Acoustic resonant scattering by an ellipsoid air bubble in a liquid

Zhen Ye
Ocean Acoustics, Institute of Ocean Sciences, Sidney, British Columbia, Canada
and Department of Physics, National Central University, Taiwan, Republic of China

(Received 29 October 1996)

This paper reports a simple analytic formula for the low frequency resonant scattering by an ellipsoid air bubble in a liquid. For a spherical bubble, the result reduces to a previous well-known formula. [S1063-651X(97)02608-1]

PACS number(s): 03.20.+i

Much effort has been devoted to acoustic resonant scattering of air-filled bubbles in past years because of the prominent role played by air bubbles in a variety of situations of great interest, such as generation of noise [1,2], wave propagation [3], and gas transfer [4] in the upper ocean surfaces, modeling scattering by fish swimbladders [5,6], and, more recently, sonoluminescence [7]. Moreover, due to their high acoustic contrast air bubbles are useful tracers for probing many important ocean processes, including turbulence generated by breaking waves, Langmuir circulation, fronts, and internal waves [8].

Previous attention has been mainly focused on air bubbles with spherical geometry. In the resonant regime, where the acoustic wavelength is much larger than the size of bubbles, i.e., low frequencies, the scattering function of such a spherical air bubble is well known, and can be expressed in an analytic form as (e.g., p. 1498 in Ref. [9])

$$f=\frac{a}{\omega_0^2/\omega^2-1-ika}, \tag{1}$$

where a is the radius of the bubble, ω_0 is the resonance frequency, ω is the angular frequency of the incident wave, and k is the acoustic wave number. In this formula, ka is identified as the radiation damping factor.

Few studies, however, have been directed to air bubbles with other geometries. This is because the spherical assumption for air bubbles is sufficient for many cases of interest. Another reason is the difficulty in dealing with nonspherical bubbles. Intuitively, the spherical assumption is only valid for sufficiently small bubbles for which the tension at the bubble-medium interface can sustain deformation. When this condition is not satisfied, investigation of acoustic scattering by nonspherical bubbles is necessary. Indeed, the study of sound scattering from deformed bubbles has become an important area. This is not only because deformed bubbles can be used to model many actual targets such as fish swimbladders, large air bubbles in upper ocean surfaces, and air bubbles embedded in the sediments in ocean bottoms, but the problem of sound scattering by deformed bubbles is theoretically challenging in its own right.

Sound scattering by nonspherical bubbles was first considered by Strasberg [10]. He derived a theoretical expression for the resonance frequency which compares favorably with experimental data. In his study, however, no mention was made about how the scattering amplitude is affected by deformation. It is only recently that more detailed studies of the problem are given [6,11]. Unfortunately, these works have either been purely numerical or involved discomforting approximations for boundary conditions, and are restricted to prolate spheroid air bubbles. The more general case of air-filled ellipsoid bubbles remains untackled.

In this Brief Report, we present a simple approximate solution to acoustic resonant scattering by an ellipsoid air bubble in a liquid. Our method, motivated by an analogy between the resonant scattering and the electrostatics, uses the Kirchhoff theorem for scattered waves and the thermodynamics relations for gases. The resonance frequency and the quality factor (Q) characterizing the broadening of resonance will also be shown in a simple form.

Consider a unit plane wave incident on an ellipsoidal gas bubble at the origin. The incident plane wave can be expressed as $p_i(\vec{r}\,')=e^{i\vec{k}'_i\cdot\vec{r}\,'}$ with suppressed time dependence $e^{-i\omega t}$. The equation for the surface of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

For convenience, the three axes of the ellipsoid are arranged in the order $a\geqslant b\geqslant c$.

The wave equation reads

$$(\nabla^2+k^2)p(\vec{r}\,')=0. \tag{2}$$

The total wave field including the scattered wave is given by the Kirchhoff theorem

$$\left.\begin{matrix}p(\vec{r}\,')\\ 0\end{matrix}\right\}=p_i(\vec{r}\,')+\frac{1}{4\pi}\int_S ds'[p_+\vec{n}\cdot\nabla G(k|\vec{r}\,'-\vec{r}\,'|) -i\omega\rho_l u_n G(k|\vec{r}-\vec{r}\,'|)] \quad \text{for } \vec{r}\,'\left\{\begin{matrix}\text{outside } S\\ \text{inside } S,\end{matrix}\right. \tag{3}$$

where the integration is performed over the surface of the object, r' is taken on S, $\vec{n}$ is a unit outward normal vector to the surface, and G is the usual three-dimensional Green's function $G(r)=\exp(ikr)/r$. In this expression, p_+ and u_n are the total pressure field and the normal velocity on the surface, approached from outside, and ρ_l is the density of the surrounding liquid.

It is clear that the scattered wave p_s is represented by the integral in Eq. (3). In the far field region, where $r \gg r'$, p_s behaves as a spherically spreading wave. And the conventional scattering function $f(\vec{k}_i,\vec{k}_s)$ can be evaluated from (e.g., Eq. (9.125) in Ref. [12])

$$f(\vec{k}_i,\vec{k}_s)=-\frac{1}{4\pi}\int_S ds' e^{-i\vec{k}_s\cdot\vec{r}'}[i\vec{k}_s\cdot\vec{n}p_+ + i\omega\rho_l u_n], \quad (4)$$

where $\vec{k}_i,\vec{k}_s$ are the incident and scattering wave vectors, respectively.

According to Eq. (4), the scattering function $f(\vec{k}_i,\vec{k}_s)$ is determined by the surface values of p_+ and u_n. In the following, we briefly outline the procedure in solving for these two quantities, while a detailed derivation is left to a forthcoming paper [13]. (1) First notice the continuity across the boundary in the absence of viscosity

$$p_g=p_+, \quad u_{g,n}=u_n. \quad (5)$$

The index ''g'' refers to the quantities inside the ellipsoid bubble. For the resonant scattering occurring at low frequencies, i.e., $ka \ll 1$, p_g can be treated as a constant [9]. (2) Analogous to the spherical case [9], the thermodynamics equation gives

$$PV^{\gamma}=\text{const}, \quad (6)$$

for ideal gases in an adiabatic process with γ being the polytropic exponent. (3) Equation (6) gives

$$\frac{dV}{dt}=\int_S u_n ds'=\frac{i\omega V_0}{\gamma P_0}p_g, \quad (7)$$

where V_0 is the volume of the ellipsoid bubble given by $4\pi abc/3$, and P_0 is the hydrostatic pressure. (4) Using Eq. (3) for $\vec{r}=0$ leads to

$$0=1+\frac{1}{4\pi}\int_S ds'[p_+\vec{n}\nabla\cdot G(k|\vec{r}'|)-i\omega\rho_l u_n G(k|\vec{r}'|)]. \quad (8)$$

With

$$G(kr')=\frac{1}{r'}e^{ikr'}, \quad \nabla G(k|\vec{r}'|)=(-1+ikr')\frac{\vec{r}'}{r'^3}e^{ikr'}$$

and a Taylor expansion for $e^{ikr'}$ at low frequencies, Eq. (8) can be further simplified to the first order of kr' as

$$\frac{i\omega\rho_l}{4\pi}\int_S \frac{1+ikr'}{r'}u_n ds'+\frac{p_g}{4\pi}\int_S \frac{\vec{r}'\cdot\vec{n}}{r'^3}ds'=1. \quad (9)$$

(5) According to Strasberg [10], u_n is mathematically equivalent to the charge density on a perfect conductor having the same shape and can be written in the following form for the ellipsoid [14]:

$$u_n=\frac{A}{\sqrt{x^2/a^4+y^2/b^4+z^2/c^4}}, \quad (10)$$

where A is a constant to be determined.

From Eqs. (7), (9), and (10), A and p_g can be solved. Plugging these into Eq. (4), the scattering function can be obtained as

$$f=\frac{aE_c/F(\Phi,m)}{\omega_0^2/\omega^2-1-ikaE_c/F(\Phi,m)}, \quad (11)$$

where

$$\omega_0^2=\frac{3\gamma P_0}{a^2\rho_l}\left[\frac{E_c}{e_b e_c F(\Phi,m)}\right], \quad (12)$$

and the two ratio aspects are defined as $e_b \equiv b/a$, $e_c \equiv c/a$, and $E_b \equiv \sqrt{1-e_b^2}$, $E_c \equiv \sqrt{1-e_c^2}$. $F(\Phi,m)$ is the first kind elliptic function [15]

$$F(\Phi,m)=\int_0^{\Phi}\frac{d\alpha}{\sqrt{1-m^2\sin^2\alpha}}. \quad (13)$$

The two variables in the elliptic function F are given as $\Phi=\sin^{-1}E_c$ and $m=E_b/E_c$.

The scattering function in Eq. (11), bearing the resonant feature, and the resonance frequency in Eq. (12) can be regarded as the general formulas for resonant scattering by the family of ellipsoids, including spheres, oblate spheroids and prolate spheroids. The term kaE_c/F in the denominator of Eq. (11) is identified as the radiation damping factor, and its inverse is defined as the quality factor Q. A striking feature from Eq. (11) is that the resonant scattering is approximately omnidirectional. This is in analogy with the static electrical field generated by an ellipsoidal conductor. At the near field, the electrical field is directional, but in the far field region the field tends to be omnidirectional.

We consider a few special cases. When $\omega \ll \omega_0$

$$f\approx\frac{\rho_l V_0}{4\pi\gamma P_0}\omega^2. \quad (14)$$

The scattering function is independent of the shape of the ellipsoid at very low frequencies, following the Rayleigh volume scattering. When $\omega=\omega_0$, the scattering amplitude, the peak amplitude, is inversely proportional to the damping factor, and equals

$$f=\frac{ic_s}{\omega_0}, \quad (15)$$

where c_s is the sound speed in the medium. Equation (15) suggests that the higher the resonance frequency, the lower the resonant amplitude. When $\omega \gg \omega_0$ and $kaE_c/F \ll 1$, the scattering function becomes

$$f\approx aE_c/F. \quad (16)$$

It is also easy to verify that for the case of a spherical bubble, Eq. (11) reduces to Eq. (1). Also, as will be reported elsewhere [16], when it is applied to prolate spheroid bubbles the numerical computation based upon the exact solution seems to support the present analytic solution, and it shows that the present solution is valid for $ka<0.3$, which covers the whole resonance region. In contrast to the present result, however, the numerical computation based on the

T-matrix method shows that the resonance scattering of a prolate spheroid bubble is directional [11]. We note the resonant scattering function of ellipsoidal bubbles may also be computed numerically by the improved T-matrix method proposed by Boström [17].

We have derived an analytic solution for acoustic scattering by an ellipsoid bubble at low frequencies. The result will be potentially useful in a number of processes such as the sonoluminescence [7] when the deformation of a spherical bubble is important, as it may guide deriving a governing equation for such a deformed bubble. This solution may be used as a check for any numerical method to be developed to calculate acoustic scattering by gas ellipsoids. Finally, we state that the present results are derived when the heat transfer and viscosity effects are ignored. When these effects are taken into account, the quality factor Q and the amplitude at resonance are both expected to reduce. In addition, the assumed adiabatic behavior of the gas may not be appropriate for some extreme cases such as very flat or slender bubbles. The inclusion of thermal and viscosity effects remains to be the subject of a future work.

This work received support from the Department of Fisheries and Oceans, Canada and the National Central University.

[1] D. M. Farmer and D. D. Lemon, J. Phys. Oceanogr. **14**, 855 (1984).

[2] Z. Ye, J. Appl. Phys. **78**, 6389 (1995).

[3] Z. Ye and L. Ding, Can. J. Phys. **74**, 92 (1996), and references therein.

[4] J. Wu, J. Geophys. Res. **86**, 457 (1981).

[5] D. E. Weston, in *Underwater Acoustics*, edited by V. M. Albers (Plenum, New York, 1967).

[6] Z. Ye, J. Acoust. Soc. Am. **99**, 785 (1996).

[7] B. P. Barber and S. J. Putterman, Nature (London) **352**, 318 (1991); R. Hiller, S. J. Putterman, and B. P. Barber, Phys. Rev. Lett. **69**, 1182 (1992).

[8] S. A. Thorpe, Proc. R. Soc. London, Ser. A **304**, 155 (1982); Prog. Oceanogr. **35**, 315 (1995).

[9] See, e.g., P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[10] M. Strasberg, J. Acoust. Soc. Am. **25**, 536 (1953).

[11] C. Feuillade and M. Werby, J. Acoust. Soc. Am. **96**, 3684 (1994).

[12] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).

[13] Z. Ye, J. Acoust. Soc. Am. **101**, 681 (1997).

[14] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).

[15] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).

[16] Z. Ye, J. Acoust. Soc. Am. **101**, 1945 (1997); Z. Ye and E. Hoskinson (unpublished).

[17] A. Bostrom, J. Math. Phys. (N.Y.) **23**, 1444 (1982).