Phase transitions in two-variable coupled map lattices

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It is shown that a two-variable coupled map lattice can reproduce certain dynamical features of oscillatory and excitable reaction-diffusion systems. Many well-known dynamics such as synchronization of spatiotemporal chaos, bistable dynamics, and traveling spatiotemporal patterns are observed in our numerical simulations. The ingredients responsible for the appearance of those phase transitions are analyzed. [S1063-651X(97)05509-8]

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Characterization of spatially extended systems is of great importance for understanding the dynamics of many physical, chemical, and biological systems.[1–4] The theoretical description of such systems is usually given in terms of the partial differential equations (PDEs), in particular, the reaction-diffusion equation. The bifurcation structure of PDEs is, in general, difficult to analyze and numerical solution is often time consuming and sometimes impossible. In order to study the spatial pattern formation and dynamical phase transitions in reaction-diffusion systems it is of interest to simplify the dynamics and consider discrete models of the spatially distributed medium. Considerable effort has been devoted to the construction of discrete models for excitable systems [5–7]. It is more difficult to construct discrete models for oscillatory systems that reproduce major well-known features of the dynamics [8,9]. Recently, phase resetting dynamics has been studied by using a coupled map model of oscillatory reaction-diffusion systems [10,26]. It is found that such a discrete model is able to reproduce faithfully the major features of the complex wave propagation processes in a relaxation oscillator reaction-diffusion system.

The dynamics of discrete model systems such as a coupled map lattice (CML) has attracted considerable attention in recent years for their combination of computational efficiency and phenomenological richness [11–15]. In this paper, we propose a two-variable coupled map lattice to model the pattern forming chemical and biological systems, where two or more coupled variables are usually involved. Our purpose is to investigate the phase transition and pattern-forming dynamics utilizing discretized space and time corresponding to the usual coarse-grained description of the reaction-diffusion processes. As will be shown later, in spite of its simplicity, our model system can capture many essential features of real reaction-diffusion systems, such as oscillatory, bistable, and excitable dynamics. On the other hand, the two-variable coupled map lattices can also be regarded as two coupled map lattices linked by common signals. Therefore, it represents a different synchronization approach, which is, in principle, drawn from a scheme for controlling spatiotemporal chaos [16]. A remarkable feature of this synchronization approach is that a chaotic orbit of one system is stabilized about a chaotic orbit of the other [17]. Thus, once synchronization is established, the coupling signals vanish, just as in the case of controlling chaos [18].

A two-variable coupled map lattice may be defined by

$$\begin{aligned} x_{n+1}(i) &= (1-\epsilon-c)f(x_n(i)) + \tfrac{1}{2}\epsilon[f(x_n(i-1)) \\ &\quad + f(x_n(i+1))] + cg(y_n(i)), \\ y_{n+1}(i) &= (1-\epsilon'-c')g(y_n(i)) + \tfrac{1}{2}\epsilon'[g(y_n(i-1)) \\ &\quad + g(y_n(i+1))] + c'f(x_n(i)), \end{aligned} \tag{1}$$

where $f(x)$ and $g(y)$ are arbitrary functions, representing the local reaction dynamics involved. The variables x and y may describe the concentrations of ''trigger'' (or ''activator'') and ''controller'' (or ''inhibitor''), depending on the chemical or biological processes determined by the reaction-diffusion equations. For the purpose of this paper, we restrict ourselves to the case of identical reaction dynamics, i.e., $g(x)=f(x)$. We further assume that $f(x)=ax(1-x)$, which is the logistic map. For convenience, we assume that the variables x and y represent the amplitudes of systems A and B, respectively. In the following, we report some interesting properties of the coupled system

$$\begin{aligned} x_{n+1}(i) &= (1-\epsilon-c)f(x_n(i)) + \tfrac{1}{2}\epsilon[f(x_n(i-1)) \\ &\quad + f(x_n(i+1))] + cf(y_n(i)), \\ y_{n+1}(i) &= (1-\epsilon'-c')f(y_n(i)) + \tfrac{1}{2}\epsilon'[f(y_n(i-1)) \\ &\quad + f(y_n(i+1))] + c'f(x_n(i)), \\ f(x) &= ax(1-x). \end{aligned} \tag{2}$$

Here n is a discrete time step and i is a lattice point ($i=1,2,\ldots,N$). N denotes the system size. The periodic boundary conditions $x_n(i+N)=x_n(i)$ and $y_n(i+N)=y_n(i)$ are assumed. It is clear that the coupled system (2) is characterized by two groups of parameters (a,ϵ,c) and (a,ϵ',c'). By varying the system parameters, a large variety of dynamical behaviors and spatiotemporal patterns may be discovered. Several special cases, however, are worth mentioning. By setting $c'=c$ and $\epsilon'=\epsilon$, Eq. (2) reduces to a symmetrically coupled system. The unidirectionally linked CML systems are only the special cases with $c'=0$ or $c=0$. On the other hand, Eq. (2) may describe a class of

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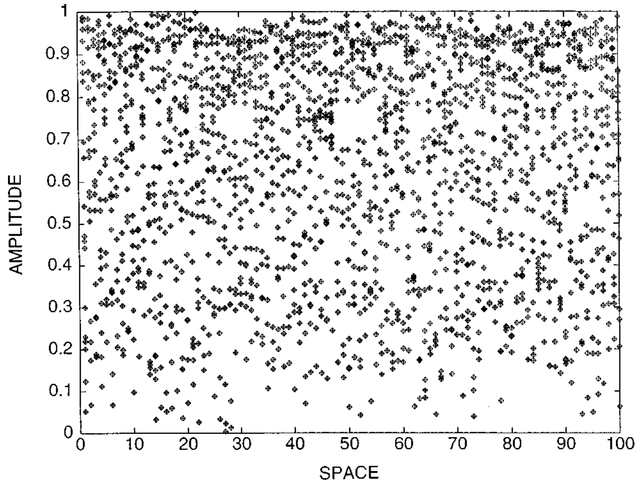

FIG. 1. Amplitude-space plot of mutual synchronization of spatiotemporal chaos, for $\epsilon=\epsilon'=0.1$ and $c=c'=0.18$. The last 20 iterations are plotted from a total 80 020 iterations.

coupled systems whose constituent elements represent different dynamical systems, if we assume that $a'\neq a$ and/or $\epsilon'\neq\epsilon$. For the purpose of illustration, we focus our attention on two different cases: (i) $a'=a$, $c'=c$, $\epsilon'=\epsilon$ and (ii) $a'=a$, $c'=c$, and $\epsilon'\neq\epsilon$, that is, we will study the influence of different diffusive couplings on the dynamics of the model system (2).

We have systematically investigated the dynamical behavior of Eq. (2). Many interesting features of such a coupled system are observed from numerical simulations. The numerical results reported in this work are performed for $a=a'=4$, which corresponds to the fully developed chaos regime. The lattice size is taken to be $N=100$. We notice that the lattice size may play a role near the pattern competition intermittency; otherwise its influence is minimal. Equation (2) with the above conditions reveals several different dynamical behaviors.

For case (i), i.e., two identical systems, the following phases are observed.

(a) Synchronized oscillation. For sufficiently small values of the coupling c, system A is synchronized with system B.

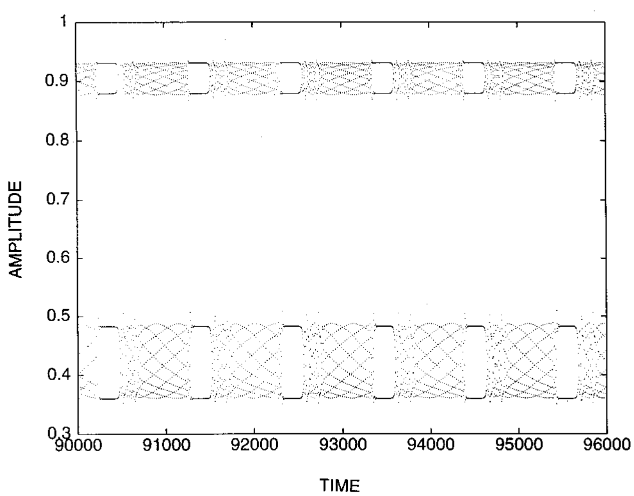

FIG. 2. Time evolution of $y_n(13)$ for $\epsilon=0.1$, $\epsilon'=0.2$, and $c=c'=0.2$. The amplitude $x_n(13)$ exhibits similar behavior. Only the last 6000 steps are plotted.

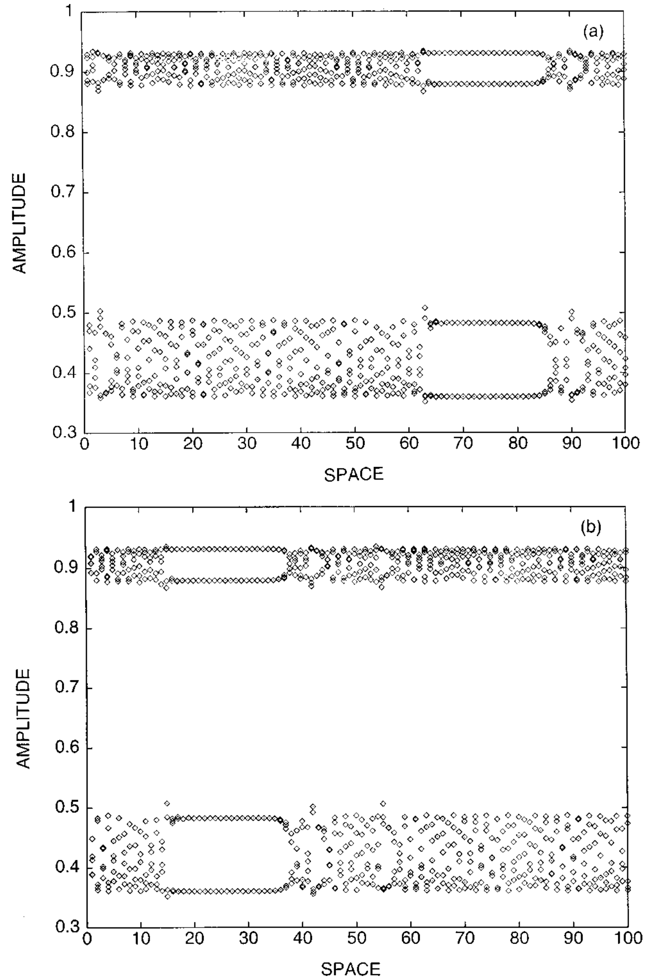

FIG. 3. Amplitude-space plot for two different time intervals: (a) $100\,100<n<1\,000\,110$ and (b) $100\,600<n<1\,000\,610$. The parameters' values are the same as in Fig. 2.

Both systems settle down in a spatially homogeneous, temporally periodic state of period 2, through pattern competition intermittency. In view of the chaotic dynamics of individual sites and the very weak coupling, this may be explained as the suppression of chaos by the selection of a spatiotemporally regular state from among many coexisting spatiotemporal orbits admitted by the system.

(b) Asynchronous chaotic state. When the coupling c is raised a little, two systems desynchronize and exhibit spatiotemporal chaos.

(c) Mutually synchronized spatiotemporal chaos. For intermediate coupling strength, in our case for approximately $0.19<c<0.84$, synchronization of spatiotemporal chaos is achieved. We found that the time required for synchronization is approximately inversely proportional to the coupling strength. In a deep synchronization regime (corresponding to large values of c), the two systems are desynchronized in a couple of iterations if the coupling is switched off (i.e., by setting $c=c'=0$). This characteristic may be useful in the communications applications of synchronization of the spatiotemporal chaos.

(d) Bistable states. As we further increase the coupling c, the coupled systems may evolve into one of the two accessible stable states, depending on the initial conditions.

The two systems behave in a complementary way such that when system A settles down in one of the two possible spatiotemporally homogeneous states, system B evolves into the other one, and vice versa.

For larger values of the coupling, the coupled system (2) shows asynchronous spatiotemporal chaos, which may be termed defect-mediated turbulence in view of the way by which the coupled system (2) becomes spatiotemporally chaotic. Figure 1 shows the synchronization of spatiotemporal chaos.

We now turn to case (ii), where $\epsilon \neq \epsilon'$ is assumed. We choose that $\epsilon = 0.1$ and $\epsilon' = 0.2$. For such a system, a complete synchronization of both amplitude and phase seems to be impossible. Nevertheless, phase synchronization with chaotic modulation of the amplitudes is observed in our numerical simulations. In addition to the usual spatiotemporal patterns found in CMLs, some different properties are uncovered. One of the most intriguing features of the coupled system (2) is the existence of traveling spatiotemporal patterns, which is similar to what is observed in the one-way coupled logistic lattice [19] and CML with local phase slips [20]. However, the mechanism leading to such a symmetry-breaking transition shown by our model system seems to be different from that discussed in Ref. [20], where the traveling wave is maintained by a local phase slip. Figure 2 shows the time evolution of a selected individual site. Clearly seen is the traveling wave of frozen spatiotemporal patterns consisting of periodic, quasiperiodic, and defect turbulence states. The period of this traveling wave is approximately $T \simeq 1000$ and the wavelength seems to be the very size of the lattice. Then the velocity of the traveling wave is estimated as $v \sim 0.1$ (sites per step). The dynamics of this traveling state may be described as periodic motion with chaotic modulation. The velocity of the traveling patterns may depend on the system parameters, especially the diffusive coupling ϵ. Figure 3 shows the spatial structure at successive iterations, which illustrate the direction and the velocity of the traveling patterns. It is interesting to note that the two systems, represented by two variables x and y, behave in a synchronized manner. In fact, they are in a state of phase synchronization.

The existence of a traveling-wave solution to Eq. (2) may be attributed to the difference of spatial couplings in the two systems. Because the difference of the diffusive couplings between the two individual systems may cause an anisotropic diffusion in the coupled system (2), this in turn leads to the traveling wave, as observed in one-way logistic lattices. As suggested in Ref. [20], it is reasonable to expect the coexistence of traveling attractors with different velocities if we change the nonlinearity a.

For intermediate values of the coupling c, we found a zigzag pattern that is spatially and temporally period 2. Both systems oscillate synchronously with a constant difference between their amplitudes. For large values of c, the coupled system (2) is found in asynchronous chaotic states.

In conclusion, we have investigated numerically a one-dimensional two-variable coupled map lattice model system. Many interesting features of such a system, such as synchronization of spatiotemporal chaos, bistable state, and a different class of traveling spatiotemporal patterns, are discovered. The symmetry-breaking transitions as manifested in the bistable dynamics and in the traveling waves with selected direction and wavelength are rather remarkable in view of the symmetrical structure of our model system (2). It should be stressed that even though the coupled system (2) is simple and abstract, it still can capture certain essential features of real reaction-diffusion systems.

In view of the recent interest in the synchronization of chaotic signals [21–25], it is worthwhile to point out that Eq. (2) also provides an approach to mutual synchronization of spatiotemporal chaos. We found that this method is very general and robust in that it works for a wide range of system parameter values. We plan to give a detailed discussion in a future work.

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