Synchronization of spatiotemporal chaos with positive conditional Lyapunov exponents

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Computer simulations show that, following the scalar driving approach, synchronization of spatiotemporal chaos can also be achieved in driven subsystem with positive conditional Lyapunov exponents. This is a result of the so-called extreme trap. In the neighborhood of the maximum value, the higher-order term of difference causes a first-order dissipative effect, which is not reflected by the conditional Lyapunov exponents. Hence the actual average divergence rate is smaller than the maximum conditional Lyapunov number and thus causes the driven system to synchronize even when the conditional Lyapunov exponents are positive.
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The synchronization of chaos [1,2] has attracted much attention in recent years. Practically, it is motivated by the potential applications in secure communications [3–8]. In the work done by Pecora and Carroll [1,2], a chaotic system is decomposed into two coupled subsystems, namely, the drive and response. When coupled with a common driving signal, the response subsystem can synchronize with the driving one if all the conditional Lyapunov exponents are negative. Recently, a number of synchronizing approaches have been proposed [9–16] and some experimental results have been reported [17–21].

It is natural to speculate that the number of variables to be transmitted should be equal to that of the positive Lyapunov exponents in order to account for the same number of unstable directions along the chaotic trajectory. Lai and Grebogi analyzed the dynamics to synchronize hyperchaotic systems [9] based on the controlling chaos method proposed by Ott, Grebogi, and Yorke [23]. In each iteration, an external control matrix is carefully calculated from the unstable base vectors. Then a control disturbance vector is computed to adjust the driven subsystem gradually. They showed that the subsystems can be in a synchronization state, while the corresponding Lyapunov exponents are positive. Xiao, Hu, and Qu [8] reported that one can use an external chaotic control key sequence to synchronize two identical spatiotemporal chaotic systems. This kind of synchronization is referred to as generalized synchronization [14,15]. To synchronize N-dimensional hyperchaos, Peng *et al.* [16] utilized a scalar signal that is a function of the sender system with N weighted parameters. To drive the response subsystem, another N-dimensional vector is required. As a result, a total of $2N$ parameters have to be found out in order to achieve the synchronization. The hyperchaotic synchronization discussed in Refs. [8, 16] is due to the fact that all the conditional Lyapunov exponents are negative.

The question raised by Pecora and Carroll [1] remains unsolved: Can self-synchronization be accomplished in the case of two or more positive exponents, but with only one drive signal? If the stability of the response systems is considered [1,2], it seems that we are forbidden to synchronize hyperchaos if the scheme of Pecora and Carroll is completely followed. However, in this paper, we will give an affirmative answer by studying the coupled logistic map lattice. Furthermore, we will investigate a different synchronizing dynamics in which, by using a driving signal only, synchronization of spatiotemporal chaos can still be achieved even when some of the conditional Lyapunov exponents are positive. It is a result of the extreme trap effect. Once the trajectory of the driven system approaches the maximum value, the higher-order term of the difference can no longer be neglected. It causes a first-order dissipative effect that counteracts, to a certain extent, the stretch effect of the linear term. This kind of dissipative effect is not reflected by the conditional Lyapunov exponent. Thus the actual average divergence rate is smaller than the maximum conditional Lyapunov number. As a result, even with positive conditional Lyapunov exponents, spatiotemporal chaos can still be synchronized by the scalar driving signal.

The dynamical behavior of the one-way coupled map lattice [22] has been investigated extensively and is now well understood. Recently, the synchronization of its spatiotemporal chaos also has been discussed [8,9]. The spatiotemporal chaotic lattice studied here is a periodic one-way coupled map lattice

$$x_0(t+1)=(1-\epsilon_0)f(x_0(t))+\epsilon_0 f(x_1(t)),$$

$$x_i(t+1)=(1-\epsilon_i)f(x_i(t))+\epsilon_i f(x_{i+1}(t)) \quad (i=1,\ldots,N),$$

$$x_{N+1}(t)=x_0(t). \tag{1}$$

The lattice can also be called the one-way coupled ring lattice [8] with length $N+1$. Here we choose $f(x)=ax(1-x)$ with $a=4.0$, $\epsilon_0=0.01$, and $\epsilon_i=\epsilon$ $(i=1,\ldots,N)$. Following Pecora and Carroll [1,2], the scalar signal x_0 is used as the driving signal.

The Jacobian matrix $D\vec{f}$ of the variational equations for the driven subsystem is a trigonometric matrix with the trigonometric variables

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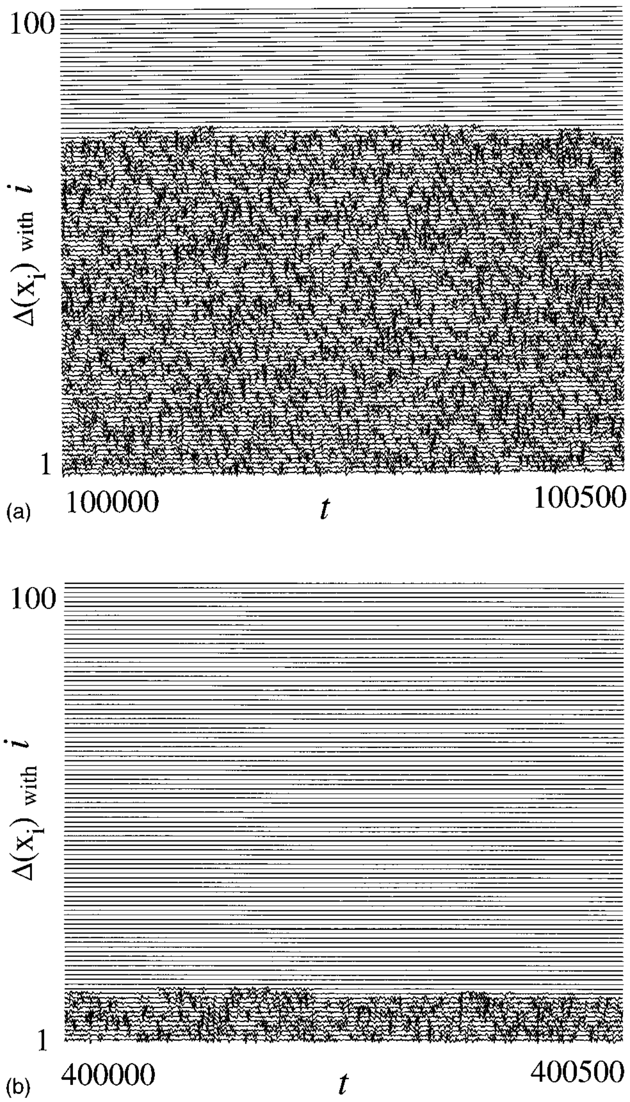

FIG. 1. Differences between two spatiotemporal chaotic coupled logistic map lattices in space i and time t with $\epsilon=0.345$ and $N=100$. (a) Time from 100 000 to 100 500 and (b) time from 400 000 to 400 500.

$$(D\vec{f})_{ii}=(1-\epsilon)f'(x_i) \quad (i=1,\ldots,N). \tag{2}$$

As a result, the conditional Lyapunov exponents are

$$\lambda_i=\ln(1-\epsilon)+\ln a+\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\ln|1-2x_i(t)| \quad (i=1,\ldots,N). \tag{3}$$

To synchronize the spatiotemporal chaos, we set $N=100$. When $\epsilon>0.358$, computer simulations show that the two subsystems are always in synchronization by using the scalar signal x_0 only. In the synchronization state, all the conditional Lyapunov exponents are negative. The two subsystems fail to be in synchronization when $\epsilon<0.318$, at which all the conditional Lyapunov exponents are positive.

Now an interesting question arises: What will happen in the range $0.318<\epsilon<0.358$? Numerical simulations show that complex and different synchronizing phenomena can be

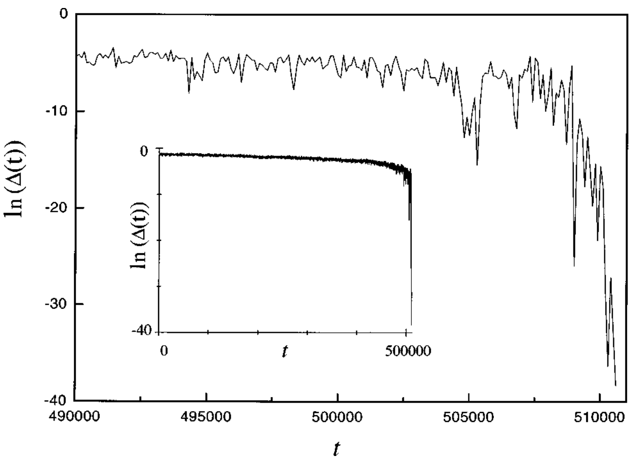

FIG. 2. Plot of time versus the logarithm of the average absolute difference $\overline{\Delta}(t)$ with $\epsilon=0.345$ and $N=100$. The outer plot is drawn with time from 490 000 to 511 000 and the inner plot is drawn with time from 0 to 511 000.

found there. For example, with $\epsilon=0.345$, the two subsystems can be in complete synchronization with 75 positive conditional Lyapunov exponents. The differences $\Delta(x_i)$ between the two spatiotemporal chaos in space i and time t are shown in Figs. 1(a) and 1(b). Simulations show that the spatiotemporal chaos in the map lattice synchronizes from x_N to x_1 in order. This is because the lattice x_i of the response system is driven by the driving signal through the cascade connection, i.e., $x_N\to x_{N-1}\to\cdots\to x_i\to\cdots\to x_1$. Eventually, all the variables of the two subsystems are in synchronization. Figure 2 shows a plot of time versus the logarithm of the average absolute difference $\overline{\Delta}(t)$, which is defined as

$$\overline{\Delta}(t)=\frac{1}{N}\sum_{i=1}^{N}|x_i'(t)-x_i(t)|. \tag{4}$$

At time $t=520\,000$, all the coupled map lattices are in synchronization states and the average absolute difference is smaller than 10^{-18}, which cannot be distinguished in our computer. However, one can see that it may take a long time to reach synchronization because of the large number of map lattices. In Fig. 3 the number of synchronizing map lattices M_1 and the number of positive conditional Lyapunov exponents M_2 are plotted against ϵ in the range 0.31–0.36.

This result conflicts with our knowledge of chaotic synchronization. To comprehend it, we now consider the simplest case of the two-dimensional coupled map lattice, i.e., $N=1$. In other words, the two subsystems are both one-dimensional, namely, x_1 and x_1'. Simulations show that when $\epsilon=0.2$, synchronization is achieved with $\lambda=0.100$, while the two Lyapunov exponents of the system x_0 and x_1 are 0.604 and 0.103, respectively; when $\epsilon=0.335$, synchronization occurs at $\lambda=0.023$ with the two Lyapunov exponents of the system x_0 and x_1 being 0.637 and 0.019, respectively.

Suppose two trajectories start at nearby points $\Delta x_1(0)=x_1'(0)-x_1(0)$. The variational equation of motion is

$$\Delta x_1(t+1)=(1-\epsilon)f'(x_1(t))\Delta x_1(t), \tag{5}$$

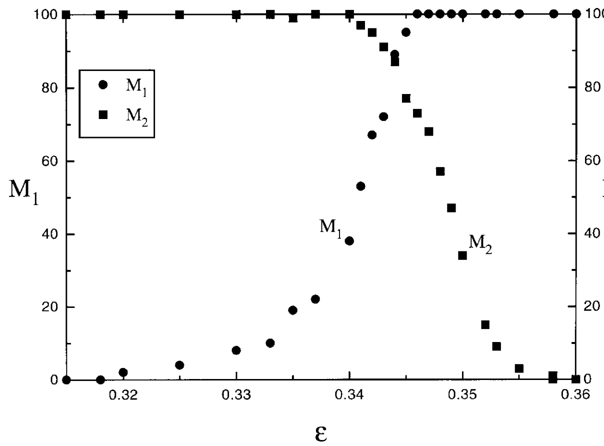

FIG. 3. Statistical results of the number of synchronizing map lattices M_1 and that of the positive conditional Lyapunov exponents M_2 against ϵ with $N=100$.

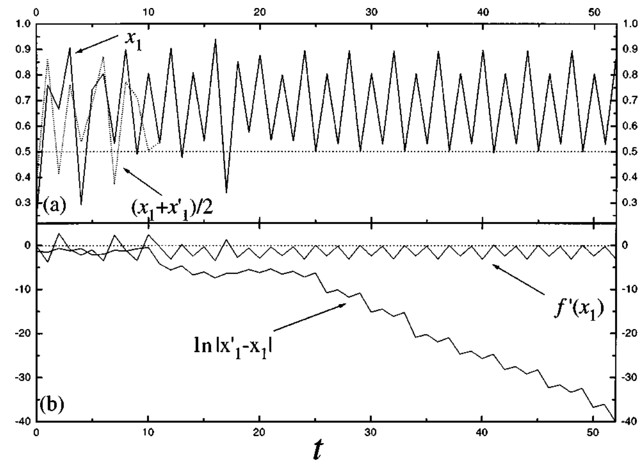

FIG. 4. Plot of time versus (a) the trajectory x_1 and the average value $(x_1+x_1')/2$ and (b) the slope $f'(x_1)$ and the logarithm of the absolute difference $|\Delta x_1|$, for the two-dimensional coupled map with $a=4.0$, $\epsilon_0=0.01$, and $\epsilon=0.2$. The initial values are $x_0=0.33$, $x_1=0.24$, and $x_1'=0.51$.

with the higher-order term of the difference

$$o(\Delta x_1(t))=-a(1-\epsilon)\Delta^2 x_1(t), \tag{6}$$

which always has been neglected by us. Now consider a special case that $x_1+x_1'=1$. In this case, we have $\Delta x=0$ in next iteration as $f(x)=f(1-x)$ for the logistic map. As a result, the systems immediately fall into the synchronization state. We call this the trap. If $x_1+x_1'\to 1$, we have $\Delta x\to 0$ in the next iteration. This is because the higher-order term $o(\Delta x_1)$ counteracts the linear term of the difference and leads to convergence. But one can see that this kind of convergence is not reflected by the conditional Lyapunov exponent defined in Eq. (3). In other words, it implies that the actual average divergence rate can be smaller than the conditional Lyapunov number e^{λ}.

However, with a set of random initial values, the relationship $x_1+x_1'=1.0$ is seldom satisfied. So it seems that the traps often lead to suppression of chaos in Δx_1 only, but cannot result in synchronization. Simulations show that the trap at the maximum value $x_1=x_1'=0.5$, which is referred to as the extreme trap, plays a different role in leading the system to synchronization. The maximum of the function $f(x)$ is at $x_m=0.5$, with $f'(x_m)=0$ and $f''(x_m)=-a$. Suppose that the two trajectories are in the neighborhood of x_m; we get $f'(x(t))=2a(x_m-x(t))$. Combining Eqs. (5) and (6), we have

$$\Delta x_1^{II}(t+1)=a(1-\epsilon)[2(x_m-x_1(t))-\Delta x_1(t)]\Delta x_1(t). \tag{7}$$

One can see that when $x_1(t)\to x_m$, the term $2(x_m-x_1(t))$ can be of the same order as $\Delta x_1(t)$, which means that the second-order nonlinear term $o(\Delta x_1)$ cannot be neglected in this case. All the possible distributions of x_1 and x_1' can be classified into two cases: (a) the lower point is at the left-hand side of x_m and (b) the upper point is at the right-hand side of x_m. In case (a), without loss of generality, suppose that $x_1(t)$ is the lower point [i.e., $\Delta x_1(t)>0$]. Equation (7) shows that the second-order term counteracts the first-order term and in fact contributes to the convergence when $2(x_m-x_1(t))$ is of the same order as $\Delta x_1(t)$. The same result can be obtained in case (b) if we let $x_1(t)$ be the upper point. As a result, we always have

$$\Delta x_1^{II}(t+1)=A(t)a(1-\epsilon)(x_m-x_1(t))\Delta x_1(t), \tag{8}$$

where the proportional parameter $A(t)$ is smaller than 1. In particular, if the distribution of x_1 and x_1' on the two sides of x_0 is close to a symmetrical manner, we have $A\ll 1$. Compare Eq. (8) with Eq. (5), one can see that if the trajectories are in the neighborhood of x_m, the nonlinear second-order term counteracts the linear term of the difference and causes a deeper convergence, which is not reflected by the conditional Lyapunov exponent defined in Eq. (3). We call this effect the extreme trap. As a consequence, if the systems approach the extreme value with high frequency, the actual average divergence rate can be smaller than the conditional Lyapunov number e^{λ}. This implies that when the conditional Lyapunov exponent is slightly larger than zero, synchronization may happen.

An example is shown in Fig. 4, in which the trajectory x_1, the average value $(x_1+x_1')/2$, the slope $f'(x_1)$, and the logarithm of the absolute difference $|\Delta x_1|$ are compared. One can see that when $x_1\approx x_1'\approx 0.5$, the slope $f'(x_1)\approx 0$ and it causes the logarithm of the absolute difference $|\Delta x_1|$ to decrease rapidly in the next iteration. In some cases, it even decreases by 10^{-2} when the states are closer to the extreme trap.

Although these are the results of the simple two-dimensional case, they can easily be extended to the case of a high-dimensional coupled logistic map lattice. We can regard the $(N+1)$-dimensional coupled logistic map lattice as the Nth-order cascade connections of one-dimensional case. Due to the extreme trap effect, once the actual average divergence rate is smaller than unity, spatiotemporal chaos can be synchronized even when the maximum conditional Lyapunov exponents are positive.

It should be emphasized that the dynamics of synchronizing hyperchaos with positive conditional Lyapunov exponent is not limited to the coupled logistic map lattice. We can also observe this phenomenon in some other maps. An example is $x \exp[\alpha(1-x)]$. Our simulations show that the logistic map seems to be the most typical example for this kind of synchronization because its higher-order term corresponds to second-order only.

In conclusion, following the scheme proposed by Pecora and Carroll, we use the coupled logistic map lattice to show that one can synchronize the hyperchaotic subsystems with a single variable only. Furthermore, we have presented a different dynamics to synchronize the spatiotemporal chaos with positive conditional Lyapunov exponents. This is a result of the extreme trap. It means that, near the extreme value, the higher-order term causes a first-order dissipative effect that cannot be reflected by the conditional Lyapunov exponent. As a result, the actual average divergence rate is smaller than the conditional Lyapunov number. Once the chaotic trajectory wanders with high frequency near the extreme trap, the driven subsystem can still synchronize with the driving system if the maximum conditional Lyapunov exponent is slightly larger than zero. Our further studies show that the dynamics of synchronizing chaos with positive conditional Lyapunov exponents is a widely existing phenomenon. For the case of synchronous chaos in coupled oscillator systems, which has been studied extensively [24], we find that such a phenomenon exists in the coupled logistic oscillator systems [25]. Furthermore, it is pointed out that noise-induced synchronization can also be obtained with a positive conditional Lyapunov exponent [26]. These results show that the conditional Lyapunov exponents cannot be used as a criterion for synchronous chaotic systems.

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