Universality in dynamic critical phenomena

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We use heat bath dynamics to evaluate the dynamic critical exponent z and the dynamic finite-size scaling function of an Ising model on square, planar triangular, and honeycomb lattices. We find convincing evidence that z is universal and, by choosing an aspect ratio and a nonuniversal metric factor for the scaled time of each lattice, we can obtain a universal dynamic finite-size scaling function for the Ising model on the planar lattices. Our results suggest many interesting problems for further research. [S1063-651X(97)15408-3]

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The physics of dynamic critical phenomena has been an interesting and important research subject in recent decades [1–8], but its progress is slower than static critical phenomena in which the universality of critical exponents was well established long ago [9] and the universality of finite-size scaling functions [10] was well established recently [11–13]. It seems that there is no strong evidence to support the universality in dynamic critical exponents [1] and there are no previous studies of universal dynamic finite-size scaling functions at all. The purpose of this paper is to fill this gap. Using heat bath dynamics, we find convincing evidence that for the Ising model on square (sq), triangular (TP), and honeycomb (hc) lattices, the dynamic critical exponent z is universal and, by choosing an aspect ratio and a nonuniversal metric factor for the scaled time of each lattice, we can obtain a universal dynamic finite-size scaling function (DFSSF) for the Ising model on the planar lattices. Our results will stimulate further researches on dynamic critical phenomena.

The dynamic critical exponent z [1,2] which characterizes the critical slowing down near the critical temperature is of much interest [4]. Since there is no exact solution for z, computer simulation plays an important role in the evaluation of z. In the past two decades, estimates of z varied in a large range between 1.7 and 2.3: see [5] for a review. Only very recently, several authors reached a consistent value for this exponent with different simulation schemes. By considering the dynamic scaling property of the relaxation time in the vicinity of the critical temperature, Miyashita and Takano [3] found that the dynamic critical exponent z is about 2.2 for the two-dimensional (2D) Ising model. From time relaxation of the magnetization and energy of the Ising model, Ito [5] found that $z=2.165\pm0.010$ for the sq lattice Ising model, which was confirmed by other calculations [6–8]. While the universality of static critical exponents was well established long ago [9], the universality of z, in the sense that z does not depend on details of local interactions and lattice structures [1], is rarely extensively studied in the literature. Almost all simulations are performed on the sq lattice Ising model [14].

Finite-size scaling function is another important concept in the theory of critical phenomena [10]. Using a histogram Monte Carlo simulation method [15], Hu, Lin, and Chen (HLC) [11] have obtained an almost perfect universal finite-size scaling function for the existence probability E_p and the percolation probability P of bond and site percolation on sq, TP, and hc lattices by choosing a very small number of nonuniversal metric factors. Using another method, Okabe and Kichuchi [13] obtained universal finite-size scaling functions for the Ising model. In this paper, we will investigate the universality of z and the DFSSF for the Ising model on sq, TP, and hc lattices with periodic boundary conditions.

The critical exponent z can be evaluated by studying the relaxation of the magnetization M on a lattice with a linear dimension L and N lattice sites, which has the following form at the critical temperature T_c [2]: $M(T_c,t)\equiv M(L\rightarrow\infty,T_c,t))\sim t^{-\beta/\nu z}$, where β and ν are universal static exponents for M and correlation length, respectively, and are 1/8 and 1 for the two-dimensional Ising model, and t is the number of Monte Carlo steps with the unit of one sweep of all lattice sites. In the damage spreading method [4] used in this paper, on a lattice of N sites we consider two spin configurations denoted by A and B. The same sequence of the random numbers $R_i(t)$ is used to update Ising spins σ_i^A and σ_i^B on the ith sites of system A and system B, respectively, where $1\leqslant i\leqslant N$. We use the following heat bath dynamics for the evolution of system A from a state at t to a state at $t+\delta t$ [16,17]:

$$\sigma_i^A(t+\delta t)=\mathrm{sgn}[P_i^A(t)-R_i(t)] \tag{1}$$

and

$$P_i^A(t)=\frac{\exp[\Sigma_k\sigma_k^A(t)/T]}{\exp[\Sigma_k\sigma_k^A(t)/T]+\exp[-\Sigma_k\sigma_k^A(t)/T]}, \tag{2}$$

where T is the temperature of the system, Σ_k sums over nearest neighbors of the ith site, and $\delta t=1/N$. Similar evolution equations may be written for system B. In the evolution process, the Hamming distance $D(L,T,t)$ of two configurations is defined by the equation

$$D(L,T,t)=\left\langle\frac{1}{N}\sum_{i=1}^{N}|\sigma_i^A(t)-\sigma_i^B(t)|/2\right\rangle. \tag{3}$$

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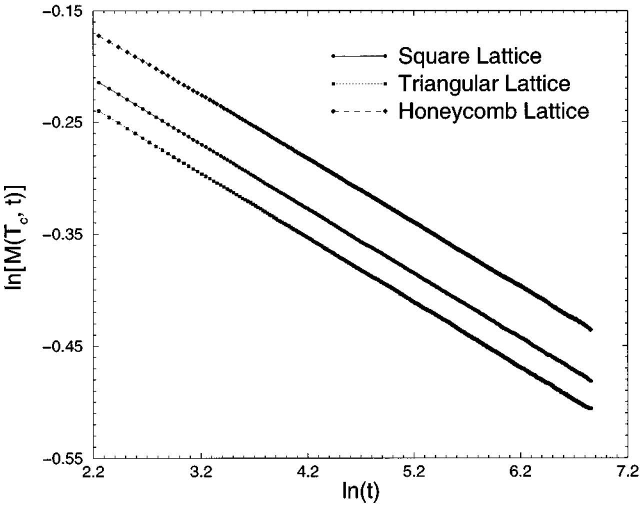

FIG. 1. ln $M(T_c,t)$ vs ln(t) for the Ising model on sq, TP, and hc lattices with $L=1000$ and $t=10-1000$ MCS.

With the heat bath algorithm, it is easy to prove that if the initial configurations of two systems satisfy the condition $\sigma_i^A(0)\geqslant\sigma_i^B(0)$ for $1\leqslant i\leqslant N$, then $\sigma_i^A(t)\geqslant\sigma_i^B(t)$ for any time $t>0$ [18]. This property is called the *monotonicity*. With this condition, we have $D(L,T,t)=[M^A(L,T,t)-M^B(L,T,t)]/2\equiv M(L,T,t)$, where $M^A(L,T,t)$ and $M^B(L,T,t)$ are magnetizations of system A and system B, respectively. The merit of this method is that $D(L,T,t)$ has less fluctuations produced by random numbers during the evolution process and shows a better power law behavior for a much longer simulation time.

We first study the Ising model on 1000×1000 sq, TP, and hc lattices and set $\sigma_i^A(0)=-\sigma_i^B(0)=1$ as the initial configurations. All simulations are performed at the critical temperatures of the lattices [19], namely, $T_c=2/\ln(\sqrt{2}+1)$, $4/\ln 3$, and $2/\ln(2+\sqrt{3})$ for sq, TP, and hc lattices, respectively. The logarithmic-scaled relaxation curves for $M(T_c,t)$ are shown in Fig. 1. The data for each curve come from the average of 2000 independent runs. Since β and ν are universal, if z is universal, then all relaxation curves should have the same slope. Figure 1 shows that this is indeed the case. Using linear least squares fitting, we estimate z to be 2.166 ± 0.007, 2.164 ± 0.007, and 2.170 ± 0.010 for sq, TP, and hc lattices, respectively, which are consistent with each other and also consistent with other calculations [5–8]. To test the reliability of our computer programs and the random number generator, we also use Glauber dynamics to calculate $M(T_c,t)$ for the Ising model on a 1000×1000 TP lattice. We find that we should have the average of 5000 runs in the Glauber dynamics in order to get precision comparable to that of the 2000 runs in the heat bath dynamics with the damage spreading technique. The agreement of $M(T_c,t)$ obtained by two methods is very well.

The universality of z provides us a good basis to study the universality of the DFSSF. Suzuki proposed that when T is near T_c, the magnetization of a system of linear dimension L at time t, $M(L,T,t)$, may be written as [2]

$$M(L,T,t)=L^{-\beta/\nu}f(L^{1/\nu}(T/T_c-1),tL^{-z}). \tag{4}$$

We first consider the case $T=T_c$ and have

$$M(L,t)\equiv M(L,T_c,t)=L^{-\beta/\nu}f(tL^{-z}), \tag{5}$$

where $L=\sqrt{N}$. In [11], HLC considered 512×512 sq, 433×500 TP, and 433×250 hc lattices, so that aspect ratios of sq, TP, and hc lattices approximately have the proportions $1:\sqrt{3}/2:\sqrt{3}$ and the existence probability E_p at critical points are identical for all lattices [20], which is a crucial step to obtain universal finite-size scaling functions for E_p and P. To obtain the universal finite-size scaling function for the Ising model, we should choose the ratios between aspect ratios of different lattices to be approximately equal to those of [11]. Therefore, in the following we consider the Ising model on 32×51 and 64×102 sq lattices, 27×50 and 54×100 TP lattices, and 27×25 and 81×75 hc lattices. The relaxation of $M(L,t)$ of the Ising model on such lattices is shown in Fig. 2(a). Using the data of Fig. 2(a), we plot $M(L,t)L^{\beta/\nu}$ as a function of tL^{-z} and show the results in Fig. 2(b), which shows that two curves of the same lattice with different L fall onto an identical dynamic finite-size scaling function and scaling functions for different lattice structures are different. Following the case of static critical phenomena [10,11], we propose the following equation for a universal DFSSF $F(x)$:

$$D_iM(L,t)=L^{-\beta/\nu}F(C_itL^{-z}). \tag{6}$$

Here D_i and C_i for i being 1, 2, and 3 are nonuniversal metric factor for sq, TP, and hc lattices, respectively. With $D_i=1$ $(i=1,2,3)$, $C_1=1$, $C_2=1.222\pm0.009$, and $C_3=0.693\pm0.018$, we get Fig. 2(c) which shows that data for different lattices fall on a universal DFSSF.

Next we evaluate $M(L,T,t)$ for the Ising model on a 32×51 sq lattice, 27×50 TP lattice, and 27×25 hc lattice for $T\neq T_c$ and at some finite scaled times, say, $C_it_iL^{-z}=1.658g$ with g being 0.5, 1, and 2, which means that $t_1=5000g$ Monte Carlo steps (MCS) for the sq lattice, $t_2=3332g$ MCS for the TP lattice, and $t_3=2773g$ MCS for the hc lattice. Following [11], we propose the following equation for a universal DFSSF F':

$$D_iM(L,T,t_i)L^{\beta/\nu}=f(E_iL^{1/\nu}(T/T_c-1),C_it_iL^{-z})$$
$$\equiv F'(E_iL^{1/\nu}(T/T_c-1)), \tag{7}$$

where D_i and E_i for $1\leqslant i\leqslant3$ are nonuniversal metric factors. With $D_i=E_i=1$ for $1\leqslant i\leqslant3$ and C_i of Fig. 2(c), we show $D_iM(L,T,t_i)L^{\beta/\nu}$ as a function of $x=E_iL^{1/\nu}(T/T_c-1)$ in Fig. 3, which shows that in the critical region and for any value of g, three lattices have universal DFSSF's for $M(L,T,t_i)$.

Figure 3 suggests that as $g\rightarrow\infty$, $D_i=E_i=1$ for $1\leqslant i\leqslant3$ still gives a universal DFSSF for $M(L,T,t_i)$ and such nonuniversal metric factors should be consistent with nonuniversal metric factors for the static finite-size scaling function (SFSSF) [11,13]. A cluster Monte Carlo method [21] which can overcome the critical slowing down is used to calculate the equilibrium magnetization M_e of the Ising model on 32×51 sq, 27×50 TP, and 27×25 hc lattices to test this idea. It has been found that $D_i'M_eL^{\beta/\nu}$ as a function of $x=E_i'L^{1/\nu}(T/T_c-1)$ for three lattices has universal SFSSF's [22] with $D_i'\approx E_i'\approx1$ for $i=1$, 2, and 3. The nonuniversal metric factors for the Ising model obtained by Okabe and

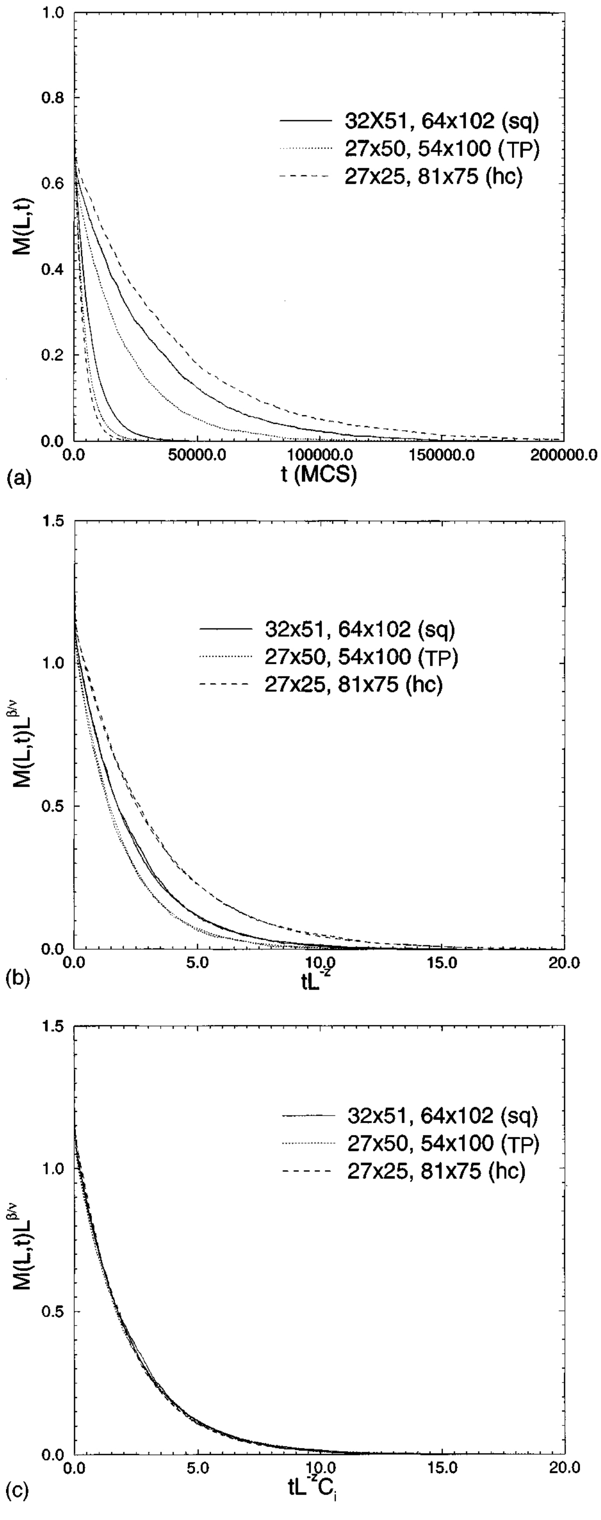

FIG. 2. (a) $M(L,t)$ of Eq. (5) vs t for the Ising model on sq, PT, and hc lattices. (b) $M(L,t)$ vs tL^{-z}. (c) $D_iM(L,t)$ vs C_itL^{-z} with nonuniversal scaling factors $D_1=D_2=D_3=1$, $C_1=1$ (sq lattice), $C_2=1.222\pm0.009$ (TP lattice), and $C_3=0.693\pm0.018$ (hc lattice)

Kikuchi [13] are corresponding to $D'_1=E'_1=1$ for the sq lattice, $D'_2=1.02\pm0.02$ and $E'_2=0.96\pm0.03$ for the TP lattice, and $D'_3=0.98\pm0.02$ and $E'_3=1.00\pm0.02$ for the hc lattice, which are very close to $D_i=E_i=1$ for $1\leqslant i\leqslant3$ used in Fig. 3 of this paper.

It is well known that the Ising model and the bond random percolation model (BRPM) correspond to the q-state Potts model [19] with q being 2 and 1, respectively, so that the bond probability p of the percolation model is related to the temperature T of the Potts model by $p=1-\exp(-2/T)$ [23]. HLC [11] obtained nonuniversal metric factors for BRPM, using $(p-p_c)L^{1/\nu}$ as a scaling variable. Recalculating their result using $(T/T_c-1)L^{1/\nu}$ as a scaling variable, we find metric factors of $D'_1=E'_1=1$ for the sq lattice, $D'_2=1.021\pm0.021$ and $E'_2=0.996\pm0.034$ for the TP lattice, and $D'_3=0.987\pm0.011$ and $E'_3=1.011\pm0.019$ for the hc lattice, which is presented in this paper. Our results imply that nonuniversal metric factors C_1 and C_2 considered in [10] are equal to 1 or very close to 1 for sq, TP, and hc lattices.

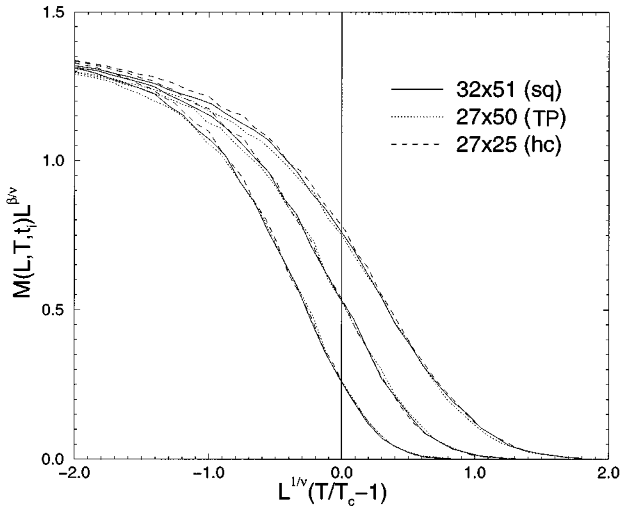

FIG. 3. $D_iM(L,T,t_i)L^{\beta/\nu}$ of Eq. (7) vs $E_iL^{1/\nu}(T/T_c-1)$ with $D_i=E_i=1$ for $1\leqslant i\leqslant3$ for the Ising model on sq, TP, and hc lattices near the critical temperature of each lattice and for the scaled times $C_it_iL^{-z}=1.658g$ with g being 0.5, 1, and 2. At $T=T_c$, the curves from top to bottom are for g being 0.5, 1.0, and 2.0, respectively.

From Eq. (5), we may define $Q(L)$ and it scales with L as

$$Q(L)\equiv\int_0^\infty M(L,t)dt=\sum_{t=1}^{\infty} M(L,t)\sim L^{z-\beta/\nu}. \qquad (8)$$

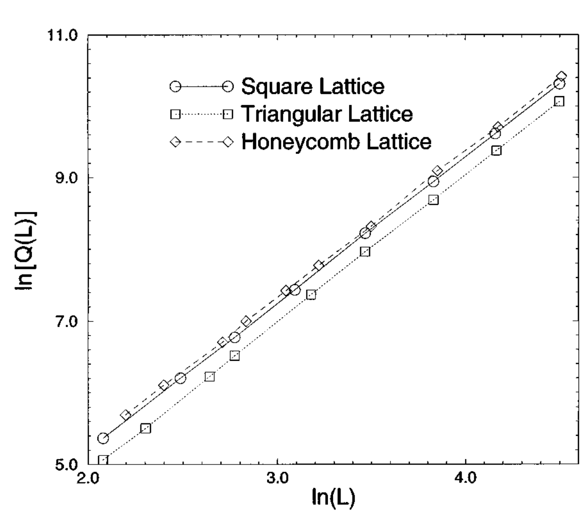

FIG. 4. ln $Q(L)$ vs ln L for $8\leqslant L\leqslant90$.

For each L, we simulate 10^3-10^6 sequences of M at T_c and continue the simulation until $M=0$. The average of M in these sequences gives $M(L,t)$ of Eq. (8). The ln-ln plots of $Q(L)$ vs L for sq, TP, and hc lattices are shown in Fig. 4. The least squares fittings of such lines give z, which are 2.168 ± 0.005, 2.180 ± 0.009, and 2.167 ± 0.008 for sq, TP, and hc lattices, respectively. Such values are consistent with each other and they are consistent with z evaluated from Fig. 1. This result implies that short time and long time behaviors of M are governed by the same dynamic critical exponent z.

In summary, Figs. 1 and 4 give convincing evidence that z is universal and Figs. 2 and 3 show that the DFSSF is universal for $T=T_c$ and $T\neq T_c$ with only one nonuniversal metric factor for scaled time of each lattice. Such results will stimulate many researches on dynamic critical phenomena, e.g., the universality of z and the DFSSF in other physical quantities, update algorithms, and dynamic systems (including surfaces, interfaces, systems quenched from $T>>T_c$ to T_c [24], etc.), renormalization group approach to the universal DFSSF, etc.

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