Power-law sensitivity to initial conditions within a logisticlike family of maps: Fractality and nonextensivity

U. M. S. Costa and M. L. Lyra
Departamento de Física, Universidade Federal de Alagoas, Maceio-AL, Brazil

A. R. Plastino and C. Tsallis*
Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil
(Received 6 January 1997)

Power-law sensitivity to initial conditions, characterizing the behavior of dynamical systems at their critical points (where the standard Liapunov exponent vanishes), is studied in connection with the family of nonlinear one-dimensional logisticlike maps $x_{t+1}=1-a|x_t|^z$ ($z>1$; $0<a\leqslant 2$; $t=0,1,2,\ldots$). The main ingredient of our approach is the generalized deviation law $\lim_{\Delta x(0)\to 0}[\Delta x(t)/\Delta x(0)]=[1+(1-q)\lambda_q t]^{1/(1-q)}$ (equal to $e^{\lambda_1 t}$ for $q=1$, and proportional, for large t, to $t^{1/(1-q)}$ for $q\neq 1$; $q\in\mathbb{R}$ is the entropic index appearing in the recently introduced nonextensive generalized statistics). The relation between the parameter q and the fractal dimension d_f of the onset-to-chaos attractor is revealed: q appears to monotonically decrease from 1 (Boltzmann-Gibbs, extensive, limit) to $-\infty$ when d_f varies from 1 (nonfractal, ergodiclike, limit) to zero. [S1063-651X(97)04907-6]

PACS number(s): 05.45.+b, 05.20.−y, 05.90.+m

I. INTRODUCTION

The standard thermostatistical formalism of Boltzmann and Gibbs (BG) constitutes one of the most successful paradigms of theoretical physics. It provides the link between microscopic dynamics and the macroscopic properties of matter. Inspired in Shannon's information theory [1], Jaynes's [2] reformulation of the BG theory greatly increased its power and scope. Jaynes provided a general prescription for the construction of a probability distribution $f(\mathbf{x})$ ($\mathbf{x}\in\mathbb{R}^d$ stands for a point in the relevant phase space), when the only available information about the system are the mean values of M quantities

$$\langle A_r(\mathbf{x})\rangle \equiv \int A_r(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \quad (r=1,\ldots,M). \tag{1}$$

According to Jaynes, the least biased distribution compatible with the data (1) is the one that maximizes Shannon's information,

$$S_1 \equiv -\int f(\mathbf{x}) \ln f(\mathbf{x}) d\mathbf{x} \tag{2}$$

(the use of the subindex 1 will become transparent later on) under the constraints imposed by the mean values (1) and appropriate normalization

$$\int f(\mathbf{x}) d\mathbf{x} = 1. \tag{3}$$

The well known answer to the above variational problem is provided by the maximum entropy (ME) distribution

$$f_{\mathrm{ME}}(\mathbf{x}) = \frac{1}{Z_1} \exp\left(-\sum_{r=1}^{M} \lambda_r A_r(\mathbf{x})\right), \tag{4}$$

where $\{\lambda_r\}$ are the M Lagrange multipliers associated with the known mean values, and the partition function Z_1 is given by

$$Z_1 \equiv \int \exp\left(-\sum_{r=1}^{M} \lambda_r A_r(\mathbf{x})\right) d\mathbf{x}. \tag{5}$$

Jaynes's prescription can be regarded as a mathematical formulation of the celebrated ''Occam's razor'' principle. In order to obtain a statistical description of a system, given by the distribution $f(\mathbf{x})$, we must employ all and only the available data (1), without assuming any further information we do not actually have.

Jaynes's informational approach allows us to consider more general statistical ensembles than the Gibbs microcanonical, canonical, and macrocanonical ensembles. Also it provides a natural way to treat nonequilibrium situations.

Despite its great success, the Boltzmann-Gibbs-Jaynes formalism is unable to deal with a variety of interesting physical problems such as the thermodynamics of self-gravitating systems, some anomalous diffusion phenomena, Lévy flights and distributions, and turbulence, among others (see [3] for a more detailed list). In order to deal with these difficulties, Jaynes's approach is compatible with exploring the possibility of building up a thermostatistics based upon an entropy functional different from the usual logarithmic entropy. Recently one of us introduced [4] the following generalized, nonextensive entropy form:

$$S_q \equiv \frac{1-\int [f(\mathbf{x})]^q d\mathbf{x}}{q-1}, \tag{6}$$

*Electronic address: tsallis@cat.cbpf.br

where q is a real parameter characterizing the entropy functional S_q. This entropy recovers S_1 as the $q=1$ particular instance and was introduced in order to describe systems where nonextensivity plays an important role; if A and B are two independent systems (in the sense that the probabilities associated with $A+B$ factorize into those of A and B) we straightforwardly verify that $S_q(A+B)=S_q(A)+S_q(B)+(1-q)S_q(A)S_q(B)$. Indeed, nonextensive behavior is the common feature among the above listed problems where the usual statistics fails. The generalized nonextensive thermostatistics has already been applied to astrophysical self-gravitating systems [5], the solar neutrino problem [6], distribution of peculiar velocities of galaxy clusters [7], cosmology [8], two-dimensional turbulence in pure-electron plasma [9], anomalous diffusions of the Lévy [10] and correlated [11] types, long-range magnetic and Lennard-Jones-like systems [12], simulated annealing and other optimization techniques [13], and dynamical linear response theory [14], among others.

The nonextensivity effects displayed by the above listed systems can arise from long-range interactions, long-range microscopic memory, or fractal space-time constraints. Even for dynamical systems that ''live'' in a Euclidean (nonfractal) space, if the subset (of this space) that the system visits (most of the time) during its evolution has a fractal geometry, the generalized thermostatistics might provide a better account of the situation than that provided by the usual statistics. Indeed, it is well known that nonlinear chaotic dynamical systems may have fractal attractors [15]. Two of the most important dynamical quantities usually employed in order to characterize such chaotic systems are the Liapunov exponents [16,17], and the Kolmogorov-Sinai (KS) entropy [18]. In recent work of Tsallis, Plastino, and Zheng [19] (TPZ from here on), generalizations for these quantities inspired in the generalized nonextensive entropy S_q (and its consequences) were introduced. The generalized Liapunov exponent λ_q and generalized KS entropy K_q provide a useful characterization of the dynamics corresponding to critical points where the usual Liapunov exponent vanishes. For these critical cases, the exponential sensitivity to initial conditions is replaced by a power-law one, and the vanishing (standard) Liapunov exponent λ_1 provides but a poor description of the concomitant dynamics. On the contrary, the generalized exponent λ_q appropriately discriminates between the different possible power-law behaviors. TPZ illustrate these ideas with the logistic map. It is of interest to explore this formalism as applied to other nonlinear dynamical systems. In particular, it is of importance to study families of dynamical systems characterized by a set of parameters. Each member of the family will have a different onset-to-chaos critical point, with a corresponding attractor characterized by a Hausdorff fractal dimension d_f and a suitable value of the entropic parameter q. The study of these families will enlighten the relation between q and d_f.

The specific aim of the present paper is to study the relation between the fractal dimension d_f of the onset-to-chaos attractor and the parameter q for the logisticlike family of maps (see [20–22], and references therein),

$$x_{t+1}=1-a|x_t|^z$$

$$(z>1;0<a\leqslant 2;t=0,1,2,\ldots;x_t\in[-1,1]). \quad (7)$$

Note that in the particular case $z=2$ we recover the standard logistic map (in its centered representation).

The paper is organized as follows. In Sec. II we briefly review the q generalizations of the Liapunov exponent and the KS entropy. In Sec. III the results for the logisticlike maps are presented. Our main conclusions are drawn in Sec. IV.

II. GENERALIZED LIAPUNOV EXPONENT AND KS ENTROPY

Let us consider, for a one-dimensional dynamical system, two nearby orbits whose initial conditions differ by the small quantity $\Delta x(0)$. We will assume that the time dependence of the distance between both orbits is given by the ansatz [19]

$$\lim_{\Delta x(0)\to 0}\frac{\Delta x(t)}{\Delta x(0)}=[1+(1-q)\lambda_q t]^{1/(1-q)} \quad (q\in\mathcal{R}), \quad (8)$$

where λ_q is our generalized Liapunov exponent, and q is a real parameter characterizing the behavior of the system. We verify that this equation is identically satisfied for $t=0$ $(\forall q)$, and that $q\neq 1$ yields, for large times, the *power law*

$$\lim_{\Delta x(0)\to 0}\frac{\Delta x(t)}{\Delta x(0)}\sim[(1-q)\lambda_q]^{1/(1-q)}t^{1/(1-q)} \quad (t\to\infty). \quad (9)$$

On the other hand, it is plain that for $q\to 1$ we recover the standard *exponential* deviation law

$$\lim_{\Delta x(0)\to 0}\frac{\Delta x(t)}{\Delta x(0)}=\exp[\lambda_1 t], \quad (10)$$

where λ_1 is just the usual Liapunov exponent. The $q=1$ scenario corresponds to situations with $\lambda_1\neq 0$. These cases describe chaotic behavior $(\lambda_1>0)$ and regular behavior $(\lambda_1<0)$. The generalized exponent λ_q is intended to provide a convenient description of the marginal situations where the usual Liapunov exponent vanishes $(\lambda_1=0)$. In these last cases, we have the power-law sensibility to initial conditions given by Eq. (9) instead of the usual exponential one. The generalized deviation law [Eq. (8)] is inspired in the form of the q-generalized nonextensive canonical distribution, given by [4]

$$p_i=\frac{[1-(1-q)/\beta\epsilon_i]^{1/(1-q)}}{Z_q}, \quad (11)$$

with the generalized partition function being given by

$$Z_q\equiv\sum_i [1-(1-q)\beta\epsilon_i]^{1/(1-q)}, \quad (12)$$

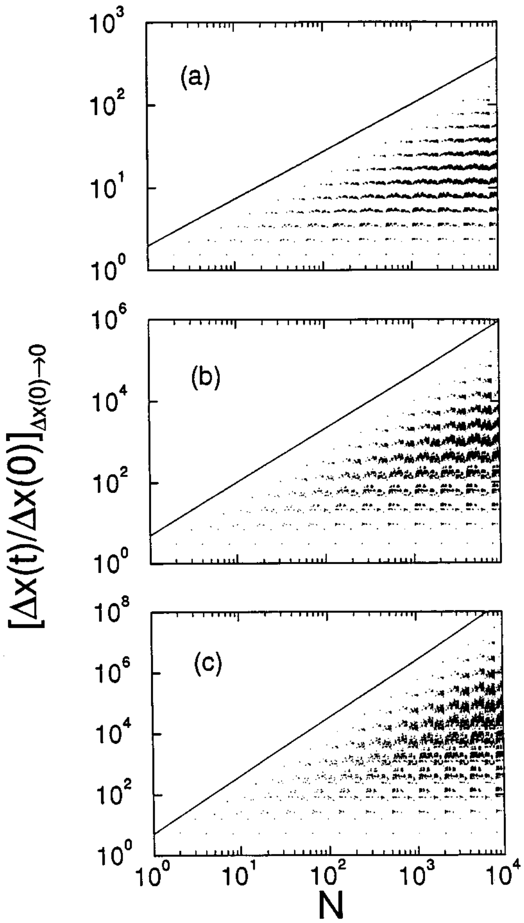

FIG. 1. Log-log plot of $\lim_{\Delta x(0)\to 0}[\Delta x(t)/\Delta x(0)]$ versus the number of iterations N calculated for $x_0=0$ [the slope $1/(1-q)$ is calculated using the upper bound points]: (a) $z=1.25$; (b) $z=2$; (c) $z=3$.

where $\beta\equiv 1/kT$ and $\{\epsilon_i\}$ is the full set of eigenvalues of the Hamiltonian of the system. Notice that, in the limit $q\to 1$, this thermal canonical equilibrium distribution reduces to the ordinary BG one,

$$p_i=\frac{\exp[-\beta\epsilon_i]}{Z_1}, \tag{13}$$

with

$$Z_1\equiv\sum_i \exp[-\beta\epsilon_i]. \tag{14}$$

It is worth remarking that the marginal case with vanishing (standard) Liapunov exponent λ_1 displays a very rich and complex behavior, reminiscent of what happens at the critical point of thermal equilibrium critical phenomena. To just say that $\lambda_1=0$ is a very poor description of its richness, intimately connected to fractality. Indeed, within our generalized formalism, the parameter q provides a characterization of the kind of power-law sensitivity to initial conditions involved, and is expected to be related to the fractal dimension d_f of the corresponding attractor.

In order to discuss the generalized KS entropy, let us consider a partition of phase space in cells with size characterized by a linear scale l. We will study the evolution of an ensemble of identical copies of our system. We assume that all the members of the ensemble start at $t=0$ with initial conditions belonging to one and the same cell. This means that the probability associated to that privileged cell is 1, while the remaining cells of the partition have vanishing initial probabilities. As time goes by, and due to the sensitivity to initial conditions, our ensemble will spread over an increasing number of cells. The standard KS entropy can be regarded as the rate of growth of the Boltzmann-Gibbs entropy associated with the partition probability distribution.

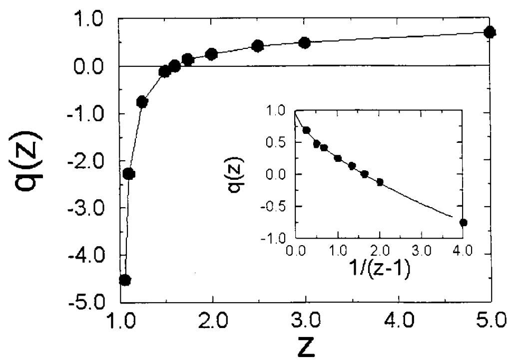

FIG. 2. z dependence of the entropic index q. The inset conveniently represents the $z\to\infty$ neighborhood [the continuous line is the best fitting with a curve $q=1-a_0/(z-1)^{a_1}$; we obtained $a_0=0.75$ and $a_1=0.60$].

Within the generalized nonextensive thermostatistics, the entropy functional for a discrete probability distribution $\{p_i\}$ is given by

$$S_q=\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \quad (q\in\mathcal{R}), \tag{15}$$

which, for equiprobability, becomes

$$S_q=\frac{W^{1-q}-1}{1-q}. \tag{16}$$

The use of Eq. (15), instead of $S_1=-\sum_{i=1}^{W} p_i \ln p_i$, yields (along Zanette's lines [23]) to the following generalization of the Kolmogorov-Sinai entropy:

$$K_q\equiv\lim_{\tau\to 0}\lim_{l\to 0}\lim_{N\to\infty}\frac{1}{N\tau}[S_q(N)-S_q(0)], \tag{17}$$

that under the assumption of equiprobability reduces to

$$K_q=\lim_{\tau\to 0}\lim_{l\to 0}\lim_{N\to\infty}\frac{1}{N\tau}\frac{[W(N)]^{(1-q)}-1}{1-q}. \tag{18}$$

In both Eqs. (17) and (18) we have maintained the traditional $\tau\to 0$ which applies for a continuous time t; it is clear, however, that, in our present case, this limit does not apply since our t is discrete. We must remark that our generalization K_q of the KS entropy is different from the generalizations $K(\beta)$ based upon Renyi information, usually called ‘‘Renyi

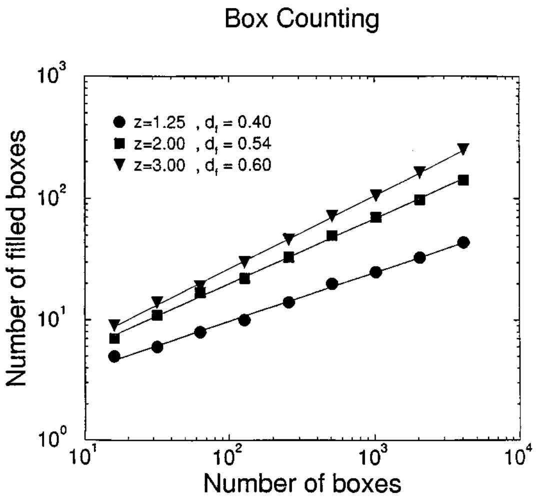

FIG. 3. Box counting procedure for determining the onset-to-chaos fractal dimension d_f for typical values of z.

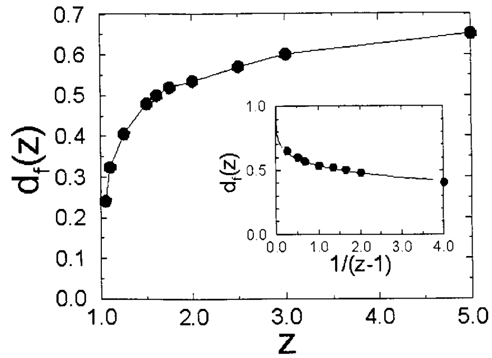

FIG. 4. z dependence of the fractal dimension d_f. The inset conveniently represents the $z\rightarrow\infty$ neighborhood (the continuous line is the best fitting with a curve $d_f = \exp -[b_0/(z-1)^{b_1}]$; we obtained $b_0=0.62$ and $b_1=0.27$).

entropies'' in the literature of thermodynamics of chaotic systems [24] (sometimes the parameter characterizing these generalizations of the KS entropies is called q instead of β [25]; this parameter q should not be confused with our q).

Consistently with the behavior indicated in Eq. (8), we have (along Hilborn's lines [17])

$$W(N)=[1+(1-q)\lambda_q N\tau]^{1/(1-q)}, \tag{19}$$

which, replaced into Eq. (18), immediately yields [for one-dimensional (1D) dynamical systems]

$$K_q=\lambda_q. \tag{20}$$

This relation holds if $\lambda_q>0$ (K_q vanishes if $\lambda_q\leqslant 0$); it constitutes a generalization of the well-known Pesin equality $K_1=\lambda_1$ (if $\lambda_1>0$; $K_1=0$ otherwise), and unifies (within a single scenario for both exponential and power-law sensitivities to initial conditions) the connection between sensitivity and rhythm of loss of information.

III. THE LOGISTICLIKE MAP

Let us now illustrate some of the above concepts by focusing the logisticlike maps (7). These maps are relatively well known and have been addressed on various occasions ([21,22], and references therein). The *topological* properties associated with them (such as the sequence of attractors while varying the parameter a) do not depend on z, but the *metrical* properties (such as Feigenbaum's exponents) do depend on z. We shall exhibit herein that the same occurs with q. Indeed, although quite a lot is known for these maps, their sensitivity to the initial conditions at the onset to chaos has never been addressed as far as we know. As we shall see, for all values of z, the sensitivity is of the *weak* type [19], i.e., power laws instead of the usual exponential ones.

We present now our main numerical results. We computed, as functions of z, parameter q and the critical fractal dimension d_f (determined within the box counting procedure).

In Fig. 1 we exhibit, for typical values of z at its chaotic threshold $a_c(z)$ and using $x_0=0$, a plot of $\ln \lim_{\Delta x(0)\rightarrow 0}\Delta x(t)/\Delta x(0)=\Sigma_{n=1}^{N}\ln[az|x_n|^{z-1}]$ versus $\ln N$, where N is the number of iterations. (Notice that for convenience we use, as argument of the logarithm, not exactly the derivative dx_{t+1}/dx_t, but rather its absolute value.) For each of these plots we see an upper bound whose slope equals $1/(1-q)$ [see Eq. (9)], from which we determine q.

In Fig. 2 we show the behavior of the parameter q as a function of the parameter z characterizing the map. The figure suggests that for $z\rightarrow 1$, q tends to $-\infty$, while in the limit $z\rightarrow\infty$, q approaches 1.

In Fig. 3 we can see, for typical values of z at its chaotic threshold $a_c(z)$, the number of filled boxes as a function of the number of boxes, corresponding to the box counting method employed in order to determine the fractal dimension d_f.

In Fig. 4 the behavior of the fractal dimension d_f of the chaotic critical attractor as a function of z is depicted. We can observe that, as $z\rightarrow 1$, d_f seems to go to 0, while, in the limit $z\rightarrow\infty$, the d_f curve approaches unity.

In Fig. 5 we show the behavior of the parameter q as a function of the fractal dimension d_f. We can see that q displays a monotonically increasing behavior with the fractal dimension d_f. The particular value $q=0$, that describes *linear* sensitivity to initial conditions, corresponds, with notable

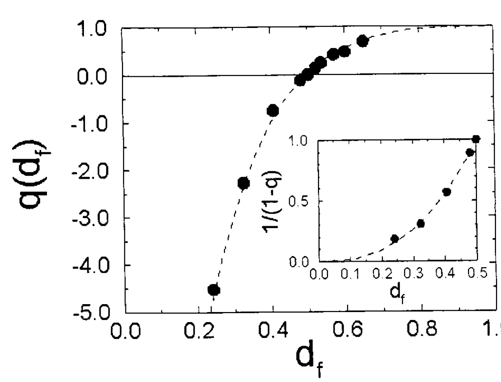

FIG. 5. d_f dependence of the entropic index q; the inset conveniently represents the $d_f\leqslant\frac{1}{2}$ region. Notice that the point $(d_f,q)=(\frac{1}{2},0)$ seems to belong to the curve. The dashed lines are guides to the eye.

numerical accuracy, to the fractal dimension $d_f = 0.5$ (numerically $d_f = 0.50 \pm 0.01$, occurring for $z = 1.609 \pm 0.001$). It is remarkable that as the fractal dimension tends towards 1, the parameter q approaches 1. If this tendency becomes confirmed by analytic results or more powerful numerical work, this would be very enlightening, because in the limit $d_f \to 1$ the attractor loses its fractal nature (in the sense that d_f coincides with the Euclidean dimension $d = 1$), and the usual statistics (i.e., the usual exponential deviation of nearby trajectories), characterized by $q = 1$, would be recovered. On the other extreme, as $d_f \to 0$, q appears to approach $-\infty$, hence $1/(1-q) = 0$, which can be considered as an indication of a possible *logarithmic* sensitivity to the initial conditions. Our results are summarized in Table I.

TABLE I. The asterisk denotes the limiting values suggested by the numerical results; $a_c = 1$ for $z = 1$ also has analytic support (see [21], and references therein); $a_c = 2$ for $z \to \infty$ also has renormalization group support [21].

z	a_c	q	d_f
1	1*	$-\infty$*	0*
1.05	1.081 648 8 . . .	-4.52 ± 0.03	0.24 ± 0.02
1.10	1.124 988 5 . . .	-2.28 ± 0.02	0.32 ± 0.02
1.25	1.209 513 7 . . .	-0.76 ± 0.01	0.40 ± 0.01
1.5	1.295 509 9 . . .	-0.12 ± 0.01	0.48 ± 0.01
1.609	1.323 643 5 . . .	0.00 ± 0.01	0.50 ± 0.01
1.75	1.355 060 7 . . .	0.13 ± 0.01	0.52 ± 0.01
2.0	1.401 155 1 . . .	0.24 ± 0.01	0.53 ± 0.01
2.5	1.470 550 0 . . .	0.41 ± 0.01	0.57 ± 0.01
3.0	1.521 878 7 . . .	0.47 ± 0.01	0.60 ± 0.01
5.0	1.645 533 9 . . .	0.62 ± 0.01	0.65 ± 0.01
∞	2*	1*	1*

IV. CONCLUSIONS

We have exhibited, for a family of logisticlike maps, the behaviors of the entropic parameter q and the fractal dimension d_f of the onset-to-chaos attractor. We showed that, at this critical point, *power* deviation laws for nearby orbits, similar to the ones appearing [19] in the logistic map, are observed. The concomitant value of q is related to the chaotic attractor fractal dimension. It would no doubt be interesting to find out if, for generic nonlinear dynamical systems, q depends only on d_f (support of the visiting frequency function) or also upon other characteristics of the critical attractor, such as the visiting frequency function itself. In order to answer this question, it would be useful to explore the behavior of families of maps whose possible chaotic critical points depend on more than one parameter. Such studies, as well as the application of these concepts to self-organized criticality [26], would be very welcome.

Finally, let us stress that the present study provides a direct and important insight into a problem which has been quite elusive up to now, namely, the microscopic interpretation of the entropic index q characterizing nonextensive statistics. The present results clearly exhibit that what determines q is not the entire phase space within which the system is allowed to evolve (the Euclidean interval $-1 \leq x_t \leq 1$ in the present examples), but the (possibly fractal) subset of it onto which the system is driven by its own dynamics. Consistently, whenever the relevant fractal dimension approaches its associated Euclidean value ($d = 1$ in the present case), extensivity (i.e., $q = 1$) and standard BG thermostatistics naturally become, as is well known, the appropriate standpoints.

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