Theory of force-free electromagnetic fields. I. General theory

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A general method to deal with the (relativistic) force-free electromagnetic field is developed. We formulate the theory without assuming symmetry of the electromagnetic field configuration. Thus we can apply it to any object where the force-free approximation is justified, e.g., the pulsar magnetosphere, the black-hole magnetosphere, and the magnetosphere around the accretion disk. We describe the force-free electromagnetic field by a classical field theory. The basic variables are the Euler potentials extended to the relativistic degenerate electromagnetic field. The basic equation is given by a two-component nonlinear equation for two Euler potentials. The theory has a close connection with geometry. It is based on the concept of the flux surface. The flux surface is a geometric entity corresponding to the world sheet of the magnetic field line. We give both the covariant and the 3+1 expression of the basic equation. By the latter form, the causal development of the force-free electromagnetic field is discussed. It is shown that the theory describes the causal development of the force-free electromagnetic field self-consistently as far as $F\cdot F>0$. The basic equation contains arbitrariness. It does not determine the solution uniquely. Although this arbitrariness originates from the gauge freedom of the electromagnetic field, it differs from the arbitrariness in the ordinary gauge field theories. Namely, the dynamics of the Euler potentials itself does not contain arbitrariness. It appears from nonuniqueness in correspondence between the Euler potentials and the electromagnetic field. Further, we discuss the breakdown of the force-free approximation. [S1063-651X(97)03008-0]

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I. INTRODUCTION

The force-free approximation (massless approximation) of the relativistic magnetohydrodynamics has been used widely in studies on the magnetosphere around the relativistic object, such as the pulsar [1–3], the black hole [4–7], and the accretion disk [8]. We can apply this approximation to the region where the electromagnetic energy density is much greater than both the rest mass and the thermal energy density of the plasma. However, most of the foregoing studies were devoted to the stationary and axisymmetric electromagnetic field configuration. Only a few works have treated the nonstationary-axisymmetric case. As examples of such works, we have studies on the obliquely rotating pulsar magnetosphere [9] and the time-dependent axisymmetric magnetosphere around the black-hole accretion-disk system [10]. Although these were pioneer works and important contributions in their specific problems, they are quite unsatisfactory in the understanding of the general properties of the force-free electromagnetic field.

The method to treat the stationary and axisymmetric force-free electromagnetic field has been almost established through the works cited above [1–8] and others. Namely, the stationary and axisymmetric force-free electromagnetic field is described by the stream function of the poloidal magnetic field lines together with two integrals on the poloidal magnetic field lines. The stream function is determined by a partial differential equation called the pulsar equation or the transfield equation. It decides the electromagnetic force balance across the poloidal magnetic field lines [1–8]. Thus, in the studies on the stationary and axisymmetric force-free configuration, major difficulty lies in solving the pulsar equation to construct realistic models.

On the contrary, the method for nonstationary-axisymmetric cases still remains in a quite underdeveloped state. In fact, the foregoing works on the nonstationary-axisymmetric configurations did not give any systematic method to deal with the force-free electromagnetic fields nor reveal dynamical properties of the force-free electromagnetic field. Accordingly, a systematic mathematical procedure treating the force-free electromagnetic field is still absent except for the stationary and axisymmetric case. As a result, we are still almost ignorant of the dynamical properties of the force-free electromagnetic field. Further, reflecting such ignorance, overall consistency of the force-free approximation has sometimes been questioned [3]. Indeed, the force-free electromagnetic field has never been studied from the field theoretical point of view as a field that has its own dynamics. Recently the present author has formulated a linear perturbation theory of the force-free electromagnetic fields and treated small disturbances in the force-free black-hole magnetosphere. In due course, we find that the force-free electromagnetic field is generally described by a field theory with two scalar variables. In this work, we will present a systematic treatment of the force-free approximation using this idea.

The basic equations for the force-free electromagnetic field are

$$\nabla_\lambda F_{\mu\nu}+\nabla_\mu F_{\nu\lambda}+\nabla_\nu F_{\lambda\mu}=0, \tag{1.1}$$

$$\nabla_\nu F^{\mu\nu}=4\pi J^\mu, \tag{1.2}$$

$$F_{\mu\nu}J^\nu=0, \tag{1.3}$$

where $F_{\mu\nu}$ is the electromagnetic field and J^μ is the four-current. Equations (1.1) and (1.2) are Maxwell's equations.

Equation (1.3) is the force-free condition that is justified when the Lorentz force acting on the plasma is much greater than the inertial force [1–8].

In the ordinary treatment of Maxwell's equations, the current J^{μ} is determined by the matter distribution. The electromagnetic field is solved so as to be consistent with the current distribution and the boundary condition. On the other hand, in the force-free approximation, the role of Eq. (1.2) is to express the current by the electromagnetic field. Namely, the current is not a quantity independent of the electromagnetic field. Accordingly, Eqs. (1.2) and (1.3) give a nonlinear equation for $F_{\mu\nu}$.

Here, some reader may wonder at the absence of equations concerning the macroscopic four-velocity of the plasma from the basic equations. In fact, traditionally the macroscopic four-velocity of the matter has been introduced based on the specific microscopic picture of the plasma. It was sometimes treated as one of the basic variables together with the electromagnetic field. However, as we will see below, the force-free approximation is independent of any picture of the constituent plasmas except for the point of whether or not the force-free approximation is justified. The dynamics of the force-free electromagnetic field is described completely without the macroscopic four-velocity of the plasma. Of course, in many cases, introduction of the macroscopic four-velocity offers useful physical information. Especially, when one compares theoretical results with the astronomical observation or with the results of other theory such as the magnetohydrodynamics (MHD), the macroscopic four-velocity of the plasma is necessary. Further, there will be the case in which the four-velocity based on a specific physical picture is incompatible with the force-free approximation. However, this is another problem to be considered apart from the dynamics of the force-free electromagnetic field. In the present work, we thus stress the point that the primary problem in the force-free approximation is in Eqs. (1.1)–(1.3). Introduction of the macroscopic four-velocity is the secondary problem in this sense. Therefore we concentrate our effort on a systematic treatment of Eqs. (1.1)–(1.3).

The plan of this paper is as follows. Among the properties of the force-free electromagnetic field, the one most fundamental to the present theory is its degeneracy. The intrinsically geometric nature of the theory results from this fact. In Sec. II properties of the degenerate electromagnetic field are discussed. The Euler potentials and the notion of the flux surface are introduced. They are the key concepts of the theory. In Sec. III the covariant form of the basic equation is given by a set of two equations for two Euler potentials. The action that yields the basic equations is obtained. Arbitrariness in solutions is also considered. In Sec. IV, leaving the main development aside, we consider description of the magnetic field line. The magnetic field line is treated as a geometric entity that has its own self-identity. In Sec. V, the causal development of the force-free electromagnetic field is studied. In the flat space-time, the basic equation is rewritten to the 3+1 form that is suitable for this purpose. Splitting the time derivatives from the spatial derivatives, we get the basic equations that are second order in the time derivative. Further, the canonical equations of motion are derived. Using these formulations, the indeterminacy in the basic equation is clarified. In Sec. VI the breakdown of the force-free approximation is discussed.

Our metric signature is $(-+++)$. We use units in which $c=G=1$.

II. EULER POTENTIALS

A. Degenerate electromagnetic field

Equation (1.3) implies that the force-free electromagnetic field is necessarily a degenerate electromagnetic field. Namely, $F_{\mu\nu}$ satisfies $\det F_{\mu\nu}=0$, as a matrix. This is also expressed as $*F^{\mu\nu}F_{\mu\nu}=0$ by means of the dual tensor of $F_{\mu\nu}$, which is given by $*F^{\mu\nu}=\varepsilon^{\mu\nu\lambda\tau}F_{\lambda\tau}$. In nonrelativistic language, this is written as $\vec{E}\cdot\vec{B}=0$ in terms of the magnetic field $\vec{B}$ and the electric field $\vec{E}$, of course. The degenerate electromagnetic fields have distinct algebraic and geometric properties that manifestly distinguish them from the nondegenerate fields. The theory explored here is largely based on the intrinsic geometrical nature of the degenerate electromagnetic fields. Thus the degeneracy of the force-free electromagnetic field plays a crucially important role in this work. Therefore it is appropriate to start our discussion with a summary of the properties of the degenerate electromagnetic field.

First, let us consider the algebraic properties of $F_{\mu\nu}$ as an antisymmetric matrix. By antisymmetry, a degenerate electromagnetic field tensor $F_{\mu\nu}$ becomes an even rank matrix. Consequently, $\det F_{\mu\nu}=0$ implies that the rank of $F_{\mu\nu}$ is two if $F_{\mu\nu}$ has nonvanishing components. Thus $F_{\mu\nu}$ has a two-dimensional vector space of zero eigenvectors. Namely, there are two linearly independent vectors that become solutions of the equation

$$F_{\mu\nu}\xi^{\nu}=0. \tag{2.1}$$

Evidently, Eq. (2.1) has nontrivial solutions only when $\det F_{\mu\nu}=0$. Further, we can show that if $\xi^{\mu}_{(1)}$ is a solution of Eq. (2.1), another solution $\xi^{\mu}_{(2)}$ orthogonal to $\xi^{\mu}_{(1)}$ is given by

$$\xi^{\mu}_{(2)}={*F}^{\mu}{}_{\nu}\xi^{\nu}_{(1)}. \tag{2.2}$$

Then any vector written by a linear combination of $\xi^{\mu}_{(1)}$ and $\xi^{\mu}_{(2)}$ also satisfies Eq. (2.1). Thus zero eigenvectors constitute a two-dimensional vector space.

These properties are illustrated easily by a simple example in the flat space. Let us set the magnetic field as $\vec{B}=B\vec{e}_y$ and the electric field $\vec{E}=E\vec{e}_z$, where $\vec{e}_y$ and $\vec{e}_z$ are unit vectors in the Cartesian coordinate. Obviously, $\vec{E}\cdot\vec{B}=0$ is satisfied. Then $F_{\mu\nu}$ and $*F_{\mu\nu}$, respectively, become

$$F_{\mu\nu}=\begin{pmatrix}0&0&0&-E\\0&0&0&-B\\0&0&0&0\\E&B&0&0\end{pmatrix},\quad *F_{\mu\nu}=\begin{pmatrix}0&0&-B&0\\0&0&E&0\\B&-E&0&0\\0&0&0&0\end{pmatrix}. \tag{2.3}$$

From the first of the above equations, we see that $\xi^{\mu}_{(1)}$ is chosen as $\xi^{\mu}_{(1)}=(0,0,1,0)$. From Eq. (2.2), we have

$\xi^\mu_{(2)}=(-B,E,0,0)$. It is easy to see that the relations $*F^{\mu\nu}F_{\mu\nu}=0$, $F_{\mu\nu}\xi^\nu_{(2)}=0$, and $\xi_{(1)\mu}\xi^\mu_{(2)}=0$ are all satisfied. We can also see that $F_{\mu\nu}$ is a rank-2 matrix. These algebraic properties are independent of the orientation of the coordinate system. Further, a similar argument is possible in the curved space-time making use of the tetrad. Thus the results hold generally.

The degenerate electromagnetic field tensor $F_{\mu\nu}$ is classified into three classes by the sign of a four-invariant quantity $F_{\mu\nu}F^{\mu\nu}$ [11]. When $F_{\mu\nu}F^{\mu\nu}>0$, there exist timelike zero eigenvectors of $F_{\mu\nu}$. Making use of a Lorentz boost, we can find a frame in which observers at rest see a pure magnetic field. In this sense, the degenerate electromagnetic field is magnetic in this case. $F_{\mu\nu}F^{\mu\nu}=0$ is the case of the null field and corresponds to the radiation field. When $F_{\mu\nu}F^{\mu\nu}<0$, all the zero eigenvectors of $F_{\mu\nu}$ are spacelike. In this case, $F_{\mu\nu}$ is electric, i.e., we can find a frame in which observers at rest see a pure electric field.

Although the force-free electromagnetic field is described apart from the picture of the underlying plasma, the physical force-free electromagnetic field should permit the existence of the velocity field of the plasma. If we demand that the velocity field U^μ satisfies $F_{\mu\nu}U^\nu=0$ in the force-free approximation as in MHD, the force-free electromagnetic field must be magnetic because U^μ is a timelike vector. Thus the physical force-free electromagnetic field must satisfy

$$F_{\mu\nu}F^{\mu\nu}>0, \tag{2.4}$$

in all over the force-free region. In the following, condition (2.4) is regarded as a basic inequality that the physically acceptable force-fee electromagnetic field must satisfy. In Sec. VI, however, we will briefly discuss this point again.

Degenerate electromagnetic fields have these algebraic properties at every point. This adds distinct geometric properties to them. If the electromagnetic field is locally well behaved, we can construct two linearly independent locally well behaved vector fields of the zero eigenvectors. By virtue of Maxwell's equation, then these vector fields generate a family of two-dimensional integral surface in the four-dimensional space-time. In the next subsection, we prove this in a somewhat restricted form. A general proof is given in the Appendix. We call this integral surface the flux surface. A vector field that satisfies Eq. (2.1) is called the generator of the flux surface. The present theory is largely based on the concept of the flux surface. The geometrical nature of the theory follows from this fact.

Further, the existence of such integral surfaces enables us to introduce the concept of the magnetic field lines into the magnetic degenerate electromagnetic field in a geometrical way. The magnetic field line on a given three-space is defined by the intersection between the three-space and the flux surface. Thus the magnetic field line has invariant meaning on each three-space. It behaves as a stringlike object that preserves its self-identity during its time evolution. Conversely, in the 3+1 viewpoint, the flux surface is a track that the magnetic field line draws in four-space-time. Namely, the flux surface is the world sheet of the magnetic field line.

From the fact that the degenerate electromagnetic tensor $F_{\mu\nu}$ is a rank-2 matrix, we immediately see that the force-free Eq. (1.3) has only two independent components. Further, we can find variables that make Eq. (1.1) trivial. Thus Eq. (1.1) is not a dynamical equation. These facts indicate that the force-free electromagnetic field has only two ($\times\infty$) degrees of freedom as a field theoretical system. This observation is also consistent with the fact that the electromagnetic field in vacuum has two degrees of freedom of the polarization.

Note that if we choose the vector potential A_μ as the basic field in Eqs. (1.2) and (1.3), we cannot handle these equations so transparently. Since four equations are necessary to determine four components of the vector potential, two independent components of the force-free equation, the degenerate condition $*F^{\mu\nu}F_{\mu\nu}=0$ and a gauge condition such as the Lorentz gauge, possibly become a closed set of the basic equations. Thus at this stage we cannot exclude a possibility that there exists a general description of the force-free electromagnetic field by means of the vector potential. However, in such a formulation, two equations are constraint. It will be difficult to solve two constraint equations simultaneously. Therefore it seems evident that the theory that describes the force-free electromagnetic field by two variables is far more preferable. Indeed, we can find such a theory.

B. Euler potentials

As mentioned above, we can find the two-dimensional flux surface at every point where the degenerate electromagnetic field is well behaved. An important theorem follows from the existence of the flux surfaces. It yields a simple expression of the degenerate electromagnetic field. Namely, the theorem asserts that a degenerate electromagnetic field is generally written as

$$F_{\mu\nu}=\partial_\mu\phi_1\partial_\nu\phi_2-\partial_\mu\phi_2\partial_\nu\phi_1, \tag{2.5}$$

by two scalars ϕ_1 and ϕ_2. This expression of the degenerate electromagnetic field provides the basis for the rest of all our discussions. Inversely, if once Eq. (2.5) is established, the flux surfaces are defined as a family of surfaces on which ϕ_1 and ϕ_2 are constant. As far as we know, Carter [12] first remarked this point in the field of relativistic astrophysics, but this is a relativistic generalization of the Euler potentials used in the nonrelativistic MHD. (See [13], for the Euler potentials in the nonrelativistic MHD.) In the nonrelativistic MHD, the Euler potentials are often used in the form as $\vec{B}=\vec{\nabla}\alpha\times\vec{\nabla}\beta$. This corresponds to $F_{ij}=\partial_i\alpha\partial_j\beta-\partial_i\beta\partial_j\alpha$ $(i,j=1-3)$. Similarity between this and Eq. (2.5) is evident. Thus we also call ϕ_1 and ϕ_2 the Euler potentials henceforth. However, the Euler potentials α and β in the traditional usage and our Euler potentials ϕ_1 and ϕ_2 are not equivalent. Therefore, leaving a general mathematical proof of Eq. (2.5) to the Appendix, here we introduce ϕ_1 and ϕ_2 deriving Eq. (2.5) from the nonrelativistic Euler potentials in the flat space. This will clarify the relation between the traditional nonrelativistic Euler potentials and our Euler potentials.

We assume $\vec{B}=\vec{\nabla}\alpha\times\vec{\nabla}\beta$ and $\vec{B}\cdot\vec{E}=0$. As is well known, a solenoidal (divergence-free) vector $\vec{B}$ is generally expressed by the Euler potentials α and β as $\vec{B}=\vec{\nabla}\alpha\times\vec{\nabla}\beta$ [13,14]. Substituting this equation into $\partial_t\vec{B}+\vec{\nabla}\times\vec{E}=\vec{0}$, we have

$$\vec{\nabla}\times[\vec{E}+\partial_t\alpha\vec{\nabla}\beta-\partial_t\beta\vec{\nabla}\alpha+\vec{\nabla}(\alpha\partial_t\beta)]=\vec{0}. \quad (2.6)$$

Thus we have

$$\vec{E}+\partial_t\alpha\vec{\nabla}\beta-\partial_t\beta\vec{\nabla}\alpha+\vec{\nabla}(\alpha\partial_t\beta)=\vec{\nabla}f, \quad (2.7)$$

where f is an arbitrary function. Then the degenerate condition $\vec{B}\cdot\vec{E}=0$ leads us to

$$\vec{B}\cdot\vec{\nabla}(f-\alpha\partial_t\beta)=0. \quad (2.8)$$

Since $\vec{B}=\vec{\nabla}\alpha\times\vec{\nabla}\beta$, this implies $f-\alpha\partial_t\beta=\psi(\alpha,\beta)$, where $\psi(\alpha,\beta)$ is an arbitrary function of α and β. Thus the electric field and the magnetic field are written as

$$\vec{E}=-\partial_t\alpha\vec{\nabla}\beta+\partial_t\beta\vec{\nabla}\alpha+\vec{\nabla}\psi(\alpha,\beta),\quad \vec{B}=\vec{\nabla}\alpha\times\vec{\nabla}\beta. \quad (2.9)$$

This expression is different from Eq. (2.5) except for the case $\vec{\nabla}\psi(\alpha,\beta)=\vec{0}$. Thus the next step of the proof is to show that it is always possible to rewrite the above expression to the form

$$\vec{E}=-\partial_t\phi_1\vec{\nabla}\phi_2+\partial_t\phi_2\vec{\nabla}\phi_1,\quad \vec{B}=\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2. \quad (2.10)$$

This step becomes somewhat transparent making use of the differential form. The electromagnetic field two-form F is $\mathsf{F}=(1/2)F_{\mu\nu}dx^\mu\wedge dx^\nu$. Then we can rewrite Eq. (2.9) as

$$\mathsf{F}=d\alpha\wedge d\beta+d\psi\wedge dt. \quad (2.11)$$

We have to derive two functions ϕ_1 and ϕ_2 that express the electromagnetic field two-form as $\mathsf{F}=d\phi_1\wedge d\phi_2$. The case in which $\partial\psi/\partial\alpha=\partial\psi/\partial\beta=0$ is trivial. By virtue of the antisymmetry of $F_{\mu\nu}$ in α and β, it suffices to consider the case in which $\partial\psi/\partial\alpha\neq0$. Since $\partial\psi/\partial\alpha\neq0$, Eq. (2.11) is rewritten as

$$\begin{aligned}\mathsf{F}&=\left(\frac{\partial\psi}{\partial\alpha}\right)^{-1}\left(\frac{\partial\psi}{\partial\alpha}d\alpha+\frac{\partial\psi}{\partial\beta}d\beta\right)\wedge d\beta+d\psi\wedge dt\\&=d\psi\wedge\left\{dt+\left(\frac{\partial\psi}{\partial\alpha}\right)^{-1}d\beta\right\}.\end{aligned} \quad (2.12)$$

Further, by virtue of $\partial\psi/\partial\alpha\neq0$, we can invert $\psi=\psi(\alpha,\beta)$ as $\alpha=\alpha(\psi,\beta)$. Thus we have

$$\mathsf{F}=d\psi\wedge\left\{dt+\frac{\partial\alpha}{\partial\psi}d\beta\right\}. \quad (2.13)$$

Defining $\Phi(\psi,\beta)$ as

$$\Phi(\psi,\beta)=\int\frac{\partial\alpha(\psi,\beta)}{\partial\psi}d\beta, \quad (2.14)$$

we have

$$\mathsf{F}=d\psi\wedge\left\{dt+\frac{\partial\Phi}{\partial\beta}d\beta+\frac{\partial\Phi}{\partial\psi}d\psi\right\}. \quad (2.15)$$

Therefore introducing two scalars ϕ_1 and ϕ_2 as

$$\phi_1=\psi,\quad \phi_2=t+\Phi(\psi,\beta), \quad (2.16)$$

we have $\mathsf{F}=d\phi_1\wedge d\phi_2$. Accordingly, we can conclude that the degenerate electromagnetic field $F_{\mu\nu}$ is written as Eq. (2.5) by two functions ϕ_1 and ϕ_2. Then the degenerate condition

$$*F^{\mu\nu}F_{\mu\nu}=4\varepsilon^{\mu\nu\lambda\tau}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\lambda\phi_1\partial_\tau\phi_2=0 \quad (2.17)$$

becomes obvious.

Once expression (2.5) for the degenerate electromagnetic field is established, the flux surfaces are defined inversely by the surfaces on which ϕ_1 and ϕ_2 are constant. This condition yields two relations among four coordinates. Thus it defines a family of two-dimensional surfaces in the four-dimensional space-time.

An expression of the vector potential that yields an electromagnetic field (2.5) is given by

$$A_\mu=\frac{1}{2}(\phi_1\partial_\mu\phi_2-\phi_2\partial_\mu\phi_1). \quad (2.18)$$

Note that the Euler potentials that yield a given degenerate electromagnetic field are not unique. For example, after fixing ϕ_1, we can change ϕ_2 as $\phi_2\rightarrow\phi_2+f(\phi_1)$ by an arbitrary function of ϕ_1. This transformation changes the vector potential (2.18) as $A_\mu\rightarrow A_\mu+\partial_\mu\lambda(\phi_1)$ with $\lambda(\phi_1)=1/2\int(\phi_1 df/d\phi_1-f)d\phi_1$. Thus this corresponds to the gauge transformation. We will later give a thorough discussion on this point.

Further, the expression of the vector potential by the fixed Euler potentials is not unique. In fact, a gauge transformation generated by an arbitrary function of ϕ_1 and ϕ_2 as

$$A_\mu\rightarrow A_\mu+\partial_\mu\lambda(\phi_1,\phi_2) \quad (2.19)$$

gives a different expression of the vector potential written by the same Euler potentials. For example, by the gauge transformation

$$A_\mu\rightarrow\tilde{A}_\mu=A_\mu+\frac{1}{2}\partial_\mu(\phi_1\phi_2), \quad (2.20)$$

we have

$$\tilde{A}_\mu=\phi_1\partial_\mu\phi_2. \quad (2.21)$$

The similar form of the vector potential is often used in the nonrelativistic MHD as $\vec{A}=\alpha\vec{\nabla}\beta$.

III. BASIC EQUATION

A. Derivation of basic equation

If $F_{\mu\nu}$ is a solution of Eqs. (1.1)–(1.3), it is necessarily a degenerate electromagnetic field. Thus we can look for solutions of these equations from degenerate electromagnetic fields. Namely, we can restrict the electromagnetic fields to the degenerate fields before solving the force-free equation. The electromagnetic field is restricted to the degenerate field expressing it as Eq. (2.5) by means of two Euler potentials. Then Eq. (1.1) is automatically satisfied. Since the force-free equation has two independent components, Eqs. (1.2) and

(1.3) give a closed set of two equations for two Euler potentials. Accordingly, the basic equations of the force-free electrodynamics are always described by two nonlinear equations for two Euler potentials. In fact, using expression (2.5) and also combining Eq. (1.2) with Eq. (1.3), the force-free equation becomes

$$(\partial_\mu\phi_1\partial_\nu\phi_2-\partial_\mu\phi_2\partial_\nu\phi_1)\nabla_\lambda(\partial^\nu\phi_1\partial^\lambda\phi_2-\partial^\nu\phi_2\partial^\lambda\phi_1)=0. \tag{3.1}$$

Since $\partial_\mu\phi_1$ and $\partial_\mu\phi_2$ are independent and nonzero as long as $F_{\mu\nu}$ does not vanish, we can separate Eq. (3.1) to

$$\partial_\nu\phi_1\partial_\lambda\{\sqrt{-g}(\partial^\nu\phi_1\partial^\lambda\phi_2-\partial^\nu\phi_2\partial^\lambda\phi_1)\}=0,$$

$$\partial_\nu\phi_2\partial_\lambda\{\sqrt{-g}(\partial^\nu\phi_1\partial^\lambda\phi_2-\partial^\nu\phi_2\partial^\lambda\phi_1)\}=0. \tag{3.2}$$

These are the general basic equations of the force-free approximation.

B. Arbitrariness in solutions

As mentioned in Sec. II, we cannot determine the Euler potentials that yield a given degenerate electromagnetic field uniquely. Consequently, our basic equations (3.2) also do not have unique solutions. Now we clarify the extent of this arbitrariness. One degenerate electromagnetic field configuration corresponds to one geometrical configuration of the flux surfaces. Thus the transformation of the Euler potentials that leaves the electromagnetic field invariant does not change the geometrical configuration of the flux surfaces. Therefore such a transformation evidently has the form

$$\tilde\phi_1=\tilde\phi_1(\phi_1,\phi_2),\quad \tilde\phi_2=\tilde\phi_2(\phi_1,\phi_2), \tag{3.3}$$

where (ϕ_1,ϕ_2) and $(\tilde\phi_1,\tilde\phi_1)$ are two sets of the Euler potentials. This is because it does not alter the equipotential surfaces of the Euler potentials. Further, by this transformation the electromagnetic field changes as

$$\partial_\mu\tilde\phi_1\partial_\nu\tilde\phi_2-\partial_\mu\tilde\phi_2\partial_\nu\tilde\phi_1=\left(\frac{\partial\tilde\phi_1}{\partial\phi_1}\frac{\partial\tilde\phi_2}{\partial\phi_2}-\frac{\partial\tilde\phi_1}{\partial\phi_2}\frac{\partial\tilde\phi_2}{\partial\phi_1}\right)\times(\partial_\mu\phi_1\partial_\nu\phi_2-\partial_\mu\phi_2\partial_\nu\phi_1). \tag{3.4}$$

Thus, so that the electromagnetic field becomes invariant, the transformation must satisfy [14]

$$\frac{\partial\tilde\phi_1}{\partial\phi_1}\frac{\partial\tilde\phi_2}{\partial\phi_2}-\frac{\partial\tilde\phi_1}{\partial\phi_2}\frac{\partial\tilde\phi_2}{\partial\phi_1}=\frac{\partial(\tilde\phi_1,\tilde\phi_1)}{\partial(\phi_1,\phi_2)}=1, \tag{3.5}$$

where $\partial(\tilde\phi_1,\tilde\phi_1)/\partial(\phi_1,\phi_2)$ is the Jacobian of transformation (3.3). We can regard (ϕ_1,ϕ_2) and $(\tilde\phi_1,\tilde\phi_1)$ as two sets of coordinates orthogonal to the flux surface. Thus geometrically this implies that the volume element of the two-dimensional subspace orthogonal to the flux surface must be invariant. In other words, the mutual spacing of the flux surfaces along the direction orthogonal to these surfaces must be invariant so that the electromagnetic field will be invariant under transformation (3.3).

The transformation defined by Eqs. (3.3) and (3.5) induces a gauge transformation to the vector potential. Let A_μ be the vector potential given by Eq. (2.18). Then the gauge transformation induced in A_μ is as follows. From Eqs. (2.18) and (3.3), we have

$$\begin{aligned}\tilde A_\mu&=\frac{1}{2}(\tilde\phi_1\partial_\mu\tilde\phi_2-\tilde\phi_2\partial_\mu\tilde\phi_1)\\&=\left\{\left[\tilde\phi_1\frac{\partial\tilde\phi_2}{\partial\phi_2}-\tilde\phi_2\frac{\partial\tilde\phi_1}{\partial\phi_2}\right]\partial_\mu\phi_2-\left[\tilde\phi_2\frac{\partial\tilde\phi_1}{\partial\phi_1}-\tilde\phi_1\frac{\partial\tilde\phi_2}{\partial\phi_1}\right]\partial_\mu\phi_1\right\}.\end{aligned} \tag{3.6}$$

Further, there are relations

$$\begin{aligned}&\frac{\partial}{\partial\phi_1}\left[\tilde\phi_1\frac{\partial\tilde\phi_2}{\partial\phi_2}-\tilde\phi_2\frac{\partial\tilde\phi_1}{\partial\phi_2}\right]\\&\quad=\frac{\partial(\tilde\phi_1,\tilde\phi_2)}{\partial(\phi_1,\phi_2)}+\tilde\phi_1\frac{\partial^2\tilde\phi_2}{\partial\phi_1\partial\phi_2}-\tilde\phi_2\frac{\partial^2\tilde\phi_1}{\partial\phi_1\partial\phi_2}\\&\quad=1+\tilde\phi_1\frac{\partial^2\tilde\phi_2}{\partial\phi_1\partial\phi_2}-\tilde\phi_2\frac{\partial^2\tilde\phi_1}{\partial\phi_1\partial\phi_2}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\frac{\partial}{\partial\phi_2}\left[\tilde\phi_2\frac{\partial\tilde\phi_1}{\partial\phi_1}-\tilde\phi_1\frac{\partial\tilde\phi_2}{\partial\phi_1}\right]\\&\quad=\frac{\partial(\tilde\phi_1,\tilde\phi_2)}{\partial(\phi_1,\phi_2)}-\tilde\phi_1\frac{\partial^2\tilde\phi_2}{\partial\phi_1\partial\phi_2}+\tilde\phi_2\frac{\partial^2\tilde\phi_1}{\partial\phi_1\partial\phi_2}\\&\quad=1-\tilde\phi_1\frac{\partial^2\tilde\phi_2}{\partial\phi_1\partial\phi_2}+\tilde\phi_2\frac{\partial^2\tilde\phi_1}{\partial\phi_1\partial\phi_2}.\end{aligned} \tag{3.8}$$

Introducing a function $\lambda(\phi_1,\phi_2)$ by

$$\lambda(\phi_1,\phi_2)=\frac{1}{2}\int\left[\tilde\phi_1\frac{\partial^2\tilde\phi_2}{\partial\phi_1\partial\phi_2}-\tilde\phi_2\frac{\partial^2\tilde\phi_1}{\partial\phi_1\partial\phi_2}\right]d\phi_1 d\phi_2, \tag{3.9}$$

then we can integrate Eqs. (3.7) and (3.8) as

$$\tilde\phi_1\frac{\partial\tilde\phi_2}{\partial\phi_2}-\tilde\phi_2\frac{\partial\tilde\phi_1}{\partial\phi_2}=\phi_1+2\frac{\partial\lambda}{\partial\phi_2},$$

$$\tilde\phi_2\frac{\partial\tilde\phi_1}{\partial\phi_1}-\tilde\phi_1\frac{\partial\tilde\phi_2}{\partial\phi_1}=\phi_2-2\frac{\partial\lambda}{\partial\phi_1}, \tag{3.10}$$

respectively. Therefore we have

$$\begin{aligned}\tilde A_\mu&=\frac{1}{2}(\phi_1\partial_\mu\phi_2-\phi_2\partial_\mu\phi_1)+\frac{\partial\lambda}{\partial\phi_2}\partial_\mu\phi_2+\frac{\partial\lambda}{\partial\phi_1}\partial_\mu\phi_1\\&=A_\mu+\partial_\mu\lambda(\phi_1,\phi_2).\end{aligned} \tag{3.11}$$

This gives the gauge transformation induced in the vector potential explicitly.

Since the force-free equation can be written by the electromagnetic field $F_{\mu\nu}$ only, clearly our equation of motion is also invariant under the transformation defined by Eqs. (3.3) and (3.5). Therefore we can determine the solution of the basic equations (3.2) within the arbitrariness arising from this transformation.

C. Action principle

Maxwell's equation is derived from the action

$$I=-\frac{1}{16\pi}\int F^{\mu\nu}F_{\mu\nu}\sqrt{-g}d^4x, \tag{3.12}$$

where the vector potential is the dynamical variable [11,16]. The action that yields our basic Eqs. (3.2) also follows from Eq. (3.12) regarding the Euler potentials as dynamical variables. Expressing $F_{\mu\nu}$ by the Euler potentials, then the Lagrangian density $\mathcal{L}$ becomes

$$\mathcal{L}=\sqrt{-g}L=-\frac{1}{16\pi}\sqrt{-g}(\partial^\nu\phi_1\partial^\lambda\phi_2-\partial^\nu\phi_2\partial^\lambda\phi_1)\times(\partial_\nu\phi_1\partial_\lambda\phi_2-\partial_\nu\phi_2\partial_\lambda\phi_1), \tag{3.13}$$

where the L is the Lagrangian scalar. Indeed, the Euler-Lagrange equations for ϕ_1 and ϕ_2 become

$$\partial_\mu\{\sqrt{-g}\partial_\nu\phi_2(\partial^\mu\phi_1\partial^\nu\phi_2-\partial^\mu\phi_2\partial^\nu\phi_1)\}=0,$$
$$\partial_\mu\{\sqrt{-g}\partial_\nu\phi_1(\partial^\mu\phi_1\partial^\nu\phi_2-\partial^\mu\phi_2\partial^\nu\phi_1)\}=0, \tag{3.14}$$

respectively. Since $\partial_\mu\partial_\nu\phi_i=\partial_\nu\partial_\mu\phi_i$ holds, these equations coincide with Eqs. (3.2). Thus action (3.12) in fact yields the basic equations of the force-free approximation.

As usual, the energy-momentum tensor $T^{\mu\nu}$ is defined by $T^{\mu\nu}=(2/\sqrt{-g})\delta(\sqrt{-g}L)/\delta g_{\mu\nu}$. Thus we have

$$T^{\mu\nu}=\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L)}{\delta g_{\mu\nu}}=\frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}_{\lambda}-\frac{1}{4}g^{\mu\nu}F^{\lambda\tau}F_{\lambda\tau}\right), \tag{3.15}$$

where the electromagnetic field is written by the Euler potentials as Eq. (2.5). Eq. (3.15) shows that the energy-momentum tensor derived from Eq. (3.13) is also consistent with that of the ordinary electromagnetic field.

Some algebraic properties of the energy-momentum tensor result from the degeneracy of the electromagnetic field. Let ξ^μ denote a generator of the flux surface. From $F_{\mu\nu}\xi^\nu=0$, we find

$$T^\mu_\nu\xi^\nu=-\frac{1}{2}(F\cdot F)\xi^\mu, \tag{3.16}$$

where $F\cdot F=F^{\mu\nu}F_{\mu\nu}$. Namely, a generator of the flux surface becomes an eigenvector of the energy-momentum tensor. Its eigenvalue is given by $-(1/2)F\cdot F$. Similarly, we can also see

$$T^{\mu\nu}\partial_\nu\phi_i=\frac{1}{2}(F\cdot F)\partial_\mu\phi_i. \tag{3.17}$$

This implies that a vector orthogonal to the flux surface also becomes an eigenvector of the energy-momentum tensor that has eigenvalue $(1/2)F\cdot F$. Accordingly, choosing an orthonormal tetrad $e^\mu_{(\nu)}$ so that $e^\mu_{(0)}$ and $e^\mu_{(1)}$ will be the generator of the flux surfaces and $e^\mu_{(2)}$ and $e^\mu_{(3)}$ will be orthogonal to the flux surface, the energy-momentum tensor of the degenerate electromagnetic field becomes a diagonal form

$$T_{(\mu)(\nu)}=T_{\lambda\tau}e^\lambda_{(\mu)}e^\tau_{(\nu)}=\frac{1}{2}(F\cdot F)\mathrm{diag}(-1,-1,1,1). \tag{3.18}$$

From these observations, we have some conclusions. Let $U^\mu_{\rm obs}$ denote the four-velocity of an arbitrary observer. Then the magnetic field observed by this observer is $B^\mu_{\rm obs}=-{}^*F^\mu_\nu U^\nu_{\rm obs}$. Since $F_{\mu\nu}B^\nu_{\rm obs}=0$ holds, Eq. (3.16) implies that every observer sees the magnetic tension $(1/2)F\cdot F$ along his magnetic field. Further, an observer comoving with the magnetic field lines, i.e., an observer whose four-velocity $U^\nu_{\rm com}$ satisfies $F_{\mu\nu}U^\nu_{\rm com}=0$, sees the electromagnetic energy density $(1/2)F\cdot F$. He also observes the magnetic pressure $(1/2)F\cdot F$ to the spatial direction orthogonal to his magnetic field.

IV. DESCRIPTION OF FORCE-FREE ELECTROMAGNETIC FIELD BY MAGNETIC FIELD LINE

The basic variables of our theory are the Euler potentials. They have distinct geometric meaning as discussed above. However, the basic equations (3.2) do not have intuitive forms. Sometimes the magnetic field line seems more intuitive than the Euler potential. As mentioned in Sec. II, the magnetic degenerate electromagnetic field introduces the magnetic field lines on an arbitrary three-space without ambiguity. Each magnetic field line behaves as a one-dimensional stringlike entity that keeps its self-identity during the causal development in the degenerate electrodynamics. Thus a description of the force-free electromagnetic field by the magnetic field line becomes possible. It will complement the description by the Euler potentials and help our understanding of the dynamics of the force-free electromagnetic field, especially in $3+1$ formalism.

A. Kinetic quantities for magnetic field lines

In this work, we consider only the space-time that has global time coordinate t. Let N^μ be the four-velocity of the fiducial observer [5,7,15,16]. Namely, N^μ is a vector orthogonal to the three-space of constant global time t. Thus it is written as $N^\mu=\alpha\partial_\mu t$ where α is the lapse function. N^μ also satisfies $N^\mu N_\mu=-1$. The magnetic field B^μ and the electric field E^μ in this three-space are defined, respectively, by

$$B^\mu=-{}^*F^{\mu\nu}N_\nu,\quad E^\mu=F^{\mu\nu}N_\nu. \tag{4.1}$$

Then $F_{\mu\nu}$ and ${}^*F^{\mu\nu}$ are decomposed as

$$F_{\mu\nu}=N_\mu E_\nu-N_\nu E_\mu+\varepsilon_{\mu\nu\lambda\tau}N^\lambda B^\tau,$$
$${}^*F_{\mu\nu}=-N_\mu B_\nu+N_\nu B_\mu+\varepsilon_{\mu\nu\lambda\tau}N^\lambda E^\tau. \tag{4.2}$$

By definition, B^μ satisfies relations

$$F_{\mu\nu}B^{\nu}=0, \quad N_{\nu}B^{\nu}=0. \tag{4.3}$$

From the first of the above equations, we see that B^{μ} is a generator of the flux surface. Further, from the second, we see that B^{μ} is tangent to the three-space of constant time. Thus Eqs. (4.3) show that the magnetic field indeed becomes tangent to the intersection of the flux surface and the three-space of constant t.

The four-velocity of the magnetic field line U_F^{μ} is introduced uniquely by the following three requirements:

$$\text{(i)}\ U_F^{\mu}U_{F\mu}=-1, \quad \text{(ii)}\ F_{\mu\nu}U_F^{\nu}=0, \quad \text{(iii)}\quad U_F^{\mu}B_{\mu}=0. \tag{4.4}$$

The first equation demands that U_F^{μ} is a timelike unit vector. Since the flux surface is a world sheet of the magnetic field line, U_F^{μ} must be tangent to the flux surface. The second equation guarantees this condition. Consequently, $U_F^{\mu}\partial_{\mu}\phi_1=U_F^{\mu}\partial_{\mu}\phi_2=0$ holds. We can regard any U_F^{μ} that satisfies (i) and (ii) as the four-velocity of the magnetic field line, because there is no physical reasoning to decide the component of U_F^{μ} along the magnetic field line. Thus the third one is an auxiliary condition so as to make the direction of U_F^{μ} unique. Later, we will see that U_F^{μ} determined adding (iii) is well incorporated with the dynamics of the magnetic field lines.

Since B^{μ} is a generator of the flux surface, another generator of the flux surface orthogonal to B^{μ} is given by $*F^{\mu\nu}B_{\nu}$ according to Eq. (2.2). From condition (iii), we see $U_F^{\mu}\propto *F^{\mu\nu}B_{\nu}$. Since

$$g_{\lambda\mu}*F^{\lambda\nu}B_{\nu}*F^{\mu\tau}B_{\tau}=-\frac{1}{2}B^2(F\cdot F)=-B^2(B^2-E^2), \tag{4.5}$$

with $B=(B^{\mu}B_{\mu})^{1/2}$ and $E=(E^{\mu}E_{\mu})^{1/2}$, we have

$$U_F^{\mu}=\frac{\sqrt{2}}{B(F\cdot F)^{1/2}}*F^{\mu\nu}B_{\nu}=\frac{1}{B(B^2-E^2)^{1/2}}*F^{\mu\nu}B_{\nu}. \tag{4.6}$$

From Eqs. (4.5) and (4.6), it turns out that U_F^{μ} can be timelike only when condition (2.4) is satisfied, i.e., when the degenerate electromagnetic field is magnetic. Further, we can decompose U_F^{μ} into the three-velocity v_F^{μ} and the component orthogonal to three-space of constant t as

$$U_F^{\mu}=\gamma_F(N^{\mu}+v_F^{\mu}), \tag{4.7}$$

where γ_F is the Lorentz factor of the magnetic field line. Then we have

$$\gamma_F=-U_F^{\mu}N_{\mu}, \quad v_F^{\mu}=\gamma_F^{-1}(N^{\mu}N_{\nu}+\delta^{\mu}_{\nu})U_F^{\nu}, \tag{4.8}$$

respectively. These quantities are also written as

$$\gamma_F=(1-v_F^2)^{-1/2}=\left(\frac{B^2}{B^2-E^2}\right)^{1/2}, \quad v_F^{\mu}=\frac{1}{B^2}N_{\lambda}\varepsilon^{\lambda\mu\nu\tau}E_{\nu}B_{\tau}. \tag{4.9}$$

The last equation agrees with the expression $\vec{v}_F=\vec{E}\times\vec{B}/B^2$ in the nonrelativistic MHD. Evidently, $v_F^{\mu}B_{\mu}=0$ also holds. Inversely, $F_{\mu\nu}$ and $*F^{\mu\nu}$ are written, respectively, as

$$F_{\mu\nu}=\gamma_F^{-1}\varepsilon_{\mu\nu\lambda\tau}U_F^{\lambda}B^{\tau}, \quad *F^{\mu\nu}=-\gamma_F^{-1}(U_F^{\mu}B^{\nu}-U_F^{\nu}B^{\mu}), \tag{4.10}$$

by means of B^{μ} and U_F^{μ}. In these equations, γ_F appears because the magnetic field is defined with respect to the fiducial observers. Let $\widetilde{B}^{\mu}$ denote the magnetic field observed by an observer who moves with U_F^{μ}. Then we have $\widetilde{B}^{\mu}=\gamma_F^{-1}B^{\mu}$. Thus Eqs. (4.10) are equivalent to $F_{\mu\nu}=\varepsilon_{\mu\nu\lambda\tau}U_F^{\lambda}\widetilde{B}^{\tau}$ and $*F^{\mu\nu}=-(U_F^{\mu}\widetilde{B}^{\nu}-U_F^{\nu}\widetilde{B}^{\mu})$, which are the equations that frequently appear in the relativistic MHD. Generally, the same relations hold for observers moving with the four-velocity tangent to the flux surface.

B. Dynamics of the magnetic field line

The basic equations are rewritten by B^{μ} and U_F^{μ}. Equation (1.2) is equivalent to $\nabla_{\mu}*F^{\lambda\mu}=0$. Substituting Eq. (4.10) into this equation, we have

$$\gamma_F^{-1}U_F^{\nu}\nabla_{\nu}B^{\mu}-B^{\nu}\nabla_{\nu}(\gamma_F^{-1}U_F^{\mu})+\nabla_{\nu}(\gamma_F^{-1}U_F^{\nu})B^{\mu} -\gamma_F^{-1}(\nabla_{\nu}B^{\nu})U_F^{\mu}=0. \tag{4.11}$$

In the above equation, the component along U_F^{μ} is

$$U_F^{\mu}U_F^{\nu}\nabla_{\nu}B_{\mu}+\gamma_F\nabla_{\nu}(\gamma_F^{-1}B^{\nu})=0. \tag{4.12}$$

The components orthogonal to U_F^{μ} are

$$h^{\mu}_{\lambda}U_F^{\nu}\nabla_{\nu}B^{\lambda}-B^{\nu}\nabla_{\nu}U_F^{\mu}+\gamma_F\nabla_{\nu}(\gamma_F^{-1}U_F^{\nu})B^{\mu}=0, \tag{4.13}$$

where h^{μ}_{λ} is the projection tensor into the directions orthogonal to U_F^{μ}, that is,

$$h^{\mu}_{\lambda}=\delta^{\mu}_{\lambda}+U_F^{\mu}U_{F\lambda}. \tag{4.14}$$

Substituting B^{μ} and U_F^{μ} into the force-free equation, we have

$$B^2(\delta^{\nu}_{\lambda}-B_{\lambda}B^{\nu}/B^2)U_F^{\mu}\{\nabla_{\mu}(\gamma_F^{-1}U_{F\nu})-\nabla_{\nu}(\gamma_F^{-1}U_{F\mu})\} +\gamma_F^{-1}h^{\nu}_{\lambda}(\nabla_{\nu}B_{\mu}-\nabla_{\mu}B_{\nu})B^{\mu}=0. \tag{4.15}$$

Here $(\delta^{\nu}_{\lambda}-B_{\lambda}B^{\nu}/B^2)$ is the projection tensor into the subspace orthogonal to B^{μ}. Thus the inner products of Eq. (4.15) with B^{μ} and U_F^{μ} both vanish identically. Thus Eq. (4.15) gives only two components of the acceleration. The remaining component follows from Eq. (4.12). Making use of $B_{\mu}U_F^{\mu}=0$, we have

$$B_{\mu}U_F^{\nu}\nabla_{\nu}U_F^{\mu}-\gamma_F\nabla_{\nu}(\gamma_F^{-1}B^{\nu})=0. \tag{4.16}$$

Equation (4.16) gives the acceleration along the magnetic field. From Eqs. (4.15) and (4.16), we have

$$B^2a_{\lambda}-[\nabla_{\nu}B^{\nu}+2\gamma_F B^{\nu}\nabla_{\nu}\gamma_F^{-1}]B_{\lambda} +h^{\mu}_{\lambda}[\gamma_F B^2\nabla_{\mu}\gamma_F^{-1}+(\nabla_{\mu}B_{\nu}-\nabla_{\nu}B_{\mu})B^{\nu}]=0, \tag{4.17}$$

where a_λ is the acceleration of the magnetic field lines defined by $a^\mu = U_F^\nu \nabla_\nu U_F^\mu$. This is the Euler equation for the magnetic field lines. Together with $U_F^\nu U_{F\nu} = -1$ and $U_F^\nu B_\nu = 0$, Eqs. (4.13) and (4.17) describe the dynamics of the force-free electromagnetic field completely. Thus these equations constitute another set of the basic equations.

From Eq. (4.1), we have $\nabla_\nu B^\nu = -{}^*F^{\mu\nu}\nabla_\nu N_\mu$. Since $N_\mu = -\alpha\partial_\mu t$, $\nabla_\nu B^\nu$ in Eq. (4.17) vanishes in many cases. Especially, in the Minkowski space-time, these equations are greatly simplified. Substituting $U_F^\mu = (\gamma_F, \gamma_F \vec{v}_F)$, and $B^\mu = (0, \vec{B})$, together with $\nabla_\nu B^\nu = \vec{\nabla}\cdot\vec{B} = 0$, Eq. (4.13) is reduced to

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\times(\vec{v}_F \times \vec{B}) = \vec{0}. \tag{4.18}$$

This is the induction equation. Further, the t component of Eq. (4.17) becomes

$$\frac{1}{2}\partial_t(B^2 v_F^2) + (\vec{v}_F\times\vec{B})\cdot(\vec{\nabla}\times\vec{B}) = (\vec{v}_F\times\vec{B})\cdot[\partial_t(\vec{v}_F\times\vec{B}) + \vec{\nabla}\times\vec{B}] = 0. \tag{4.19}$$

As seen from the second expression of the above equation, this corresponds to $\vec{J}\cdot\vec{E} = 0$, i.e., absence of the Joule heating. The spatial components of Eq. (4.17) are

$$\gamma_F^2 B^2 D_t \vec{v}_F + \gamma_F(\vec{B}\cdot\vec{\nabla}v_F^2)\vec{B} + \frac{1}{2}(\partial_t B^2 - \gamma_F^2 B^2 \partial_t v_F^2)\vec{v}_F - \frac{1}{2}\gamma_F^2 B^2 \vec{\nabla} v_F^2 + \vec{B}\times(\vec{\nabla}\times\vec{B}) = \vec{0}, \tag{4.20}$$

where D_t is $\partial_t + (\vec{v}_F\cdot\vec{\nabla})$. The third term is rewritten by Eqs. (4.18) and (4.19). After some manipulations, we have the equation of motion for the magnetic field lines,

$$D_t\vec{v}_F = -\left\{\frac{1}{2}(\vec{e}_B\cdot\vec{\nabla})v_F^2\right\}\vec{e}_B + \left\{\frac{1}{2}(\vec{e}_v\cdot\vec{\nabla})v_F^2 + \frac{|\vec{v}_F|}{B^2}\vec{\nabla}\cdot(B^2\vec{v}_F) + \frac{1}{B^2}(1+v_F^2)\vec{e}_v\cdot[(\vec{\nabla}\times\vec{B})\times\vec{B}]\right\}\vec{e}_v + \left\{\frac{1}{2}(\vec{e}_E\cdot\vec{\nabla})v_F^2 + \frac{1}{B^2}(1-v_F^2)\vec{e}_E\cdot[(\vec{\nabla}\times\vec{B})\times\vec{B}]\right\}\vec{e}_E. \tag{4.21}$$

Here $\vec{e}_B$, $\vec{e}_v$, and $\vec{e}_E$ are three spatial basis vectors orthogonal to each other, i.e.,

$$\vec{e}_B = \frac{1}{|\vec{B}|}\vec{B}, \quad \vec{e}_v = \frac{1}{|\vec{v}_F|}\vec{v}_F, \quad \vec{e}_E = \frac{1}{|\vec{E}|}\vec{E} = -\frac{1}{|\vec{v}_F\times\vec{B}|}\vec{v}_F\times\vec{B}. \tag{4.22}$$

Namely, $\vec{e}_B$ is a unit vector along the magnetic field, $\vec{e}_v$ is a unit vector along the velocity of the magnetic field line, and $\vec{e}_E$ is a unit vector along the electric field. From Eqs. (4.18) and (4.21), it is easily verified that $D_t(\vec{B}\cdot\vec{v}_F) = 0$ if $\vec{B}\cdot\vec{v}_F = 0$ holds. Here the right-hand side of Eqs. (4.18) and (4.21) are written by $\vec{B}$, $\vec{v}_F$, and their spatial derivatives. Consequently, if $\vec{B}\cdot\vec{v}_F = 0$ is satisfied initially, $\vec{B}\cdot\vec{v}_F = 0$ holds through the evolution. Thus together with the initial condition $\vec{B}\cdot\vec{v}_F = 0$, Eqs. (4.18) and (4.21) determine the evolution of the force-free electromagnetic field completely in the flat space.

Equation (4.18) is identical with the induction equation. Further, mathematical structure of Eq. (4.21) is very similar to the Euler equation in the nonrelativistic MHD. Probably numerical method in the nonrelativistic MHD will be applicable to the force-free electromagnetic field modifying the force term in the Euler equation.

The right-hand side of Eq. (4.21) corresponds to the force acting on the magnetic field lines. It arises from three different effects: $\vec{\nabla}v_F^2$, $\vec{\nabla}\cdot(B^2\vec{v}_F)$ and $(\vec{\nabla}\times\vec{B})\times\vec{B}$. The force due to $\vec{\nabla}v_F^2$ has the opposite effect in the direction along the magnetic field and in the directions orthogonal to the magnetic field. The force from $\vec{\nabla}\cdot(B^2\vec{v}_F)$ acts in the direction along the velocity of the magnetic field line only. Further, $(\nabla\times\vec{B})\times\vec{B}$ force has a similar form to $\vec{j}\times\vec{B}$ force in the nonrelativistic MHD. However, it is enhanced by a factor $(1+v_F^2)$ in the direction of the velocity of the magnetic field line; on the other hand, it is reduced by $(1-v_F^2)$ in the direction of the electric field. When $v_F^2 \ll 1$, i.e., in ''nonrelativistic'' limit, the right-hand side of Eq. (4.21) tends to $(1/B^2)(\vec{\nabla}\times\vec{B})\times\vec{B}$. Thus B^2 corresponds to the mass. Further, from Eq. (4.17), we also have

$$\frac{1}{2}\partial_t(B^2 v_F^2 + B^2) + \vec{\nabla}\cdot(B^2\vec{v}_F) = 0. \tag{4.23}$$

This is equivalent to the energy conservation. We treat another form of this equation in the next section.

V. EVOLUTION OF FORCE-FREE ELECTROMAGNETIC FIELD

A. Initial value problem

In order to gain further insight into the contents and the structure of the basic equations, we are going to consider the causal development of the force-free electromagnetic field. For this purpose, we separate the time derivatives of the Euler potentials from the spatial derivatives in Eq. (3.2). Although we confine our consideration to the flat Minkowski space-time for simplicity, this does not bring any essential restriction to our analysis. Extension to the 3+1 formulation in the curved space-time is straightforward. In the flat three-space, we assume a curvilinear orthogonal coordinate and use the ordinary vector analysis to the spatial derivatives of quantities.

The magnetic field $\vec{B}$ and the electric field $\vec{E}$ are given by Eq. (2.10). Let us split the time derivatives of $\phi_i (i=1,2)$ explicitly denoting $\partial_t\phi_i$ as $\dot{\phi}_i$. Straightforward but somewhat tedious calculations show us that Eqs. (3.2) become

$$\begin{pmatrix} \vec\nabla\phi_2\cdot\vec\nabla\phi_2 & -\vec\nabla\phi_1\cdot\vec\nabla\phi_2 \\ -\vec\nabla\phi_1\cdot\vec\nabla\phi_2 & \vec\nabla\phi_1\cdot\vec\nabla\phi_1 \end{pmatrix}\begin{pmatrix}\ddot\phi_1\\ \ddot\phi_2\end{pmatrix}+\begin{pmatrix} \vec\nabla\dot\phi_2\cdot\vec\nabla\phi_2 & \vec\nabla\phi_1\cdot\vec\nabla\dot\phi_2-2\vec\nabla\dot\phi_1\cdot\vec\nabla\phi_2 \\ \vec\nabla\dot\phi_1\cdot\vec\nabla\phi_2-2\vec\nabla\phi_1\cdot\vec\nabla\dot\phi_2 & \vec\nabla\dot\phi_1\cdot\vec\nabla\phi_1 \end{pmatrix}\begin{pmatrix}\dot\phi_1\\ \dot\phi_2\end{pmatrix}$$
$$\pm(\dot\phi_2\vec\nabla^2\phi_1-\dot\phi_1\vec\nabla^2\phi_2)\begin{pmatrix}\dot\phi_2\\ \dot\phi_1\end{pmatrix}\pm\vec\nabla\times(\vec\nabla\phi_1\times\vec\nabla\phi_2)\cdot\begin{pmatrix}\vec\nabla\phi_2\\ \vec\nabla\phi_1\end{pmatrix}=0. \tag{5.1}$$

Thus if the condition

$$\begin{vmatrix} \vec\nabla\phi_2\cdot\vec\nabla\phi_2 & -\vec\nabla\phi_1\cdot\vec\nabla\phi_2 \\ -\vec\nabla\phi_1\cdot\vec\nabla\phi_2 & \vec\nabla\phi_1\cdot\vec\nabla\phi_1 \end{vmatrix}=(\vec\nabla\phi_1\times\vec\nabla\phi_2)^2=|\vec B|^2\neq 0, \tag{5.2}$$

is satisfied, i.e., if $\vec B\neq\vec 0$ holds everywhere, we can invert Eq. (5.1) to the form

$$\begin{pmatrix}\ddot\phi_1\\ \ddot\phi_2\end{pmatrix}=-\frac{(\vec\nabla\phi_1\times\vec\nabla\phi_2)}{(\vec\nabla\phi_1\times\vec\nabla\phi_2)^2}\cdot\begin{pmatrix}\vec\nabla\phi_1\times\vec\nabla\dot\phi_2 & \vec\nabla\phi_1\times\vec\nabla\dot\phi_1\\ \vec\nabla\phi_2\times\vec\nabla\dot\phi_2 & \vec\nabla\phi_2\times\vec\nabla\dot\phi_1\end{pmatrix}\begin{pmatrix}\dot\phi_1\\ \dot\phi_2\end{pmatrix}-\frac{\vec\nabla\cdot(\dot\phi_1\vec\nabla\phi_2-\dot\phi_2\vec\nabla\phi_1)}{(\vec\nabla\phi_1\times\vec\nabla\phi_2)^2}\begin{pmatrix}\vec\nabla\phi_1\cdot\vec\nabla\phi_2 & -\vec\nabla\phi_1\cdot\vec\nabla\phi_1\\ \vec\nabla\phi_2\cdot\vec\nabla\phi_2 & -\vec\nabla\phi_1\cdot\vec\nabla\phi_2\end{pmatrix}\begin{pmatrix}\dot\phi_1\\ \dot\phi_2\end{pmatrix}$$
$$+\frac{1}{(\vec\nabla\phi_1\times\vec\nabla\phi_2)^2}\begin{pmatrix}\vec\nabla\phi_1\times(\vec\nabla\phi_1\times\vec\nabla\phi_2)\\ \vec\nabla\phi_2\times(\vec\nabla\phi_1\times\vec\nabla\phi_2)\end{pmatrix}\cdot\vec\nabla\times(\vec\nabla\phi_1\times\vec\nabla\phi_2). \tag{5.3}$$

This equation has a typical form of the Cauchy problem. As far as $|\vec B|\neq 0$ holds, the second time derivatives of ϕ_i in the left-hand side of the above equation are expressed by $\phi_i,\dot\phi_i$, and their spatial derivatives. Accordingly, if we prescribe the Euler potentials and their first time derivatives all over the initial three-space as initial data, the second time derivatives are given at every point of this three-surface from the initial data. This is because the spatial derivatives of these quantities can be evaluated by differentiating them on the initial three-surface. Further, if this is done at a time t, we can decide ϕ_i and $\dot\phi_i$ at $t+\Delta t$ by Eq. (5.3) so far as the Euler potentials admit smooth, i.e., power-series-like, causal development. Therefore, continuing this process, we can trace a causal development of the Euler potentials by Eq. (5.3) in principle until $|\vec B|=0$ occurs. Thus our basic equation gives complete and self-consistent description of the force-free electromagnetic field as far as $|\vec B|\neq 0$. For a while, let us assume that $|\vec B|\neq 0$ is satisfied in the whole region considered and study the contents of Eq. (5.3) further.

B. Arbitrariness in solutions

The covariant basic equation (3.2) does not determine the Euler potentials uniquely. We now clarify how this arbitrariness appears in the causal development of the Euler potentials. Without any loss of generality, we can set the initial time as $t=0$. We abbreviate $\phi_i(t,\vec x)$ as $\phi_i(t)$ below. Further, we assume that $|\vec B|=0$ does not occur.

First, we should note that the initial data of the Euler potentials and their first time derivatives that yield a given initial electromagnetic field configuration are not unique. In fact, two sets of initial data: $(\phi_i(0),\dot\phi_i(0))$ and $(\tilde\phi_i(0),\dot{\tilde\phi}_i(0))$ $(i=1,2)$, give the same electromagnetic field if they relate as

$$\tilde\phi_1(0)=f_1(\phi_1(0),\phi_2(0)),\quad \tilde\phi_2(0)=f_2(\phi_1(0),\phi_2(0)), \tag{5.4}$$

$$\frac{\partial(\tilde\phi_1(0),\tilde\phi_2(0))}{\partial(\phi_1(0),\phi_2(0))}=\frac{\partial(f_1,f_2)}{\partial(\phi_1(0),\phi_2(0))}=1, \tag{5.5}$$

and

$$\dot{\tilde\phi}_1(0)=\frac{\partial f_1}{\partial\phi_1}\dot\phi_1(0)+\frac{\partial f_1}{\partial\phi_2}\tilde\phi_2(0),$$
$$\dot{\tilde\phi}_2(0)=\frac{\partial f_2}{\partial\phi_1}\tilde\phi_1(0)+\frac{\partial f_2}{\partial\phi_2}\dot\phi_2(0), \tag{5.6}$$

on the initial three-space. In fact, we find

$$\vec B=\vec\nabla\tilde\phi_1\times\vec\nabla\tilde\phi_2=\vec\nabla\phi_1\times\vec\nabla\phi_2,$$
$$\vec E=-(\dot{\tilde\phi}_1\vec\nabla\tilde\phi_2+\dot{\tilde\phi}_2\vec\nabla\tilde\phi_1)=-(\dot\phi_1\vec\nabla\phi_2-\dot\phi_2\vec\nabla\phi_1). \tag{5.7}$$

Thus we should consider the relation between two sets of the Euler potentials that have evolved from two different sets of the initial condition satisfying relations (5.4)–(5.6).

Let $\tilde\phi_1(t)$ and $\tilde\phi_2(t)$ denote the Euler potentials starting from the initial data $(\tilde\phi_i(0),\dot{\tilde\phi}_i(0))$. Thus $\tilde\phi_1(t)$ and $\tilde\phi_2(t)$ obey the same equation as Eq. (5.3) in which ϕ_i are all replaced by $\tilde\phi_i$. For a while, we refer to this equation as Eq. (5.3′). Suppose that $(\tilde\phi_i(0),\dot{\tilde\phi}_i(0))$ and $(\phi_i(0),\dot\phi_i(0))$ relate as Eqs. (5.4)–(5.6) in the initial three-space. Then our task is to clarify the relation between $(\phi_i(t),\dot\phi_i(t))$ and $(\tilde\phi_i(t),\dot{\tilde\phi}_i(t))$ $(i=1,2)$ for all future time. The relation after the infinitesimal interval Δt suffices for this purpose. Since

$$\phi_i(\Delta t)=\phi_i(0)+\dot\phi_i(0)\Delta t,\quad \tilde\phi_i(\Delta t)=\tilde\phi_i(0)+\dot{\tilde\phi}_i(0)\Delta t, \tag{5.8}$$

and

$$\dot{\phi}_i(\Delta t)=\dot{\phi}_i(0)+\ddot{\phi}_i(0)\Delta t,\quad \dot{\tilde{\phi}}_i(\Delta t)=\dot{\tilde{\phi}}_i(0)+\ddot{\tilde{\phi}}_i(0)\Delta t \tag{5.9}$$

hold, we need to express $\dot{\tilde{\phi}}_i(0)$ and $\ddot{\tilde{\phi}}_i(0)$ as functions of $\phi_i(0)$ and $\dot{\phi}_i(0)$. From Eqs. (5.6) and (5.8), we find

$$\begin{aligned}\tilde{\phi}_i(\Delta t)&=f_i(\phi_1(0),\phi_2(0))+\left.\frac{\partial f_i}{\partial \phi_1}\right|_{t=0}\dot{\phi}_1(0)\Delta t+\left.\frac{\partial f_i}{\partial \phi_2}\right|_{t=0}\\&\quad\times\dot{\phi}_2(0)\Delta t\\&\cong f_i(\phi_1(\Delta t),\phi_2(\Delta t))\quad (i=1,2).\end{aligned}\tag{5.10}$$

Thus at $t=\Delta t$, $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are the same functions of ϕ_1 and ϕ_2 as at $t=0$. Consequently, we also have

$$\frac{\partial(\tilde{\phi}_1(\Delta t),\tilde{\phi}_2(\Delta t))}{\partial(\phi_1(\Delta t),\phi_2(\Delta t))}=\frac{\partial(f_1,f_2)}{\partial(\phi_1,\phi_2)}=1. \tag{5.11}$$

Here we use the fact that Eq. (5.5) is a functional identity between two sets of functions: (f_1,f_2) and (ϕ_1,ϕ_2). This is easily seen from the following consideration. $\partial(f_1,f_2)/\partial(\phi_1(0),\phi_2(0))$ is generally a function of $\phi_1(0)$ and $\phi_2(0)$. Denoting $\partial(f_1,f_2)/\partial(\phi_1(0),\phi_2(0))$ as $F(\phi_1(0),\phi_2(0))$, on the initial three-space we have

$$\vec{\nabla}F(\phi_1(0),\phi_2(0))=\frac{\partial F}{\partial \phi_1}\vec{\nabla}\phi_1+\frac{\partial F}{\partial \phi_2}\vec{\nabla}\phi_2=\vec{0}. \tag{5.12}$$

From $\vec{B}\neq\vec{0}$, we see $\vec{\nabla}\phi_1\neq\vec{0}$ and $\vec{\nabla}\phi_2\neq\vec{0}$. Further, we can also see that $\vec{\nabla}\phi_1$ and $\vec{\nabla}\phi_2$ are not parallel. Thus we have

$$\frac{\partial F(\phi_1,\phi_2)}{\partial \phi_1}=\frac{\partial F(\phi_1,\phi_2)}{\partial \phi_2}=0. \tag{5.13}$$

Thus F does not depend on $\phi_1(0)$ and $\phi_2(0)$. Therefore $\partial(f_1,f_2)/(\phi_1(0),\phi_2(0))=1$ identically holds irrespective of the functional forms of $\phi_1(0)$ and $\phi_2(0)$ as functions of the position.

Next we must rewrite the right-hand side of Eq. (5.3′). Using Eq. (5.7), the second and the third terms of the right-hand side of Eq. (5.3′) are expressed easily by $\phi_1(0)$ and $\phi_2(0)$. On the other hand, the first term is tedious. From the spatial derivatives of Eq. (5.6) on the initial three-surface, and further using

$$\frac{\partial}{\partial \phi_1}\frac{\partial(\tilde{\phi}_1,\tilde{\phi}_2)}{\partial(\phi_1,\phi_2)}=\frac{\partial}{\partial \phi_2}\frac{\partial(\tilde{\phi}_1,\tilde{\phi}_2)}{\partial(\phi_1,\phi_2)}=0, \tag{5.14}$$

we have

$$\begin{aligned}&(\vec{\nabla}\tilde{\phi}_1\times\vec{\nabla}\dot{\phi}_2)\dot{\tilde{\phi}}_1-(\vec{\nabla}\tilde{\phi}_1\times\vec{\nabla}\dot{\phi}_1)\dot{\tilde{\phi}}_2\\&\quad=\frac{\partial\tilde{\phi}_1}{\partial\phi_1}[(\vec{\nabla}\phi_1\times\vec{\nabla}\dot{\phi}_2)\dot{\phi}_1-(\vec{\nabla}\phi_1\times\vec{\nabla}\dot{\phi}_1)\dot{\phi}_2]\\&\qquad+\frac{\partial\tilde{\phi}_1}{\partial\phi_2}[(\vec{\nabla}\phi_2\times\vec{\nabla}\dot{\phi}_2)\dot{\phi}_1-(\vec{\nabla}\phi_2\times\vec{\nabla}\dot{\phi}_1)\dot{\phi}_2]\\&\qquad-\left[\frac{\partial^2\tilde{\phi}_1}{\partial\phi_1^2}\dot{\phi}_1{}^2+2\frac{\partial^2\tilde{\phi}_1}{\partial\phi_1\partial\phi_2}\dot{\phi}_1\dot{\phi}_2+\frac{\partial^2\tilde{\phi}_1}{\partial\phi_2^2}\dot{\phi}_2{}^2\right]\\&\qquad\times\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2,\end{aligned}\tag{5.15}$$

and

$$\begin{aligned}&(\vec{\nabla}\tilde{\phi}_2\times\vec{\nabla}\dot{\phi}_2)\dot{\tilde{\phi}}_1-(\vec{\nabla}\tilde{\phi}_2\times\vec{\nabla}\dot{\phi}_1)\dot{\tilde{\phi}}_2\\&\quad=\frac{\partial\tilde{\phi}_2}{\partial\phi_1}[(\vec{\nabla}\phi_1\times\vec{\nabla}\dot{\phi}_2)\dot{\phi}_1-(\vec{\nabla}\phi_1\times\vec{\nabla}\dot{\phi}_1)\dot{\phi}_2]\\&\qquad+\frac{\partial\tilde{\phi}_2}{\partial\phi_2}[(\vec{\nabla}\phi_2\times\vec{\nabla}\dot{\phi}_2)\dot{\phi}_1-(\vec{\nabla}\phi_2\times\vec{\nabla}\dot{\phi}_1)\dot{\phi}_2]\\&\qquad-\left[\frac{\partial^2\tilde{\phi}_2}{\partial\phi_1^2}\dot{\phi}_1{}^2+2\frac{\partial^2\tilde{\phi}_2}{\partial\phi_1\partial\phi_2}\dot{\phi}_1\dot{\phi}_2+\frac{\partial^2\tilde{\phi}_2}{\partial\phi_2^2}\dot{\phi}_2{}^2\right]\\&\qquad\times\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2.\end{aligned}\tag{5.16}$$

In Eqs. (5.14)–(5.16), $\phi_i(0)$ is abbreviated as ϕ_i. Using Eqs. (5.15) and (5.16) in the first term of Eq. (5.3′), we have

$$\begin{aligned}\ddot{\tilde{\phi}}_i(0)&=\left.\frac{\partial f_i}{\partial\phi_1}\right|_{t=0}\ddot{\phi}_1(0)+\left.\frac{\partial f_i}{\partial\phi_2}\right|_{t=0}\ddot{\phi}_2(0)+\left.\frac{\partial^2 f_i}{\partial\phi_2^2}\right|_{t=0}\dot{\phi}_1{}^2(0)\\&\quad+2\left.\frac{\partial^2 f_i}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_1(0)\dot{\phi}_2(0)+\left.\frac{\partial^2 f_i}{\partial\phi_2^2}\right|_{t=0}\dot{\phi}_2{}^2(0).\end{aligned}\tag{5.17}$$

The second of Eqs. (5.9) then becomes

$$\begin{aligned}\dot{\tilde{\phi}}_i(\Delta t)&=\dot{\tilde{\phi}}_i(0)+\left[\left.\frac{\partial f_i}{\partial\phi_1}\right|_{t=0}\ddot{\phi}_1(0)+\left.\frac{\partial f_i}{\partial\phi_2}\right|_{t=0}\ddot{\phi}_2(0)\right.\\&\quad+\left.\frac{\partial^2 f_i}{\partial\phi_2^2}\right|_{t=0}\dot{\phi}_1{}^2(0)+2\left.\frac{\partial^2 f_i}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_1(0)\dot{\phi}_2(0)\\&\quad\left.+\left.\frac{\partial^2 f_i}{\partial\phi_2^2}\right|_{t=0}\dot{\phi}_2{}^2(0)\right]\Delta t\\&\cong\left[\left.\frac{\partial f_i}{\partial\phi_1}\right|_{t=0}+\left.\frac{\partial^2 f_i}{\partial\phi_1^2}\right|_{t=0}\dot{\phi}_1(0)\Delta t\right.\\&\quad\left.+\left.\frac{\partial^2 f_i}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_2(0)\Delta t\right][\dot{\phi}_1(0)+\ddot{\phi}_1(0)\Delta t]\\&\quad+\left[\left.\frac{\partial f_i}{\partial\phi_2}\right|_{t=0}+\left.\frac{\partial^2 f_i}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_1(0)\Delta t\right.\\&\quad\left.+\left.\frac{\partial^2 f_i}{\partial\phi_2^2}\right|_{t=0}\dot{\phi}_2(0)\Delta t\right][\dot{\phi}_2(0)+\ddot{\phi}_2(0)\Delta t],\end{aligned}\tag{5.18}$$

where Eq. (5.6) is also used. Therefore finally we have

$$\dot{\tilde{\phi}}_i(\Delta t)\cong\left.\frac{\partial f_i}{\partial\phi_1}\right|_{t=\Delta t}\dot{\phi}_1(\Delta t)+\left.\frac{\partial f_i}{\partial\phi_2}\right|_{t=\Delta t}\dot{\phi}_2(\Delta t), \tag{5.19}$$

up to the first order in Δt. Thus $\dot{\phi}_i$ and $\dot{\tilde{\phi}}_i$ have the same relation as Eq. (5.6) at $t=\Delta t$. From Eqs. (5.10), (5.11), and (5.19), we can conclude that the functional relation between two sets of the Euler potentials is preserved through their evolution as specified at the initial time. We can expect this result to some extent. This is because the covariant form of the basic equation (3.2) has solutions that yield the same electromagnetic field and relate as Eqs. (3.3) and (3.4) in the whole space-time. Further, conditions (5.4) and (5.5) are nothing but the restriction of Eqs. (3.3) and (3.4) to the initial three-surface. Furthermore, condition (5.6) is the restriction of the time derivatives of Eq. (3.3) to the initial surface. However, these arguments manifestly lead us to the following conclusions. (i) Arbitrariness in the solutions of the Euler potentials can always be reduced to the arbitrariness in the initial datum of the Euler potentials. Namely, specifying one initial datum from all the possible initial data, the subsequent causal development of the Euler potentials is determined uniquely. (ii) Thus this indeterminacy does not lie in the dynamics of the Euler potentials but lies in the nonuniqueness of correspondence between the Euler potentials and the electromagnetic field. (iii) We need not add the gauge condition further, although the arbitrariness comes from the gauge invariance of the electromagnetic field.

C. Canonical formulation

Time evolution of the force-free electromagnetic field is described more transparently by the Hamiltonian form. Probably this formulation is also useful in the numerical simulation of the time evolution of the force-free electromagnetic field.

Let π_1 and π_2 be the canonical momentum conjugate to ϕ_1 and ϕ_2, respectively, i.e., π_1 and π_2 are defined by

$$\pi_1=\frac{\partial L}{\partial \dot{\phi}_1}, \quad \pi_2=\frac{\partial L}{\partial \dot{\phi}_2}. \tag{5.20}$$

Usually, the canonical momentum is defined from the Lagrangian density as $\partial \mathcal{L}/\partial \dot{\phi}_i$. In the non-Cartesian coordinate, this definition differs from the present one by the three-volume element. Both conventions describe the canonical equation of motion consistently. However, when the vector analysis in a curvilinear orthogonal coordinate is used for the spatial derivatives, the present one is simpler.

Equation (5.20) gives

$$\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}=\frac{1}{4\pi}\begin{pmatrix}\vec{\nabla}\phi_2\cdot\vec{\nabla}\phi_2 & -\vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2\\ -\vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2 & \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_1\end{pmatrix}\begin{pmatrix}\dot{\phi}_1\\ \dot{\phi}_2\end{pmatrix}. \tag{5.21}$$

Therefore, when the Hessian matrix $\partial^2 L/\partial\dot{\phi}_1\partial\dot{\phi}_2$ satisfies

$$\frac{\partial^2 L}{\partial\dot{\phi}_1\partial\dot{\phi}_2}=\frac{1}{(4\pi)^2}|\vec{B}|^2\neq 0 \tag{5.22}$$

in the entire force-free region, we can invert Eq. (5.21) as

$$\begin{pmatrix}\dot{\phi}_1\\ \dot{\phi}_2\end{pmatrix}=\frac{4\pi}{B^2}\begin{pmatrix}\vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_1 & \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2\\ \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2 & \vec{\nabla}\phi_2\cdot\vec{\nabla}\phi_2\end{pmatrix}\begin{pmatrix}\pi_1\\ \pi_1\end{pmatrix}. \tag{5.23}$$

Here we become aware of the difference between the force-free electromagnetic field and the ordinary gauge fields again. As is well known, in the gauge field theory, the determinant of the Hessian matrix vanishes. The time derivatives of some of the field variables cannot be expressed by the conjugate momenta. This demands introduction of constraint conditions. Consequently, the gauge field is generally described as a constraint system [17]. On the contrary, in the force-free electromagnetic field, the time derivatives of the Euler potentials are written by their conjugate momenta without a constraint condition as Eq. (5.23) so far as $|\vec{B}|^2\neq 0$ holds. This happens in spite of the existence of the arbitrariness originating from the gauge freedom. This feature of the basic equations is consistent with the conclusion of the preceding subsection. We can also explain it noting that the arbitrariness of the Euler potentials results only from the nonuniqueness of the correspondence between the Euler potentials and the electromagnetic field in the initial data.

The Hamiltonian scalar H is defined by

$$H=\pi_1\dot{\phi}_1+\pi_2\dot{\phi}_2-L. \tag{5.24}$$

Then we have

$$H=\frac{1}{8\pi}\left\{\frac{(4\pi)^2}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)^2 +(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2\right\}. \tag{5.25}$$

From the relation between $\dot{\phi}_i$ and π_i, we can also find that the electric field and the magnetic field are, respectively, expressed as

$$\vec{E}=\frac{4\pi(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\times(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2),$$

$$\vec{B}=\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2, \tag{5.26}$$

by the canonical variables. Thus the first term in the curly brackets of H, Eq. (5.25), is $|\vec{E}|^2$ and the second term corresponds to $|\vec{B}|^2$. Further, the Poynting vector $\vec{S}$ is given by

$$\vec{S}=\frac{1}{4\pi}\vec{E}\times\vec{B}=-(\pi_1\vec{\nabla}\phi_2+\pi_2\vec{\nabla}\phi_2). \tag{5.27}$$

By Eq. (5.25), the canonical equations of motion

$$\dot{\phi}_i=\frac{\delta H}{\delta\pi_i}=\frac{\partial H}{\partial\pi_i}-\vec{\nabla}\cdot\frac{\partial H}{\partial(\vec{\nabla}\pi_i)},$$

$$\dot{\pi}_i=-\frac{\delta H}{\delta\phi_i}=-\left(\frac{\partial H}{\partial\phi_i}-\vec{\nabla}\cdot\frac{\partial H}{\partial(\vec{\nabla}\phi_i)}\right) \tag{5.28}$$

become

$$\dot{\phi}_1=\frac{4\pi(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\cdot\vec{\nabla}\phi_1,$$

$$\dot{\phi}_2=\frac{4\pi(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\cdot\vec{\nabla}\phi_2, \tag{5.29}$$

and

$$\dot{\pi}_1=\vec{\nabla}\cdot\left\{\frac{4\pi(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\pi_1\right.$$
$$+\frac{1}{4\pi}\left[1-\frac{(4\pi)^2(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)^2}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^4}\right]$$
$$\left.\times\vec{\nabla}\phi_2\times(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)\right\},$$

$$\dot{\pi}_2=\vec{\nabla}\cdot\left\{\frac{4\pi(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\pi_2\right.$$
$$-\frac{1}{4\pi}\left[1-\frac{(4\pi)^2(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)^2}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^4}\right]$$
$$\left.\times\vec{\nabla}\phi_1\times(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)\right\}. \tag{5.30}$$

Perhaps these are the most compact expressions of the canonical equations for $\dot{\pi}_1$ and $\dot{\pi}_2$. However, sometimes another form is far more convenient. Decomposing the square brackets in the right-hand side and also rearranging the terms, we have

$$\dot{\pi}_1=-\frac{1}{4\pi}\vec{\nabla}\phi_2\cdot\vec{\nabla}\times(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)-4\pi\vec{\nabla}\cdot\left\{\frac{(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)\cdot\vec{\nabla}\phi_2}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\left[\frac{(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)\cdot\vec{\nabla}\phi_2}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\vec{\nabla}\phi_1\right.\right.$$
$$\left.\left.-\frac{(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)\cdot\vec{\nabla}\phi_1}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\vec{\nabla}\phi_2\right]\right\},$$

$$\dot{\pi}_2=-\frac{1}{4\pi}\vec{\nabla}\phi_1\cdot\vec{\nabla}\times(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)+4\pi\vec{\nabla}\cdot\left\{\frac{(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)\cdot\vec{\nabla}\phi_1}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\left[\frac{(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)\cdot\vec{\nabla}\phi_2}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\vec{\nabla}\phi_1\right.\right.$$
$$\left.\left.-\frac{(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)\cdot\vec{\nabla}\phi_1}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\vec{\nabla}\phi_2\right]\right\}. \tag{5.31}$$

For purposes such as checking the consistency of the canonical equations of motion with the equation of motion (5.1) or the proof of the energy conservation the latter expression is more useful.

Time evolution of any quantity written by the canonical variables is then obtained from Eqs. (5.29) and (5.30). For example, it is straightforward to show

$$\partial_t H+\vec{\nabla}\cdot\vec{S}=0. \tag{5.32}$$

In Sec. II we introduced the three-velocity of the magnetic field line. By the canonical variables, the three-velocity and the Lorentz factor are written, respectively, as

$$\vec{v}_F=\frac{4\pi(\pi_1\vec{\nabla}\phi_2+\pi_2\vec{\nabla}\phi_2)}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2},$$

$$\gamma_F=\left[1-\frac{(4\pi)^2(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)^2}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^4}\right]^{-1/2}. \tag{5.33}$$

Substituting these equations into Eqs. (5.25) and (5.27), we can easily see the equivalence between Eq. (4.23) and Eq. (5.32). We can also derive equations for the time derivatives of $\vec{B}$ and $\vec{v}_F$. After somewhat tedious calculation, it turns out that we have Eqs. (4.18) and (4.21) indeed.

Using Eq. (5.33), Eqs. (5.29) are written as

$$(\partial_t+\vec{v}_F\cdot\vec{\nabla})\phi_1=0,\quad(\partial_t+\vec{v}_F\cdot\vec{\nabla})\phi_2=0. \tag{5.34}$$

This manifests that the Euler potentials are indeed constant on a given magnetic field line. Further, Eqs. (5.30) are written as

$$\partial_t\pi_1+\vec{\nabla}\cdot(\pi_1\vec{v}_F)=-\frac{1}{4\pi}\vec{\nabla}\phi_2\cdot\vec{\nabla}\times\{\gamma_F^{-2}\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2\},$$

$$\partial_t\pi_2+\vec{\nabla}\cdot(\pi_2\vec{v}_F)=\frac{1}{4\pi}\vec{\nabla}\phi_1\cdot\vec{\nabla}\times\{\gamma_F^{-2}\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2\}. \tag{5.35}$$

Thus Eqs. (5.30) have similar structure to the equation of continuity with the source or the sink.

D. Arbitrariness in canonical variables

The arbitrariness appears also in the canonical formalism. Let ϕ_i and $\widetilde{\phi}_i$ be two sets of the Euler potentials relating as Eqs. (3.3) and (3.5). Further, let $\widetilde{\pi}_i$ be the canonical momentum conjugate to $\tilde{\phi}_i$. Then a similar relation to Eq. (5.21) holds between $\tilde{\pi}_i$ and $\tilde{\phi}_i$. From Eq. (5.6), then we can show that π_i and $\tilde{\pi}_i$ relate as

$$\tilde{\pi}_1=\frac{\partial\tilde{\phi}_2}{\partial\phi_2}\pi_1-\frac{\partial\tilde{\phi}_2}{\partial\phi_1}\pi_2,\quad \tilde{\pi}_2=-\frac{\partial\tilde{\phi}_1}{\partial\phi_2}\pi_1+\frac{\partial\tilde{\phi}_1}{\partial\phi_1}\pi_2. \tag{5.36}$$

Equations (3.3), (3.5), and (5.36) define a transformation from one set of the canonical variables (ϕ_i,π_i) to another set $(\tilde{\phi}_i,\tilde{\pi}_i)$. Since there are relations

$$\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2=\vec{\nabla}\tilde{\phi}_1\times\vec{\nabla}\tilde{\phi}_2,$$

$$\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2=\tilde{\pi}_1\vec{\nabla}\tilde{\phi}_1+\tilde{\pi}_2\vec{\nabla}\tilde{\phi}_2, \tag{5.37}$$

the magnetic field and the electric field are invariant under this transformation.

The canonical equations of motion are similarly written as Eqs. (5.29) and (5.30) in terms of $\tilde{\pi}_i$ and $\tilde{\phi}_i$. Thus the solution of the canonical equations of motion is determined within the arbitrariness arising from the transformation given by Eqs. (3.3), (3.5), and (5.36). Here, note that we can restrict this transformation to a three-space of constant time. Thus it will be reduced to the arbitrariness in the initial data.

Let $(\phi_i(0),\pi_i(0))$ and $(\tilde{\phi}_i(0),\tilde{\pi}_i(0))$ be two initial data corresponding to the same electromagnetic field. Namely, they relate as Eqs. (5.4), (5.5), and

$$\tilde{\pi}_1(0)=\frac{\partial f_2}{\partial\phi_2}\pi_1(0)-\frac{\partial f_2}{\partial\phi_1}\pi_2(0),$$

$$\tilde{\pi}_2(0)=-\frac{\partial f_1}{\partial\phi_2}\pi_1(0)+\frac{\partial f_1}{\partial\phi_1}\pi_2(0). \tag{5.38}$$

Further, let $(\tilde{\phi}_i(t),\tilde{\pi}_i(t))$ denote the Euler potentials and their conjugate momenta that have evolved from the initial data $(\tilde{\phi}_i(0),\tilde{\pi}_i(0))$. From Eq. (5.4), we have Eq. (5.6) again. Further, from Eq. (5.38), we get

$$\begin{aligned}\dot{\tilde{\pi}}_1(0)=&\left.\frac{\partial f_2}{\partial\phi_2}\right|_{t=0}\dot{\pi}_1(0)-\left.\frac{\partial f_2}{\partial\phi_1}\right|_{t=0}\dot{\pi}_2(0)\\&+\left(\left.\frac{\partial^2 f_2}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_1(0)+\left.\frac{\partial^2 f_2}{\partial^2\phi_2}\right|_{t=0}\dot{\phi}_2(0)\right)\pi_1(0)\\&-\left(\left.\frac{\partial^2 f_2}{\partial^2\phi_1}\right|_{t=0}\dot{\phi}_1(0)+\left.\frac{\partial^2 f_2}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_2(0)\right)\pi_2(0),\end{aligned}$$

$$\begin{aligned}\dot{\tilde{\pi}}_2(0)=&\left.\frac{\partial f_1}{\partial\phi_1}\right|_{t=0}\dot{\pi}_2(0)-\left.\frac{\partial f_1}{\partial\phi_2}\right|_{t=0}\dot{\pi}_1(0)\\&+\left(\left.\frac{\partial^2 f_1}{\partial^2\phi_1}\right|_{t=0}\dot{\phi}_1(0)+\left.\frac{\partial^2 f_1}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_2(0)\right)\pi_2(0)\\&-\left(\left.\frac{\partial^2 f_1}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_1(0)+\left.\frac{\partial^2 f_1}{\partial^2\phi_1}\right|_{t=0}\dot{\phi}_2(0)\right)\pi_1(0).\end{aligned} \tag{5.39}$$

Using Eqs. (5.31) and (5.37), together with $\tilde{\phi}_i(\Delta t)=\tilde{\phi}_i(0)+\dot{\tilde{\phi}}_i(0)\Delta t$ and $\tilde{\pi}_i(\Delta t)=\tilde{\pi}_i(0)+\dot{\pi}_i(0)\Delta t$, we arrive at the relations between two sets of the canonical variables after the infinitesimal time interval Δt. Namely, at $t=\Delta t$, two sets of the Euler potentials relate as

$$\begin{aligned}\tilde{\phi}_i(\Delta t)&=\tilde{\phi}_i(0)+\left[\left.\frac{\partial f_i}{\partial\phi_1}\right|_{t=0}\dot{\phi}_1(0)+\left.\frac{\partial f_i}{\partial\phi_2}\right|_{t=0}\dot{\phi}_2(0)\right]\Delta t\\&=f_i(\phi_1(\Delta t),\phi_2(\Delta t)).\end{aligned} \tag{5.40}$$

The canonical momenta relate as

$$\begin{aligned}\tilde{\pi}_1(\Delta t)\cong&\left\{\left.\frac{\partial f_2}{\partial\phi_2}\right|_{t=0}+\left[\left.\frac{\partial^2 f_2}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_1(0)\Delta t\right.\right.\\&\left.\left.+\left.\frac{\partial^2 f_2}{\partial^2\phi_2}\right|_{t=0}\dot{\phi}_2(0)\Delta t\right]\right\}[\pi_1(0)+\dot{\pi}_1(0)\Delta t]\\&-\left\{\left.\frac{\partial f_2}{\partial\phi_1}\right|_{t=0}+\left[\left.\frac{\partial^2 f_2}{\partial^2\phi_1}\right|_{t=0}\dot{\phi}_1(0)\Delta t\right.\right.\\&\left.\left.+\left.\frac{\partial^2 f_2}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_2(0)\Delta t\right]\right\}[\pi_2(0)+\dot{\pi}_2(0)\Delta t]\\=&\left.\frac{\partial f_2}{\partial\phi_2}\right|_{t=\Delta t}\pi_1(\Delta t)-\left.\frac{\partial f_2}{\partial\phi_1}\right|_{t=\Delta t}\pi_2(\Delta t),\end{aligned}$$

$$\begin{aligned}\tilde{\pi}_2(\Delta t)\cong&\left\{\left.\frac{\partial f_1}{\partial\phi_1}\right|_{t=0}+\left[\left.\frac{\partial^2 f_1}{\partial^2\phi_1}\right|_{t=0}\dot{\phi}_1(0)\Delta t\right.\right.\\&\left.\left.+\left.\frac{\partial^2 f_1}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_2(0)\Delta t\right]\right\}[\pi_2(0)+\dot{\pi}_2(0)\Delta t]\\&-\left\{\left.\frac{\partial f_1}{\partial\phi_2}\right|_{t=0}+\left[\left.\frac{\partial^2 f_1}{\partial\phi_1\partial\phi_2}\right|_{t=0}\dot{\phi}_1(0)\Delta t\right.\right.\\&\left.\left.+\left.\frac{\partial^2 f_1}{\partial^2\phi_2}\right|_{t=0}\dot{\phi}_2(0)\Delta t\right]\right\}[\pi_1(0)+\dot{\pi}_1(0)\Delta t]\\=&-\left.\frac{\partial f_1}{\partial\phi_2}\right|_{t=\Delta t}\pi_1(\Delta t)+\left.\frac{\partial f_1}{\partial\phi_1}\right|_{t=\Delta t}\pi_2(\Delta t).\end{aligned} \tag{5.41}$$

Therefore the functional relation between two sets of the canonical variables that yield the same initial electromagnetic field does not change during the evolution of the system. Consequently, the arbitrariness in the canonical variables is also reduced to the arbitrariness in the initial data as expected.

VI. BREAKDOWN OF FORCE-FREE APPROXIMATION

We have assumed $\vec{B}\neq\vec{0}$ so far. We have also stated that the force-free approximation breaks down as $\vec{B}\to\vec{0}$. However, this statement is somewhat crude. First, strictly speaking, we did not decide whether the force-free approximation actually breaks down or not as $\vec{B}\to\vec{0}$ yet. Indeed, $|\vec{B}|=0$ implies that two canonical momenta cannot be independent and the system becomes a constrained one. Probably, $|\vec{B}|=0$ will happen in some region or at a point. If this is true, both the regions in which $|\vec{B}|=0$ and $|\vec{B}|\neq 0$ will coexist in the same three-space. As far as we know, such a case has never been treated in the field theory. The author cannot go into further detail at this point.

There is, however, a more important point. As mentioned in Sec. II, a physical force-free electromagnetic field must satisfy $F\cdot F=2(B^2-E^2)>0$. This condition is necessary as long as we assume the existence of the macroscopic four-velocity field of the matter U_M^μ satisfying $F_{\mu\nu}U_M^\nu=0$. Since $B^2\geqslant 0$ and $E^2\geqslant 0$ hold, generally $F\cdot F>0$ is more stringent than $\vec{B}\neq\vec{0}$.

When $F\cdot F$ vanishes, we can no longer find a timelike generator of the flux surface. U_M^μ is a unit timelike generator of the flux surface pointing future. Accordingly, it satisfies

$$F_{\mu\nu}U_M^\nu=0,\ g_{\mu\nu}U_M^\mu U_M^\nu=-1,\ U_M^\mu N_\mu<0. \tag{6.1}$$

Any generator of the flux surface is written by a linear combination of U_F^μ and B^μ. Thus U_M^μ is written as

$$U_M^\mu=a_1U_F^\mu+a_2e_B^\mu, \tag{6.2}$$

where $e_B^\mu=B^\mu/|B|$. Here a_1 and a_2 may be functions of position. Further, they satisfy $a_1^2-a_2^2=1$ and $a_1>1$ by Eqs. (6.1). From Eqs. (4.7) and (4.9), we have

$$U_M^\mu=a_1\left(\frac{B^2}{B^2-E^2}\right)^{1/2}N^\mu+\left\{a_1\left(\frac{B^2}{B^2-E^2}\right)^{1/2}v_F^\mu+a_2e_B^\mu\right\}. \tag{6.3}$$

Decomposing U_M^μ into the Lorentz factor γ_M and the three-velocity v_M^μ measured by a fiducial observer, we have

$$\gamma_M=a_1\left(\frac{B^2}{B^2-E^2}\right)^{1/2},$$

$$v_M^\mu=v_F^\mu+(1-1/a_1^2)^{1/2}\left(\frac{B^2-E^2}{B^2}\right)^{1/2}e_B^\mu, \tag{6.4}$$

respectively, where we eliminate a_2 by $a_1^2-a_2^2=1$. From these equations, we can see that if $B^2-E^2\to 0$ happens leaving B^2 finite, γ_M and v_M^μ tend to

$$\gamma_M\to\infty,\quad v_M^\mu\to v_F^\mu, \tag{6.5}$$

respectively. Namely, Eq. (6.4) shows that the three-velocity of any four-velocity field tangent to the flux surfaces approaches the three-velocity of the magnetic field line when B^2-E^2 approaches 0. The limiting behavior (6.5) is irrelevant to the choice of a_1 and a_2. At the same time, the three-velocity of the magnetic field line itself tends to the speed of light as B^2-E^2 vanishes, unless condition $(B^2-E^2)/B^2>0$ is kept when B^2-E^2 approaches zero.

The special case mentioned above is possible only when both $\vec{B}$ and $\vec{E}$ simultaneously tend to $\vec{0}$ satisfying the condition $(B^2-E^2)/B^2>0$. Thus this happens only when the rank of $F_{\mu\nu}$ as a matrix becomes zero as $\vec{B}\to\vec{0}$. By the canonical variables, these conditions are written as

$$\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2\to\vec{0},\ \ \vec{\nabla}\phi_1\times\vec{\nabla}\phi_2\to\vec{0},$$

$$1-\frac{(4\pi)^2(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)^2}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^4}>0. \tag{6.6}$$

Thus the condition

$$\left|\frac{(4\pi)(\pi_1\vec{\nabla}\phi_1+\pi_2\vec{\nabla}\phi_2)}{(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2}\right|<1 \tag{6.7}$$

is guaranteed even if $(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2\to 0$. Therefore all the apparently singular terms in the canonical equations of motion (5.29) and (5.30) remain regular in this case. Thus the basic equations are still applicable when $\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2=\vec{0}$ arises.

In order that Eq. (6.7) is satisfied, the denominator and the numerator in the left-hand side of this equation approach zero at least at the same rate. Roughly speaking, the denominator vanishes when $\vec{\nabla}\phi_1=\vec{0}$, $\vec{\nabla}\phi_2=\vec{0}$, or $\vec{\nabla}\phi_1\|\vec{\nabla}\phi_2$ happens. In any case, fine tuning of the canonical momentum is necessary so that the numerator vanishes with the denominator. Thus this is indeed a very special case. Judging from this, it seems that we can exclude this case from consideration.

Probably the breakdown of the condition $F\cdot F>0$ happens fairly universally in the real magnetospheres. Further, it seems very likely that it plays an important role in the magnetospheres. This is the reason for paying much attention to this point. Indeed, in a magnetosphere that consists of both the open and the closed field lines, such as the pulsar magnetosphere, the magnetic neutral point (line, or sheet) arises at the boundary of the open field lines and the closed field lines. More generally, such a region appears in the configurations in which the magnetic field lines having different topology coexist. At the magnetic neutral point (line, or sheet), the magnetic field vanishes. In the neighborhood of the magnetic neutral point, the force-free approximation will thus break down in twofold ways. Namely, in some region around the magnetic neutral point, the condition $F\cdot F>0$ will break down. This causes disappearance of the timelike generators of the flux surface. Further, at the magnetic neutral point $|\vec{B}|=0$ occurs. This alters the structure of the basic equation except for the special case mentioned above.

Unfortunately, the complexity of the basic equations prevents us from having any definite picture of the breakdown of the force-free approximation within the scope of this work. What physical or initial conditions do cause the breakdown of the force-free approximation? What physical process takes place if the force-free approximation breaks down? Further, do the basic equations really become singular or remain apparent singularity near the magnetic neutral point? These questions remain open.

Concerning this point, however, it is suggestive to recall the solutions of the stationary and axisymmetric configuration of the force-free electromagnetic field or the ideal MHD known so far. Almost all of them are made up of the open field lines only. The examples are Michel's split-monopole solution of the force-free electromagnetic field [2], the paraboloidal force-free electromagnetic field by Blandford [8], Macdonald's numerical solutions of the force-free black-hole magnetosphere [6], and the numerical solutions of the nonrelativistic ideal MHD flow by Sakurai [18]. On the contrary, numerical construction of the force-free electromagnetic field configuration that has both the open and the closed field lines (cited in [3]) indicates difficulty at the light cylinder. Evidently, there is much difficulty in the treatment of the force-free electromagnetic configurations that consist of both the open and the closed field lines. It seems very likely that the force-free approximation cannot describe the boundary between the topologically different field lines.

In addition, we should also note that if $F\cdot F=0$ happens, any theory based on the degenerate electromagnetic field, such as the ideal MHD approach, equally breaks down as far as the existence of the four-velocity tangent to the flux surface is assumed. Consequently, inclusion of the effect of the finite inertia of the plasma does not resolve the difficulty. Thus the breakdown of the force-free approximation discussed above is not a defect proper to the force-free approximation.

Hitherto, we have demanded that the physical force-free electromagnetic fields must have a four-velocity field satisfying $F_{\mu\nu}U^{\nu}=0$. This requires $F\cdot F>0$. However, $B^2-E^2=0$ does not make the basic equations singular as far as $|\vec{B}|\neq 0$. Since the force-free electromagnetic field is described without the four-velocity field of the matter, we have a possibility to extend the force-free approximation to the region where $F\cdot F\leqslant 0$ allowing $F_{\mu\nu}U_M^{\nu}\neq 0$. This implies that the degeneracy of the electromagnetic field still holds but the magnetic flux freezing to the matter is abandoned. Thus the flow of the matter across the magnetic field lines appears. Such flows necessarily accompany dissipation. Thus, if the fraction of the dissipative energy is much smaller than the whole electromagnetic energy, we will possibly extend the force-free approximation to the region where the condition $F\cdot F>0$ does not hold. In this sense, the force-free approximation may become more flexible than the theory based on the ideal MHD. Of course, the question of whether such a description is actually possible or not must be examined in each specific physical context. This is beyond the scope of this work.

VII. CONCLUDING REMARKS

In this paper we have presented a method to deal with the force-free electromagnetic field. We have shown that the force-free electromagnetic field is described as a self-consistent field theory so long as $F\cdot F>0$ is satisfied. In principle, the formulation presented here enables us to treat many problems that have not been studied systematically yet. The structure of the obliquely rotating pulsar magnetosphere and the evolution of the axisymmetric magnetosphere around the black hole or the accretion disk are a few examples. We hope and think that our formalism offers a concrete base for researches on these objects. Further, we have also formulated the initial value problem of the force-free electromagnetic field. We hope that this stimulates the numerical investigations of the force-free electromagnetic field and the relativistic magnetosphere.

Although we believe that this work has clarified the essential features of the force-free electromagnetic field, several important questions remain open to future work. Especially, the question on the breakdown of the force-free approximation near the magnetic neutral point will be important. Existence of the magnetic neutral point (region) will be quite universal, as already mentioned. Further, it seems that various energetic phenomena in the universe take place in association with the magnetic neutral point. The analysis on this point given in this work is quite insufficient and preliminary. It calls for further investigation.

A topic that is not considered in this work is concerned with the method of treating the configuration with symmetry. We discuss this point in the accompanying work.

Another problem that is not considered here is introduction of the macroscopic four-velocity field of plasmas into the force-free approximation. As mentioned in the Introduction, this is because the four-velocity of the plasma is an auxiliary variable in the force-free electrodynamics, and it should be distinguished from the dynamical variables of the force-free electromagnetic field. However, it becomes necessary when we relate solutions of the electromagnetic field with the motion of matter. Traditionally, the four-velocity of the plasma is introduced by means of the electric current and the charge density as $\vec{j}=\rho_e\vec{v}$. However, this definition is inadequate in the magnetospheres around the black hole or the accretion disk, because the fluid description of the plasma is more adequate in these objects. We will treat this problem elsewhere and propose another way to introduce the four-velocity of the plasma to the force-free approximation.

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APPENDIX

In this appendix, we shall prove the existence of the flux surface generally. We show that the zero eigenvectors of the degenerate electromagnetic field generate a family of two-dimensional integral surfaces. In addition to this point, we also prove that a degenerate electromagnetic field is expressed as Eq. (2.5). The proof is based on Frobenius theorem [19].

Let $F_{\mu\nu}$ denote a magnetic degenerate electromagnetic field. Then there is a vector field $\mathbf{U}=U^{\mu}\partial_{\mu}$ satisfying $F_{\mu\nu}U^{\nu}=0$ and $U^{\mu}U_{\mu}=-1$. (In this appendix, four-vectors are denoted by boldface.) We can define a vector field

$\mathbf{B}=B^\mu\partial_\mu$ by $B^\mu=-*F^{\mu\nu}U_\nu$. In terms of U^μ and B^μ, $*F^{\mu\nu}$ is written as

$$*F^{\mu\nu}=U^\mu B^\nu-B^\mu U^\nu. \tag{A1}$$

Then Maxwell's equation $\nabla_\nu*F^{\mu\nu}=0$ yields

$$U^\mu\nabla_\mu B^\nu-B^\mu\nabla_\mu U^\nu+(\nabla_\mu U^\mu)B^\nu-(\nabla_\mu B^\mu)U^\nu=0. \tag{A2}$$

This is also written as

$$[\mathbf{U},\mathbf{B}]=-(\boldsymbol{\nabla}\cdot\mathbf{U})\mathbf{B}+(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{U}, \tag{A3}$$

where $[\mathbf{U},\mathbf{B}]$ is the commutator of **U** and **B**. This is equivalent to the Lie derivative of **B** with respect to **U**. We can apply Frobenius theorem to Eq. (A3).

Let us summarize the Frobenius theorem. Let $T_x(M)$ be the tangent space of n-dimensional manifold M at $x\in M$ and Δ_x be an m-dimensional subset of $T_x(M)(m<n)$. Further, let a set $\{\mathbf{X}_{(i)}\}$ $(i=1,\ldots,m)$ be m-linearly independent vector fields constituting a local basis of Δ_x. Then the Frobenius theorem asserts if and only if, for every local basis $\{\mathbf{X}_{(i)}\}$ $(i=1,\ldots,m)$, each commutator $[\mathbf{X}_{(i)},\mathbf{X}_{(j)}]$ is written as

$$[\mathbf{X}_{(i)},\mathbf{X}_{(j)}]=c_{ij}^k\mathbf{X}_{(k)}, \tag{A4}$$

by differentiable functions c_{ij}^k, there is an m-dimensional integral submanifold passing through each point in M.

Further, this leads to the following corollaries.

(1) A Pfaffian system $X_{(i)}^\mu\partial_\mu f=0$ $(i=1,\ldots,m)$ is completely integrable. We have $n-m$ independent integral functions $f^k(k=m+1,\ldots,n)$. Slices on which $f^k(k=m+1,\ldots,n)$ are constant are the integral submanifolds for Δ.

(2) There is a local coordinate system $x^j(j=1,\ldots,n)$, such that $\partial/\partial x^i(i=1,\ldots,m)$ becomes a local basis vector for Δ, and any integral function $f^k(k=m+1,\ldots,n)$ is written as a function of $x^k(k=m+1,\ldots,n)$.

Since Eq. (A3) has the form of Eq. (A4), from the Frobenius theorem it turns out that the zero eigenvectors of the degenerate electromagnetic field generate a family of two-dimensional integral surfaces passing through each point. This is the flux surface.

Further, the first order partial differential equations

$$U^\mu\partial_\mu\phi=0,\quad B^\mu\partial_\mu\phi=0 \tag{A5}$$

yield two integrals ϕ_1 and ϕ_2 from (1). Then the flux surface becomes a surface on which two scalars ϕ_1 and ϕ_2 are constant. Furthermore, from (1) and (2), we can regard ϕ_1 and ϕ_2 as two coordinates orthogonal to the flux surfaces. Thus $d\phi_1$ and $d\phi_2$ are two independent one-forms orthogonal to **U** and **B**. Therefore the electromagnetic field two-form F is written as

$$\mathsf{F}=f d\phi_1\wedge d\phi_2. \tag{A6}$$

Then $d\mathsf{F}=df\wedge d\phi_1\wedge d\phi_2=0$ implies $f=f(\phi_1,\phi_2)$. Thus we have

$$\mathsf{F}=f(\phi_1,\phi_2)d\phi_1\wedge d\phi_2. \tag{A7}$$

However, by the transformation

$$\tilde{\phi}_1=\int f(\phi_1,\phi_2)d\phi_1, \tag{A8}$$

F is rewritten to the form as $\mathsf{F}=d\tilde{\phi}_1\wedge d\phi_2$. Therefore a degenerate electromagnetic field is always expressed as Eq. (2.5).

Further, from the definition of **U** and Maxwell's equations, we immediately see

$$\pounds_U\mathsf{F}=0,\quad \pounds_U d\mathsf{F}=0. \tag{A9}$$

This implies

$$\int_c\mathsf{F}=\int_{c'}\mathsf{F}, \tag{A10}$$

where c and c' denote two-surfaces connected by the tube of the trajectory of **U** (i.e., two two-surfaces on the same fluid element). This corresponds to the flux freezing in the ideal magnetohydrodynamics.

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