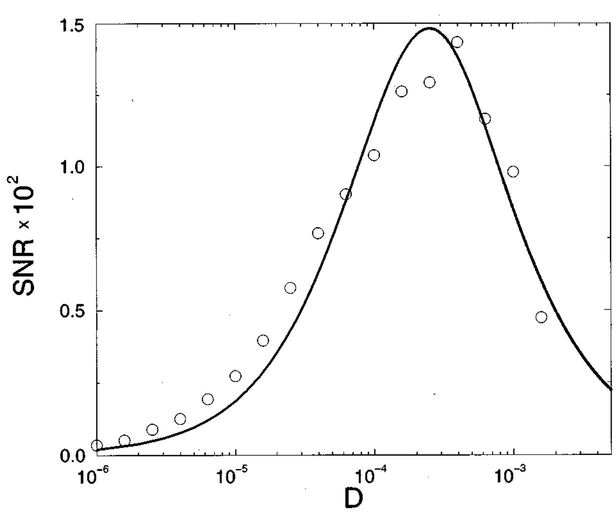

FIG. 6. SNR vs D at the noise-induced frequency (left subharmonic) of the Rössler system. The solid line represents the approximation given by Eq. (2). The parameter values are the same as in Fig. 5.

revealed that increasing noise intensity in Eq. (3) results in exactly the same effects observed for the preceding case of the period-doubling bifurcation. Figure 4 makes clear the dependence of the SNR at one of the noise-induced peaks as a function of noise intensity. The SR-like behavior is clearly seen again.

Qualitatively the same behavior is also observed in flow systems. To demonstrate this we present a numerical study of the noisy Rössler system [13] near a period-doubling bifurcation. With additive white noise the system is governed by the three-dimensional stochastic differential equations

$$\dot{x}=-(y+z)+\sqrt{2D}\xi_1(t), \quad \dot{y}=x+ay+\sqrt{2D}\xi_2(t)$$

$$\dot{z}=b+z(x-c)+\sqrt{2D}\xi_3(t), \tag{4}$$

where a, b, and c are the parameters and D is the intensity of the statistically independent white noises $\xi_i(t)$. In the absence of noise $(D=0)$, and with the parameter values $a=b=0.2$, the first period-doubling bifurcation occurs at $c=c_1\approx 2.835$. For $c=2.7<c_1$ and in the absence of noise, there is a stable cycle of period 1. With noise switched on, the precursors of period-doubling become visible. The power spectra of the x coordinate (4) is shown for three different noise intensities (Fig. 5). As can be seen, the evolution of the power spectrum is qualitatively equivalent to that of the Feigenbaum map (cf. Fig. 1). The dependence of the SNR versus D is shown in Fig. 6 and again displays the SR-like behavior.

The effect we report here is very simple, and, in fact, extremely general. Nonlinear dynamical systems when excited by noise give rise to precursors of instabilities of periodic motion. These noisy precursors, which are prominently close to points of bifurcation, manifest themselves as noise-induced peaks in the power spectrum of the system. The peaks are most strongly expressed at an optimal level of noise, and thus the SNR at the noise-induced peaks pass through a maximum as noise intensity increases. This phenomenon has a simple physical interpretation that can be stated in terms of two competing mechanisms. The first, is the increase of the height of the noise-induced peak as the noise strength increases; a tendency which makes the precursor more visible above the noise background. The growth of the height is, nevertheless, bounded by the nonlinearity of the system. The second mechanism is the increase of the width of the peak with noise, which tends to create difficulties in resolving the peak. In short these two quantities, the peak's height and width, vary differently with noise intensity. The competition in the growth of these quantities yields an optimal noise intensity at which the SNR takes its maximal value. The effect reported here appears to be quite general and provides an interesting interpretation of coherence resonance in autonomous noisy systems.

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[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981); R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, Tellus **34**, 10 (1982).

[2] F. Moss, D. Pierson, and D. O'Gorman, Int. J. Bifurc. Chaos **4**, 1383 (1994); K. Wiesenfeld and F. Moss, Nature (London) **373**, 33 (1995); Proceedings of the NATO Advanced Research Workshop on Stochastic Resonance in Physics and Biology, edited by F. Moss, A. Bulsara, and M. Shlesinger [J. Stat. Phys. **70**, 1 (1992)]; Proceedings of the International Workshop on Fluctuations in Physics and Biology: Stochastic Resonance, Signal Processing and Related Phenomena, edited by A. Bulsara, S. Chillemi, L. Kiss, P.V.E. McClintock, R. Mannella, F. Marchesoni, K. Nikiolos, and K. Wiesenfeld [Nuovo Cimento D **17**, 1 (1995)].

[3] M.I. Dykman and M.A. Krivoglaz, in Sov. Phys. Rev. **5**, 265 (1984).

[4] S.M. Soskin, Physica A **155**, 401 (1989).

[5] Hu Gang, T. Ditzinger, C. Ning, and H. Haken, Phys. Rev. Lett. **71**, 807 (1993); W.J. Rappel and S.H. Strogatz, Phys. Rev. E **50**, 3249 (1994).

[6] A.S. Pikovsky and J. Kurths, Phys. Rev. Lett. **78**, 775 (1997).

[7] K. Wiesenfeld, J. Stat. Phys. **38**, 1071 (1985).

[8] R.L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967), Vol. 2.

[9] M.J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978).

[10] J.P. Crutchfield, M. Nauenberg, and J. Rudnic, Phys. Rev. Lett. **46**, 933 (1981); S. Shraiman, C.E. Wayne, and P.C. Martin, *ibid.* **46**, 935 (1981); M.J. Feigenbaum and B. Hasslacher, *ibid.* **49**, 605 (1982); J.P. Crutchfield, J.D. Farmer, and B.A. Hubermann, Phys. Rep. **92**, 45 (1982); E.B. Vul, Ya.G. Sinai, and K.M. Khanin, Russ. Math. Surveys **39**, 1 (1984).

[11] V.S. Anishchenko, *Dynamical Chaos – Models and Experiments* (World Scientific, Singapore, 1995).

[12] V.I. Arnold, in *Nonlinear Waves*, edited by A.V. Gaponov-Grechov (Nauka Publisher, Moscow, 1979), pp. 116–130; K. Kaneko, *Collapse of Tori and Genesis of Chaos in Dissipative Systems* (World Scientific, Singapore, 1986).

[13] O.E. Rössler, Phys. Lett. A **57**, 397 (1976).