# Burst avalanches in solvable models of fibrous materials

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We review limiting models for fracture in bundles of fibers, with statistically distributed thresholds for breakdown of individual fibers. During the breakdown process, avalanches consisting of simultaneous rupture of several fibers occur, and the distribution $D(\Delta)$ of the magnitude $\Delta$ of such avalanches is the central characteristic in our analysis. For a bundle of parallel fibers two limiting models of load sharing are studied and contrasted: the *global* model, in which the load carried by a bursting fiber is equally distributed among the surviving members; and the *local* model, in which the nearest surviving neighbors take up the load. For the global model we investigate in particular the conditions on the threshold distribution which would lead to anomalous behavior, i.e., deviations from the asymptotics $D(\Delta)\sim\Delta^{-5/2}$, known to be the generic behavior. For the local model no universal power-law asymptotics exists, but we show for a particular threshold distribution how the avalanche distribution can nevertheless be explicitly calculated in the large-bundle limit.
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## I. INTRODUCTION

When a weak structural element in a material with stochastically distributed strengths fails, the increased load on the remaining elements may cause further ruptures, and thus induce a burst avalanche of a certain size $\Delta$, i.e., one in which $\Delta$ elements fail simultaneously. When the load is further increased, new avalanches occur. The distribution of avalanche sizes, either at a fixed load, or the cumulative distribution from zero load until complete breakdown of the material, depends on several factors, in particular the threshold strength distribution and the mechanism for load sharing between the elements.

Due to the complex interplay of failures and redistributions of local stresses, few analytical results are available in this field; computer simulations are commonly applied — see Herrmann and Roux [1] for a review. However, firm analytical results, albeit on simplified models, are important in order to develop a deeper understanding of universal properties and general trends. In the present paper we therefore review and study burst events in models of fibrous materials that are sufficiently simple to allow theoretical treatment.

The models we consider are bundles of $N$ parallel fibers, clamped at both ends, and stretched by a force $F$ (Fig. 1). The individual fibers in the bundle are assumed to have strength thresholds $f_i$, $i=1,2,\ldots,N$, which are independent random variables with the same cumulative distribution function $P(f)$ and corresponding density function $p(f)$:

$$\mathrm{Prob}(f_i<f)=P(f)=\int_0^f p(u)du. \quad (1)$$

Whenever a fiber experiences a force equal to or greater than its strength threshold, it breaks immediately and does not contribute to the strength of the bundle thereafter. The models differ, apart from differences in the threshold distribution, in how stress is redistributed on the surviving fibers when a fiber fails. A central quantity to be studied in the following is the expected number $D(\Delta,N)$ of bursts of size $\Delta$ when the fiber bundle is stretched until complete breakdown.

The model of this kind with the longest history [2] is one in which it is assumed that the fibers obey Hookean elasticity right up to the breaking point, and that the load distributes itself equally among the surviving fibers. The model with

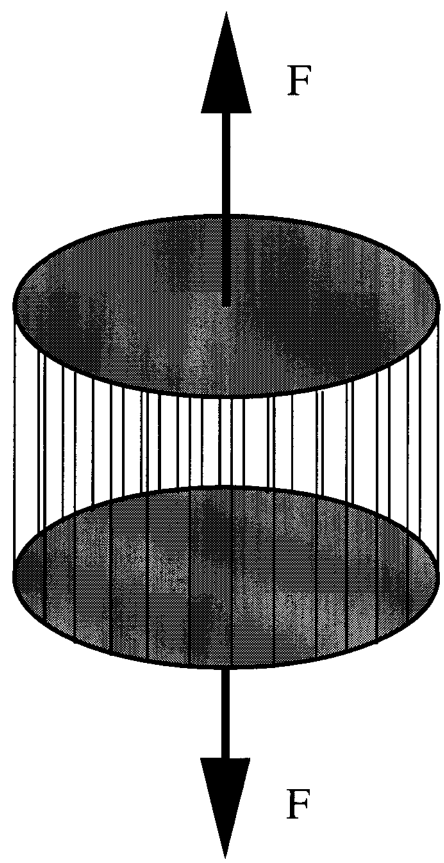


FIG. 1. A fiber bundle with periodic boundary conditions. The externally applied force $F$ is the control parameter.

 

this democratic load redistribution is similar to mean-field models in statistical physics, and is called here the *global model*. For large $N$, Daniels [3] was able to determine the asymptotic distribution for the bundle strength, a result that has been refined later [4–6]. The distribution of burst avalanches was first studied by Hemmer and Hansen [7]. Their main result was that for a large class of threshold distributions $P(f)$ the bursts were distributed according to an asymptotic power law,

$$\lim_{N\to\infty}\frac{D(\Delta)}{N}\simeq\frac{C}{\Delta^{\xi}}, \tag{2}$$

with a universal exponent

$$\xi=\frac{5}{2}. \tag{3}$$

In Sec. II we show that for special threshold distributions the power law (2) is not obeyed.

The assumption of global load sharing among surviving fibers is often unrealistic, and it is natural to consider models in which the extra stresses by a fiber rupture are taken up by the fibers in the immediate vicinity. The extreme version is to assume that only the *nearest-neighbor* surviving fibers take part in the load-sharing. In a one-dimensional geometry, as in Fig. 1, precisely two fibers, one on each side, share the extra stress. When the strength thresholds take only two values, the bundle strength distribution has been found analytically [8–10]. One interesting result is that the average bundle strength has a logarithmic size effect. The distribution of burst avalanches for such models with local load sharing has not yet been determined, but simulations [11,12] show that this model is *not* in the same universality class as the global model. The challenge to determine the burst distribution by other means than simulations remains, and that this is possible, at least in a special case (Sec. III), is one of the main results of the present paper.

## II. GLOBAL MODEL

In the global model the total force on a fiber bundle is distributed evenly on the surviving fibers. At a force $f$ per surviving fiber, the total force on the bundle is

$$F(f)=Nf[1-\phi(f)], \tag{4}$$

where $\phi(f)$ is the fraction of failed fibers. In Fig. 2 we show an example of a $F$ vs $f$. We have in mind an experiment in which the force $F$, our control parameter (Fig. 1), is steadily increasing. This implies that not all parts of the $F(f)$ curve are physically realized. The experimentally relevant function is

$$F_{ph}(f)=\text{LMF } F(f), \tag{5}$$

the least monotonic function (LMF) not less than $F(f)$. A horizontal part of $F_{\rm ph}(f)$ corresponds to an avalanche, the size of which is characterized by the number of maxima of $F(f)$ within the corresponding range of $f$.

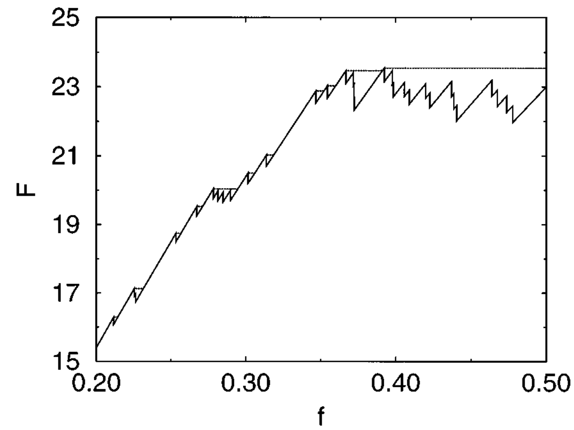


FIG. 2. The solid curve indicates the total force $F(f)$ as a function of $f$ — the force per surviving fiber, Eq. (4). However, when our control parameter is $F$ rather than $f$, the system will follow the broken line, $F_{\rm ph}=\text{LMF }F(f)$, defined in Eq. (5). The avalanches are the horizontal parts of $F_{\rm ph}(f)$. Here $N=100$.

It is the fluctuations in $F(f)$ that create avalanches. For a large sample the fluctuations will be small deviations from the average macroscopic characteristics $\langle F\rangle$. This average total force is given by

$$\langle F\rangle(f)=Nf[1-P(f)]. \tag{6}$$

Let us for the moment assume that $\langle F\rangle(f)$ has a single maximum. This maximum corresponds then to the value $f=f_c$ for which $d\langle F\rangle/df$ vanishes. This gives

$$1-P(f_c)-f_cp(f_c)=0. \tag{7}$$

In Ref. [7] the burst distribution was derived using the fiber elongation $x$ as the independent variable, under the assumption that Hooke's law holds up to the threshold for breaking. Here, however, we formulate everything in terms of the force per fiber, $f$, and simplify the derivation by using directly the fact that the thresholds in a small interval of $f$ are Poisson distributed.

### A. Burst distribution

Let us consider a small force-per-fiber interval $(f,f+df)$ in a range where the average force $\langle F\rangle(f)$ increases with $f$. For a large number $N$ of fibers the expected number of surviving fibers is $N[1-P(f)]$. The thresholds in the interval, of which there are $Np(f)df$, will be Poisson distributed. When $N$ is arbitrarily large, the burst sizes can be arbitrary large in any finite interval of $f$.

Assume that an infinitesimal increase in the external force results in a break of a fiber with threshold $f$. Then the load that this fiber suffered, will be redistributed on the $N[1-P(f)]$ remaining fibers; thus they experience a load increase

$$\delta f=\frac{f}{N[1-P(f)]}. \tag{8}$$

The *average* number of fibers that break as a result of this load increase is

$$a=a(f)=Np(f)\delta f=\frac{fp(f)}{1-P(f)}. \quad (9)$$

For a burst of size $\Delta$ the increase in load per fiber will be a factor $\Delta$ larger than the quantity (8), and an average number $a(f)\Delta$ will break. The probability that precisely $\Delta-1$ fibers break as a consequence of the first failure is given by a Poisson distribution with this average, i.e., it equals

$$\frac{(a\Delta)^{\Delta-1}}{(\Delta-1)!}e^{-a\Delta}. \quad (10)$$

This is not sufficient, however. We must ensure that the thresholds for these $\Delta-1$ fibers are not so high that the avalanche stops before reaching size $\Delta$. This requires that at least $n$ of the thresholds are in the interval $(f,f+n\delta f)$, for $1\leqslant n\leqslant\Delta-1$. In other words, if we consider the $\Delta$ intervals $(f,f+\delta f)$, $(f+\delta f,f+2\delta f)$, $\ldots$, $(f+(\Delta-1)\delta,f+\Delta\delta f)$, we must find at most $n-1$ thresholds in the $n$ last intervals. There is the same *a priori* probability to find a threshold in any interval. The solution to this combinatorial problem is given in Appendix A. The resulting probability to find all intermediate thresholds weak enough equals $1/\Delta$. Combining this with Eq. (10), we have, for the probability $\phi(\Delta,f)$ that the breaking of the first fiber results in a burst of size $\Delta$,

$$\phi(\Delta,f)=\frac{\Delta^{\Delta-1}}{\Delta!}a(f)^{\Delta-1}e^{-a(f)\Delta}. \quad (11)$$

This gives the probability of a burst of size $\Delta$, as a consequence of a fiber burst due to an infinitesimal increase in the external load. However, we still have to ensure that the burst actually *starts* with the fiber in question and is not part of a larger avalanche starting with another, weaker, fiber. Let us determine the probability $P_b(f)$ that this initial condition is fulfilled.

For that purpose consider the $d-1$ fibers with the largest thresholds below $f$. If there is no strength threshold in the interval $(f-\delta f,f)$, at most one threshold value in the interval $(f-2\delta f,f)$, $\ldots$, and at most $d-1$ values in the interval $(f-d\delta f,f)$, then the fiber bundle cannot, at any of these previous $f$ values, withstand the external load that forces the fiber with threshold $f$ to break. The probability that there are precisely $h$ fiber thresholds in the interval $(f-\delta fd,f)$ equals

$$\frac{(ad)^h}{h!}e^{-ad}.$$

Dividing the interval into $d$ subintervals each of length $\delta f$, the probability $p_{h,d}$ that these conditions are fulfilled is exactly given by the solution of the combinatorial problem in Appendix A: $p_{h,d}=1-h/d$. Summing over the possible values of $h$, we obtain the probability that the avalanche cannot have started with the failure of a fiber with any of the $d$ nearest-neighbor threshold values below $f$:

$$P_b(f|d)=\sum_{h=0}^{d-1}\frac{(ad)^h}{h!}e^{-ad}\left(1-\frac{h}{d}\right)$$

$$=(1-a)e^{-ad}\sum_{h=0}^{d-1}\frac{(ad)^h}{h!}+\frac{(ad)^d}{d!}e^{-ad}. \quad (12)$$

Finally we take the limit $d\to\infty$, for which the last term vanishes. For $a>1$ the sum must vanish since the left-hand side of Eq. (12) is non-negative, while the factor $(1-a)$ is negative. For $a<1$, on the other hand, we find

$$P_b(f)=\lim_{d\to\infty}P_b(f|d)=1-a, \quad (13)$$

where $a=a(f)$. The physical explanation of the different behavior for $a>1$ and $a\leqslant1$ is straightforward: The maximum of the total force on the bundle occurs at $f_c$ for which $a(f_c)=1$ [see Eqs. (7) and (9)], so that $a(f)>1$ corresponds to $f$ values almost certainly involved in the final catastrophical burst. The region of interest for us is therefore when $a(f)\leqslant1$, where avalanches on a microscopic scale occur. This is in accordance with what we found in the beginning of this section, viz. that the burst of a fiber with threshold $f$ leads immediately to an average number $a(f)$ of additional failures.

Summing up, we obtain the probability that the fiber with threshold $f$ is the first fiber in an avalanche of size $\Delta$ as the product

$$\Phi(f)=\phi(\Delta,f)P_b(f)=\frac{\Delta^{\Delta-1}}{\Delta!}a(f)^{\Delta-1}e^{-a(f)\Delta}[1-a(f)], \quad (14)$$

where $a(f)$ is given by Eq. (9),

$$a(f)=\frac{fp(f)}{1-P(f)}.$$

Since the number of fibers with threshold in $(f,f+\delta f)$ is $Np(f)df$, the burst distribution is given by

$$\frac{D(\Delta)}{N}=\frac{1}{N}\int_0^{f_c}\Phi(f)p(f)df$$
$$=\frac{\Delta^{\Delta-1}}{\Delta!}\int_0^{f_c}a(f)^{\Delta-1}e^{-a(f)\Delta}[1-a(f)]p(f)df. \quad (15)$$

For large $\Delta$ the maximum contribution to the integral comes from the neighborhood of the upper integration limit, since $a(f)e^{-a(f)}$ is maximal for $a(f)=1$, i.e., for $f=f_c$. Expansion around the saddle point and integration yields the asymptotic behavior

$$D(\Delta)/N\propto\Delta^{-5/2}, \quad (16)$$

universal for those threshold distributions for which the assumption of a single maximum of $\langle F\rangle(f)$ is valid.

Note that if the experiment had been stopped before complete breakdown, at a force per fiber $f_m<f_c$, the asymptotic behavior would have been *exponential* rather than a power law:

$$D(\Delta)/N\propto\Delta^{-5/2}e^{-[a(f_m)-1-\ln a(f_m)]\Delta}. \quad (17)$$

In the form

$$D(\Delta)\propto\Delta^{-\eta}e^{-\Delta/\Delta_0}\quad \text{with}\quad \Delta_0\propto(f_c-f)^{-\nu}, \tag{18}$$

the breakdown process is similar to a critical phenomena with a critical point at total breakdown [13]. The distribution follows a power law with index $\eta=\frac{5}{2}$ with a cutoff that diverges at total failure with an index $\nu=\frac{1}{2}$.

What happens when the average strength $\langle F\rangle(f)$ curve does not have a unique maximum? If it has several parabolic maxima, and the absolute maximum does not come first (i.e., at the lowest $f$ value), then there will be several avalanches of macroscopic size in the sense that a finite fraction of the $N$ fibers break simultaneously [14]. The asymptotics (16) is thereby unaffected, however. We turn next to threshold distributions that are more interesting because they lead to different asymptotics.

### B. Strong threshold distributions

Rather than consider bundle strength functions $\langle F\rangle(f)$ with several parabolic maxima, we study now cases in which there is no such maximum. We are particularly interested in the asymptotics of the burst distributions.

Model examples of such threshold distributions are

$$P(f)=\begin{cases}0 & \text{for}\quad f\leqslant f_0\\ 1-[1+(f-f_0)/f_r]^{-\alpha} & \text{for}\quad f>f_0.\end{cases} \tag{19}$$

Here $\alpha$ and $f_0$ are positive parameters, and $f_r$ is a reference quantity which for simplicity we set equal to unity in the following. These distributions are all characterized by diverging moments. When $\alpha\leqslant 1$, even the first moment — the mean — as well as all other moments diverge. This class of threshold distributions are rich enough to exhibit several qualitatively different avalanche distributions.

The corresponding macroscopic bundle strength per fiber is, according to Eq. (6),

$$\frac{\langle F\rangle(f)}{N}=\begin{cases} f & \text{for}\quad f\leqslant f_0\\ \dfrac{f}{(1+f-f_0)^{\alpha}} & \text{for}\quad f>f_0.\end{cases} \tag{20}$$

In Fig. 3 some threshold distributions $p(f)$ and the corresponding macroscopic force curves $\langle F\rangle(f)$ are sketched. We note that when $\alpha\to 1$, the plateau in Eq. (20) becomes infinitely wide.

The distribution of avalanche sizes is given by Eq. (15). In the present case the function $a(f)$ takes the form

$$a(f)=\frac{fp(f)}{1-P(x)}=\frac{\alpha f}{1+f-f_0}. \tag{21}$$

A simple special case is $f_0=1$, corresponding to

$$p(f)=\alpha f^{-\alpha-1}\quad \text{for}\quad f\geqslant 1,$$

since then function (21) is independent of $f$:

$$a(f)=\alpha.$$

This at once gives

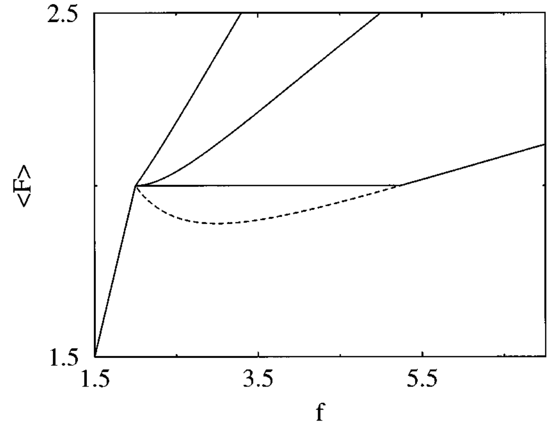


FIG. 3. The threshold distribution density $p(f)$ and the macroscopic bundle strength $\langle F\rangle(f)$ for distribution (17), with $f_0=2f_r$, and for $\alpha=\frac{1}{3}$ (upper curve), $\frac{1}{2}$ (middle curve), and $\frac{2}{3}$ (lower curve). The broken part of the $\alpha=2/3$-curve is unstable and the macroscopic bundle strength will follow the solid line.

$$\frac{D(\Delta)}{N}=\frac{1-\alpha}{\alpha}\frac{\Delta^{\Delta-1}}{\Delta!}[\alpha e^{-\alpha}]^{\Delta}\simeq\frac{1-\alpha}{\alpha\sqrt{2\pi}}\Delta^{-3/2}[\alpha e^{1-\alpha}]^{\Delta}. \tag{22}$$

In other cases it is advantageous to change integration variable in Eq. (15) from $f$ to $a$:

$$\frac{D(\Delta)}{N}=\frac{\Delta^{\Delta-1}}{e^{\Delta}\Delta!}\frac{1}{\alpha^{\alpha-1}(1-f_0)^{\alpha}}\int_{\alpha f_0}^{\alpha}(\alpha-a)^{\alpha-1}\times(1-a)a^{-1}(ae^{1-a})^{\Delta}da. \tag{23}$$

The asymptotics for large $\Delta$, beyond the $\Delta^{-3/2}$ dependence of the prefactor, is determined by the $\Delta$-dependent factor in the integrand. The maximum of $ae^{1-a}$ is unity, obtained for $a=1$, and the asymptotics depends crucially on whether $a=1$ falls outside the range of integration, or inside (including the border). If the maximum falls inside the range of integration the $D(\Delta)\propto\Delta^{-5/2}$ dependence remains. A special case of this is $\alpha=1$, for which the maximum of the integrand is located at the integration limit and the macroscopic force has a ''quadratic'' maximum *at infinity*.

Another special case is $\alpha f_0=1$ (and $\alpha<1$), for which again the standard asymptotics $\Delta^{-5/2}$ is valid. In this instance the macroscopic force has a quadratic *minimum* at $f=f_0$ (see Fig. 3 for $\alpha=\frac{1}{2}$), and critical behavior arises just as well from a minimum as from a maximum.

In the remaining cases, in which $a=1$ is not within the range of integration in Eq. (23), the avalanche distribution is always a power law with an exponential cutoff,

$$\frac{D(\Delta)}{N}\simeq\Delta^{-\xi}A^{\Delta}. \tag{24}$$

Here, however, $\xi$ and $A$ depend on the parameter values $f_0$ and $\alpha$. This is easy to understand. Since

$$\frac{da(f)}{df}=\frac{\alpha(1-f_0)}{(1+f-f_0)^2}, \tag{25}$$

TABLE I. Asymptotic behavior of the burst distribution for strong threshold distributions in the global model.

| Parameters | Asymptotics |
|---|---|
| $0\leq f_0<1,\alpha<1$ | $\Delta^{-(3/2)-\alpha}(\alpha e^{1-\alpha})^\Delta$ |
| $0\leq f_0<1,\alpha=1$ | $\Delta^{-5/2}$ |
| $f_0=1,\alpha<1$ | $\Delta^{-3/2}(\alpha e^{1-\alpha})^\Delta$ |
| $1<f_0<\alpha^{-1}$ | $\Delta^{-5/2}(\alpha f_0 e^{1-\alpha f_0})^\Delta$ |
| $1<f_0=\alpha^{-1}$ | $\Delta^{-5/2}$ |
| $1<\alpha^{-1}<f_0$ | $\Delta^{-5/2}e^{-\Delta/\Delta_0}$ |

we see that $a(f)$ is a monotonically decreasing function for $f_0>1$, so that the maximum of $ae^{1-a}$ is obtained at the lower limit $f=f_0$, where $a=\alpha f_0$. The asymptotics

$$D(\Delta)\propto\Delta^{-5/2}(\alpha f_0 e^{1-\alpha f_0})^\Delta \tag{26}$$

follows.

This is true merely for $\alpha f_0<1$, however. For $\alpha f_0>1$ the macroscopic force $\langle F\rangle(f)$ *decreases* near $f=f_0$, so that a macroscopic burst takes place at a force $f_0$ per fiber, and stabilization is obtained at a larger force $f_1$ (Fig. 3). The subsequent bursts have an asymptotics

$$D(\Delta)\propto\Delta^{-5/2}(a(f_1)e^{1-a(f_1)})^\Delta \tag{27}$$

determined by the neighborhood of $f=f_1$.

For $f_0<1$, the maximum of $ae^{1-a}$ is obtained at $f=\infty$, leading to the asymptotics

$$D(\Delta)\propto\Delta^{-(3/2)-\alpha}(\alpha e^{1-\alpha})^\Delta, \tag{28}$$

reflecting the power-law behavior of the integrand at infinity.

The results are summarized in Table I. Note that the $f_0=1$ result (22) cannot be obtained by putting $f_0=1$ in Eq. (26), since in Eq. (23) the order of the limits $\Delta\to\infty$ and $f_0\to1$ is crucial.

## III. LOCAL MODEL

The assumption of global load sharing among surviving fibers is often unrealistic, since fibers in the neighborhood of the failed fiber are expected to take most of the load increase. The extreme form for local load redistribution is that all extra stresses caused by a fiber failure are taken up by the *nearest-neighbor* surviving fibers.

The simplest geometry is one-dimensional so that the $N$ fibers are ordered linearly, without or with periodic boundary conditions (Fig. 1). In this case precisely two fibers, one on each side, take up, and divide equally, the extra stress. At a total force $F_{\rm tot}$ on the bundle the force on a fiber surrounded by $n_l$ previously failed fibers on the left-hand side, and $n_r$ on the right-hand side, is then

$$\frac{F_{\rm tot}}{N}\left(1+\frac{1}{2}(n_l+n_r)\right)=x(2+n_l+n_r). \tag{29}$$

Here

$$x=\frac{F_{\rm tot}}{2N}, \tag{30}$$

one-half the force-per-fiber, is a convenient variable to use as the driving *force parameter*. This model has been discussed previously [8–10,15–17] for a different purpose. Preliminary studies [11,12] of the avalanche distribution for some threshold strength distributions have not yielded analytical results but simulation results that show convincingly that the local model is *not* in the same univerality class as the global model.

In order to obtain explicit results we assume for the fiber strengths the simplest possible case, a *uniform* threshold distribution. In units of the maximum threshold,

$$P(f)=\begin{cases} f & \text{for } 0\leq f<1 \\ 1 & \text{for } f\geq 1. \end{cases} \tag{31}$$

Avalanches in the local and the global models have different characters. In the local model an avalanche unroll with one failure acting as the seed. If many neighboring fibers have failed, the load on the fibers on each side is high, and if they burst the load on the new neighbors will be even higher, etc. In this way a weak region in the bundle may be responsible for the failure of the whole bundle. For a large number $N$ of fibers the probability of a weak region somewhere is high, and this explains in a qualitative way that the maximum load the bundle are able to carry does not increase proportional to $N$, but slower than linear.

The load distribution rule (29) implies that an avalanche of size $\Delta$ does necessarily lead to a complete breakdown of the whole bundle if the external force is too high, i.e., if $x$ exceeds a critical value $x_{\rm max}$. Since here a fiber can at most take a load of unity, we have

$$x_{\rm max}=\frac{1}{\Delta+2}. \tag{32}$$

The strategy of the derivation is to first establish a set of recursion relations between quantities that give probabilities of certain configurations at fixed external force, i.e., at fixed $x$. Below (Sec. III B), we connect this with the size distribution of avalanches for all $x$ up to the critical value $x_{\rm max}$.

### A. Recursion relations

We will use the terminology that the *magnitude* of an avalanche is the number of failing fibers in the avalanche, and the *length* of an avalanche is the number of fibers between the nearest surviving fibers on each side of the avalanche. The length can be larger than the magnitude since it may include fibers that have failed in previous avalanches.

We define $S(l;x)$, the *gap probability*, to be the probability (at given force parameter $x$) that in a selected region of $l$ consecutive fibers all fibers have failed, assuming the two fibers on each side to be intact. We let $S(0;x)=1$ by definition.

Another central quantity is the probability density $p(l,a;x)$. We define it by selecting a region of $l$ consecutive fibers, and let $p(l,a;x)dx$ be the probability that a force increase from $x$ to $x+dx$ leads to an avalanche of this length $l$ and of magnitude $a$.

The state at force parameter $x$ that all $L$ fibers have failed must have appeared for some force parameter in the range

$(0,x)$, and by a burst of some magnitude $a$ in the range $(1,L)$. Thus

$$S(L;x)=\sum_{a=1}^{L}\int_0^x p(L,a;y)dy. \tag{33}$$

Let us now obtain expressions for the probability density $p(L,a;x)$, first for the special case that the magnitude $a$ is unity. Just one fiber fails in this burst, and in an avalanche of length $L$ therefore $L-1$ of the neighboring fibers must already have failed. By Eq. (29), the force on the fiber just before it fails is $(L+1)x$, and for the uniform distribution the probability that it fails due to a force parameter increase $dx$ is just $(L+1)dx$. The probability of the burst of magnitude 1 to occur when $x\rightarrow x+dx$ is this probability of failure of the single remaining fiber, $(L+1)dx$, times the probability that the $L-1$ neighbors have already failed. The latter is given by the appropriate gap probabilities. Since the position of the failing fiber is arbitrary in the interval, we have

$$p(L,1;x)dx=\sum_{i=0}^{L-1} S(L-i-1;x)(L+1)dxS(i;x). \tag{34}$$

We next consider expressions for the probability $p(L,a;x)$ with an internal avalanche of magnitude $a$ larger than unity. For that purpose we introduce two new quantities: Let $p_l(L+1,a;x)dx$ be the probability that a fiber fails because a force parameter increase $x\rightarrow x+dx$ starts, on its right-hand side, an avalanche of magnitude $a$ (not counting the ultimate fiber on the left-hand side) and of length $L$. Similarly $p_r(L+1,a;x)dx$ is the probability that a fiber fails because the force parameter increase $x\rightarrow x+dx$ starts, on its left-hand side, an avalanche of magnitude $a$ and of length $L$.

Consider the event described by $p(L,a;x)$, and let the last of the $a$ fibers that fail be fiber $\mathcal{F}$. The force distribution mechanism in the local model implies that $\mathcal{F}$ is either the leftmost or the rightmost of the $a$ fibers. The first possibility implies that the increase $x\rightarrow x+dx$ induces the first failure to the right of $\mathcal{F}$, which starts an avalanche of magnitude $a-1$ and length $i$, say, to the right of $\mathcal{F}$. Here $a-1\leq i\leq L-1$, of course, and $\mathcal{F}$ must have $L-i-1$ previously failed fibers on its left-hand side.

Including all possibilities, we have

$$p(L,a;x)=\sum_{i=a-1}^{L-1}[S(L-i-1;x)p_l(i+1,a-1;x)+p_r(i+1,a-1;x)S(L-i-1;x)], \tag{35}$$

where the second term represents events in which the first failure occurs to the *left* of $\mathcal{F}$.

On the other hand we want to express $p_l(l+1,a;x)$ and $p_r(l+1,a;x)$ in terms of previously defined quantities. For magnitude $a=1$ this is relatively simple. Let the single fiber (call it $\mathcal{G}$) that starts the process have $n$ already failed fibers to the right and $l-n-1$ fibers to the left, with $0\leq n\leq l-1$. The probability that fiber $\mathcal{G}$ fails under the load increase $x\rightarrow x+dx$ is $(l+1)dx$ for the uniform threshold distribution. Since the failure of fiber $\mathcal{G}$ causes a load increase $(n+1)x$ on the left fiber, the probability of its failure is $(n+1)x$. When all possible positions of $\mathcal{G}$ are taken into account, we have

$$p_l(l+1,1;x)dx=\sum_{n=0}^{l-1} S(l-n-1;x)(l+1)dx\times S(n;x)(n+1)x. \tag{36}$$

Similarly,

$$p_r(l+1,1;x)=\sum_{n=0}^{l-1} S(l-n-1;x)(l+1)S(n;x)(l-n)x=p_l(l+1,1;x). \tag{37}$$

The corresponding expressions for $p_r(L+1,a;x)$ and $p_l(L+1,a;x)$, with $a$ larger than unity, are more complicated. The internal avalanche started by $x\rightarrow x+dx$ proceeds so that the final failure is either the leftmost or the rightmost of the $a$ fibers, or both. If both go simultaneously, we make the arbitrary definition that in such a case the right-hand neighbor fails first. This secures a unique sequential ordering of failures.

Consider first $p_r(L+1,a+1;x)$, and denote the last surviving fiber on the right-hand side as $\mathcal{F}$. Assume first that the rightmost of the $a$ internal fibers fails last, and let this fiber have $i$ fibers on its left-hand side and $L-i-1$ failed fibers on the right-hand side. The probability that this right-hand side fiber fails under $x\rightarrow x+dx$ with an internal avalanche of magnitude $a$ and length $i$ is just $p_r(i+1,a;x)dx$, and the probability of finding $L-i-1$ failed fibers on the right-hand side is given by the gap probability $S(L-i-1;x)$. After the rightmost internal fiber has failed the load increase on the $\mathcal{F}$ is $(i+1)x$, which also equals its probability of failure. The other alternative is that the leftmost of the $a$ internal fibers fails last, with, say, $i$ fibers on its right-hand side, and $L-i-1$ failed fibers on its left. Then the extra load increase on $\mathcal{F}$, and hence its probability of failure, is $(L-i)x$. Including all possible positions $i$ we end up with

$$p_r(L+1,a+1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)[p_r(i+1,a;x)(i+1)x+p_l(i+1,a;x)(L-i)x]. \tag{38}$$

By a similar argument the corresponding expression for $p_l(L+1,a+1;x)$ is built up. However, when in this case the rightmost of the internal fibers fails last we must add the probability that the leftmost and the rightmost fibers are simultaneously overburdened. Letting $p_2(i+2,a;x)dx$ be the probability that an avalanche of length $i$ and size $a$ makes *both* neighbor fibers fail, we have

$$p_l(L+1,a+1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)[p_l(i+1,a;x)(i+1)x+p_r(i+1,a;x)(L-i)x+p_2(i+2,a;x)]. \tag{39}$$

Finally, the recursion relations for $p_2$ close the set. One sees easily that when the failure of the two end fibers is caused by a single internal fiber burst due to the force increase from $x$ to $x+dx$, we have

$$p_2(L+2,1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)(L+1)\times S(i;x)(i+1)x(L-i)x. \tag{40}$$

Here $(L+1)dx$ is the probability that the single fiber fails, $(i+1)x$ is the probability that the $(i+1)$ new failures on the right makes the left-hand-side fiber break, while $(L-i)x$ is the probability that the $(L-i)$ new failures on the left makes the right-hand-side fiber break.

When the failure of the two end fibers are caused by an internal avalanche involving $a>1$ fibers we may argue along the same lines as for $p_l$, with the result

$$p_2(L+2,a+1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)[p_l(i+1,a;x)(i+1)x(L-i)x+(p_r(i+1,a;x)(L-i)x+p_2(i+2,a;x))(i+1)x]. \tag{41}$$

We can simplify the set of equations somewhat by introducing the sum $p_s=p_l+p_r$, with the results

$$S(L;x)=\sum_{a=1}^{L} \int_0^x p(L,a;y)dy, \tag{42}$$

$$p(L,1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)(L+1)S(i;x), \tag{43}$$

$$p(L,a+1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)p_s(i+1,a;x), \tag{44}$$

$$p_s(L+1,1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)(L+1)S(i;x)(L+1)x, \tag{45}$$

$$p_s(L+1,a+1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)[p_s(i+1,a;x)+p_2(i+2,a;x)] \tag{46}$$

$$p_2(L+2,1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)(L+1)S(i;x)\times(i+1)x(L-i)x, \tag{47}$$

$$p_2(L+2,a+1;x)=\sum_{i=0}^{L-1} S(L-i-1;x)(i+1)x[p_s(i+1,a;x)\times(L-i)x+p_2(i+2,a;x)]. \tag{48}$$

Starting with $S(0;x)=1$, one easily proves by induction the following $x$ dependence of all quantities involved [18]:

$$S(L;x)=S(L)x^L, \tag{49}$$

$$p(L,a;x)=p(L,a)x^{L-1}, \tag{50}$$

$$p_s(L+1,a;x)=p_s(L+1,a)x^L, \tag{51}$$

$$p_\Delta(L+2,a;x)=p_\Delta(L+2,a)x^{L+1}. \tag{52}$$

In $x$-independent form the recursion relations then take the form (with $a>0$)

$$S(L)=L^{-1}\sum_{a=1}^{L} p(L,a), \tag{53}$$

$$p(L,1)=\sum_{i=0}^{L-1} S(L-i-1)(L+1)S(i), \tag{54}$$

$$p(L,a+1)=\sum_{i=0}^{L-1} S(L-i-1)p_s(i+1,a), \tag{55}$$

$$p_s(L+1,1)=\sum_{i=0}^{L-1} S(L-i-1)(L+1)S(i)(L+1), \tag{56}$$

$$p_s(L+1,a+1)=\sum_{i=0}^{L-1} S(L-i-1)[p_s(i+1,a)+p_2(i+2,a)], \tag{57}$$

$$p_2(L+2,1)=\sum_{i=0}^{L-1} S(L-i-1)(L+1)S(i)(i+1)(L-i), \tag{58}$$

$$p_2(L+2,a+1)=\sum_{i=0}^{L-1} S(L-i-1)(i+1)[p_s(i+1,a)(L-i)+p_2(i+2,a)]. \tag{59}$$

Let us finally note that the feature of the uniform distribution that makes the derivation simpler than for other distributions is that the probability for failure of a fiber is given by the load increase, *independent* of the actual load level.

We can now calculate recursively $S(L;x)$ and $p(L,a;x)$ for integer $L$ and $a$. By Eqs. (49)–(52) the $x$ dependence is trivial.

### B. Asymptotic burst distribution

In order to use the quantitative information obtained above we must first determine the survival probability $P_s(N,x)$ that a fiber bundle is able to tolerate a force per fiber equal to $2x$. Noting that in this model avalanches are local phenomena, and that two failed fibers are only correlated when all fibers in between have failed, the survival probability $P_s(N,x)$ is expected to depend *exponentially* on the length $N$ for large $N$, so that

$$\lim_{N\to\infty} N^{-1}\ln P_s(N,x) = -t(x) \tag{60}$$

is finite. The exponential form of the survival probability is discussed and confirmed in other studies [15,17,19].

We assume periodic boundary conditions, and number the fibers from an arbitrary starting point. We define $P_f(n,L;x)$ to be the probability, at force parameter $x$, that among the $n$ first fibers there is no fatal burst, and that the last $L$ fibers of these have all failed. Fiber number $n+1$ is assumed to hold. We will now establish a recursion relation between the $P_f(n,L;x)$.

Consider a region of $n+1+L$ fibers in which no fatal burst has occurred, and where the last $L$ fibers have failed. The probability of this configuration is $P_f(n+1+L,L;x)$. In the region to the left of fiber number $n+1$, let the length of the region of broken fibers that contain fiber number $n$ be $i$, where $i$ may take all values between zero (if fiber number $n$ is intact) and $M(x)=[x^{-1}-2]$. The region to the right of fiber number $n+1$ has $L$ broken fibers, and the probability of this is $S(L;x)$. This gives the recursion relation

$$P_f(n+1+L,L;x) = \sum_{i=0}^{M(x)-L} P_f(n,i;x)[1-(i+L+2)x]S(L;x). \tag{61}$$

The last factor is the probability that fiber number $n+1$, which has $i+L$ failed neighbors, holds.

Insertion of the product form $P_f(n,L;x)\simeq t(x)^n P_f(L;x)$ into Eq. (61) yields the following equations for $P_f(i;x)$:

$$P_f(L;x) - \sum_{i=0}^{M(x)} [1-(i+L+2)x]S(L;x)t(x)^{-L-1}P_f(i;x). \tag{62}$$

It is consistent to let $P_f(0;x)=1$. Since $L$ may take the values $0,1,\ldots,M(x)$, Eq. (62) is a set of $M(x)+1$ homogeneous equations for the $M(x)+1$ quantities $P_f(i;x)$. The system determinant of the equation set must vanish, and this determines $t(x)$ for a given force parameter $x$. With $P_f(0;x)=1$ all quantities can then be determined. The practical solution procedure is by iteration.

From the definitions of $P_f(n,L;x)$ and $S(L;x)$ it follows that the ratio

TABLE II. The burst distribution $D(\Delta)$ for the local model with a bundle of $N=20\,000$ fibers. The simulation results are based on 4 000 000 samples. The calculated values are based on Eq. (65), and have been multiplied by 4 000 000.

| $\Delta$ | Simulation | Calculation |
|---|---|---|
| 1 | 8 327 378 752 | 8 327 331 808 |
| 2 | 491 305 573 | 491 331 178 |
| 3 | 72 126 803 | 72 114 644 |
| 4 | 17 179 080 | 17 180 414 |
| 5 | 5 590 887 | 5 591 243 |
| 6 | 2 243 916 | 2 243 012 |
| 7 | 1 030 833 | 1 031 678 |
| 8 | 515 309 | 515 310 |
| 9 | 268 589 | 268 139 |
| 10 | 140 911 | 140 751 |
| 11 | 72 251 | 72 701 |
| 12 | 36 525 | 36 277 |
| 13 | 17 523 | 17 285 |
| 14 | 8015 | 7835 |
| 15 | 3352 | 3392 |
| 16 | 1442 | 1418 |
| 17 | 559 | 579 |
| 18 | 223 | 233 |
| 19 | 90 | 93.8 |
| 20 | 40 | 37.5 |
| 21 | 18 | 15.0 |
| 22 | 10 | 6.0 |
| 23 | 1 | 2.4 |
| 24 | 2 | 1.0 |
| 25 | 0 | 0.4 |

$$\frac{P_f(n,L;x)}{S(L;x)}$$

is the probabilility, at force parameter $x$, that among the first $n$ fibers there is no fatal burst, *given* that there are $L$ failed fibers on the right-hand side. Then

$$\frac{P_f(n,L;x)}{S(L;x)}p(L,\Delta;x)dx \tag{63}$$

is the probability that an increase of the force parameter from $x$ to $x+dx$ starts an avalanche of size $\Delta$ and length $L$, so that afterwards there is no fatal burst among the $n$ fibers on the left-hand side.

Finally we want to determine the probability for a burst of size $\Delta$ in a system of $N$ fibers in a ring configuration (Fig. 1). On the left of a selected fiber **f** we consider a region of $n$ fibers, and on the right a region of $N-n-1$ fibers. The probability that the force parameter increase $x\to x+dx$ induces a burst of size $\Delta$ and length $L_1$ to the left of **f** that holds is given by Eq. (63). Here $\Delta < L_1 \leq n$, of course. On the right-hand side of **f** a number $L_2$ fibers adjacent to **f** may have failed. (Here $L_2$ is less than the remaining number $N-n-1$ of fibers.) The probability of such a configuration (with no fatal burst) is $P_f(N-n-1,L_2;x)$. We must also

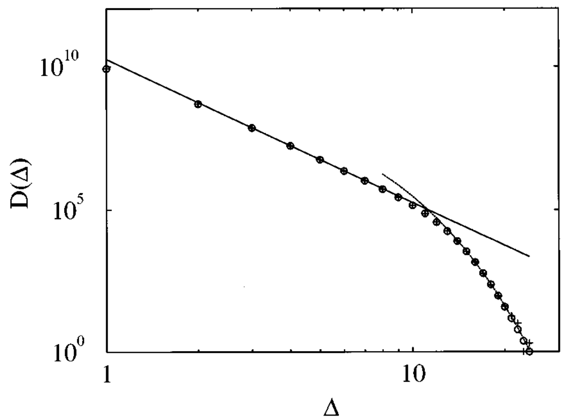


FIG. 4. Burst distribution in local model as found numerically for 4 000 000 samples with $N=20000$ fibers (+), and calculated from Eqs. (65) (○). The straight line shows the power law $\Delta^{-5}$, and the broken curve the function $\exp(-\Delta/\Delta_0)$ with $\Delta_0=1.1$. Note the small value of $\Delta_0$.

take into account that the fiber **f** itself, with $L_1+L_2$ failed neighboring fibers, must hold, the probability of which is $[1-(L_1+L_2+2)x]$.

When we take this together, sum over the possible values of $L_1$, $L_2$ and $n$, and integrate over $x$, we obtain

$$D(\Delta)=\int_0^{1/(\Delta+2)}\sum_{n=1}^{N}\sum_{L_1=\Delta}^{M(x)}\sum_{L_2=0}^{M(x)-L_1}\frac{P_f(n,L_1;x)}{S(L_1;x)}\times p(L_1,\Delta;x)P_f(N-n-1,L_2;x)\times[1-(L_1+L_2+2)x]dx. \tag{64}$$

Using the product property $P_f(n,L;x)\simeq t(x)^N P_f(L;x)$ the sum over $n$ simply yields a factor $N$, and we find

$$\frac{D(\Delta)}{N}=\int_0^{1/(\Delta+2)}\sum_{L_1=0}^{M(x)}\sum_{L_2=0}^{M(x)-L_1}\frac{P_f(L_1;x)}{S(L_1;x)}\times p(L_1,\Delta;x)P_f(L_2;x)t(x)^{N-1}\times[1-(L_1+L_2+2)x]dx. \tag{65}$$

This may now be evaluated. The results for a bundle of $N=20\,000$ are shown in Table II, together with simulation results for 4 000 000 bundles, each having 20 000 fibers. The agreement between the simulation data and the theoretical data is, as we see, extremely satisfactory.

An analysis of the burst distribution obtained for this local model shows that the distribution does not follow a power law except for small values of $\Delta$ (Fig. 4). If one nevertheless does a linear regression analysis on this part of the data set, the effective power would be of the order 5, considerably larger than the ''mean-field'' value $\frac{5}{2}$ for the global model [11,12]. This effective exponent increases with increasing $N$ [12].

### C. Size-dependent bundle strength

Let us now attempt to find a simple estimate for the maximal force per fiber that the fiber bundle can tolerate. In order to do this, we assume that the fatal burst occurs in a region where no fibers have previously failed so that the burst has the same magnitude and length. We know that a single burst of length $\Delta=x^{-1}-2$ is fatal, Eq. (32), so our criterion is simply

$$D(x^{-1}-2)=1. \tag{66}$$

If we take into account that the two fibers adjacent to the burst should hold, and ignore the rest of the bundle, the gap distribution would be

$$N^{-1}D(\Delta)\approx\int_0^{1/(\Delta+2)}[1-(2+\Delta)x]^2p(\Delta,\Delta;x)dx=\frac{2p(\Delta,\Delta)}{\Delta(\Delta+1)(\Delta+2)^{\Delta+1}}. \tag{67}$$

With the abbreviation

$$R_\Delta=\frac{p(\Delta,\Delta)}{(\Delta-1)!},$$

we have

$$D(\Delta)/N\approx\frac{2(\Delta+2)!}{\Delta^2(\Delta+1)^2(\Delta+2)^{\Delta+2}}R_\Delta\simeq\frac{\sqrt{8\pi(\Delta+2)}}{\Delta^2(\Delta+1)^2}e^{-\Delta-2}R_\Delta, \tag{68}$$

using Stirling's formula.

Taking logarithms we have

$$\ln D(\Delta)-\ln N=-(\Delta+2)\left[1+\frac{\ln R_\Delta}{\Delta+2}+O\left(\frac{\ln\Delta}{\Delta}\right)\right]\simeq-(\Delta+2), \tag{69}$$

using result (B9) of Appendix B for $R_\Delta$ when $\Delta$ is large.

The failure criterion (66) then takes the form

$$\ln N\simeq\frac{1}{x}.$$

Since $x=F/2N$, we have the following estimate for the maximum force $F$ that the fiber bundle can tolerate before complete failure:

$$F\simeq\frac{2N}{\ln N}. \tag{70}$$

Due to the assumption that the fatal burst occurs in a region with no previously failed fibers, the numerical prefactor is an overestimate. The size dependence

$$F \propto \frac{N}{\ln N} \tag{71}$$

shows that the maximum load the fiber bundle can carry does not increase proportionally to the number of fibers, but slower. This is to be expected since the probability of finding somewhere a stretch of weak fibers that start a fatal avalanche increases when the number of fibers increases. The $N/\ln N$ dependence agrees with a previous estimate by Zhang and Ding [20] and is also seen in the model with thresholds zero or unity [10,9].

## IV. CONCLUDING REMARKS

In this paper we have discussed burst distributions in fiber bundles with two different mechanisms for load distribution when fibers rupture, viz. global or extremely local load redistributions. The main results are the following.

(i) For the global model the burst distribution follows a universal power law $\Delta^{-5/2}$.

(ii) Deviations from this power-law dependence may, however, occur for exceptional distributions of fiber strengths.

(iii) For the local model and for a uniform distribution of fiber thresholds we show that it is possible, although complicated, to carry through a theoretical analysis of the burst distribution.

(iv) A simulation study for a bundle of 20 000 fibers confirms convincingly the theoretical results.

(v) For the local model the burst distribution falls off with increasing burst size much faster than for the global model, and does not follow a power law.

(vi) The expected maximum load that a bundle with global redistribution mechanism can tolerate increases proportional to the number $N$ of fibers, and proportional to $N/\ln N$ for the local redistribution mechanism.

## APPENDIX A

The combinatorial problem in Sec. II A can be formulated more generally as follows: Let $p_{h,n}$ be the probability that by distributing $h$ nonidentical particles among $n$ numbered boxes, box number 1 will contain no particles, box number 2 will contain at most 1 particle, and in general box number $i$ will contain at most $i-1$ particles.

Since the probability that there are $h-k$ particles in box number $n$ is equal to

$$\binom{h}{k}\left(\frac{1}{n}\right)^{h-k}\left(\frac{n-1}{n}\right)^{k},$$

we must have

$$p_{h,n}=\sum_{k=0}^{h}\binom{h}{k}\left(\frac{1}{n}\right)^{h-k}\left(\frac{n-1}{n}\right)^{k}p_{k,n-1}. \tag{A1}$$

We now prove by induction that

$$p_{h,n}=1-\frac{h}{n}. \tag{A2}$$

Assume that this holds for $p_{h,n-1}$, all $h$. Insertion into the right-hand side of Eq. (A1) gives

$$\begin{aligned}p_{h,n}&=\sum_{k=0}^{h}\binom{h}{k}\left(\frac{1}{n}\right)^{h-k}\left(\frac{n-1}{n}\right)^{k}\left(1-\frac{k}{n-1}\right)\\&=1-\frac{h}{n}\sum_{k=1}^{h}\binom{h-1}{k-1}\left(\frac{1}{n}\right)^{h-k}\left(\frac{n-1}{n}\right)^{k-1}=1-\frac{h}{n},\end{aligned} \tag{A3}$$

in accordance with Eq. (A2). Since Eq. (A2) is valid for $n=2$, the induction is complete. For the application in the text,

$$p_{n-1,n}=\frac{1}{n}$$

is needed.

## APPENDIX B

In Sec. III C an estimate for $p(n,n,)$ was needed. We base it on the recursion relations (55), (57), and (59) for $L=n+1$, $a=n$:

$$p(n+1,n+1)=p_s(n+1,n), \tag{B1}$$

$$p_s(n+2,n+1)=(n+2)p_s(n+1,n)+p_2(n+2,n), \tag{B2}$$

$$p_2(n+3,n+1)=(n+1)p_s(n+1,n)+(n+1)p_2(n+2,n). \tag{B3}$$

We have used that $p_s(n+1,a)$ and $p_2(n+2,a)$ vanish for $a>n$.

It is easy to eliminate $p_s$ by Eq. (B1), and $p_2$ by Eq. (B2), with the result

$$p(n+1,n+1)=2np(n,n)-(n-1)^2p(n-1,n-1). \tag{B4}$$

This is a three-term recursion starting off with $p(1,1)=2$ and $p(2,2)=2p(1,1)$ by Eq. (B4).

With

$$R_n=\frac{p(n,n)}{(n-1)!}, \tag{B5}$$

the recursion takes the form

$$R_{n+1}=2R_n-\left(1-\frac{1}{n}\right)R_{n-1}. \qquad \text{(B6)}$$

Introducing the generating function

$$G(z)=\sum_{n=1}^{\infty} R_n z^n, \qquad \text{(B7)}$$

the recursion (B6) may be transformed to the differential equation

$$\frac{\partial}{\partial z}[G(z)(1-z^2)/z]=G(z),$$

with solution

$$G(z)=\frac{2z}{1-z}e^{z/(1-z)}. \qquad \text{(B8)}$$

Thus the radius of convergence of the power series (B7) is unity, and therefore

$$\lim_{n\to\infty} R_n^{1/n}=1. \qquad \text{(B9)}$$

In fact $R_n \propto n^{-1/4}e^{2\sqrt{n}}$ for large $n$ [21].


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