Scattering amplitude for a plane angular sector

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The exact solution of the wave equation for an incident plane wave in the spheroconal coordinate system is used to obtain formulas for scattering amplitude for a plane angular sector subject to Dirichlet or Neumann boundary conditions. These formulas are obtained by performing line integrals along the edges of the sector. The arguments of these integrals are the coefficients of the field singularity along the path of integration. [S1063-651X(97)10807-8]

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I. INTRODUCTION

There are only a few objects for which the scattering amplitude can be calculated analytically. The reason for this of course is due to the fact that calculation of scattering amplitude for a given object requires the solution of the wave equation by satisfying boundary conditions on the surface of the object. If the surface on which the boundary condition is specified corresponds to a coordinate surface, the wave equation can be separated into ordinary differential equations, which in most cases results in closed form solutions. The infinite cylinder, the half plane, and the sphere are examples of objects for which the scattering amplitude can be calculated analytically. In addition to the above objects for which an analytical solution is possible in the familiar cylindrical and polar coordinate systems, there are those objects for which this solution is possible in less familiar coordinate systems. For example, an analytic expression for scattering amplitude can be obtained for the parabolic cylinder and the elliptic cylinder in the parabolic and elliptic coordinate systems, respectively [1]. An analytic solution of the wave equation satisfying boundary conditions on the surface of an elliptic cone is also possible. Kraus and Levine [2] used the method of separation of variables in the spheroconal coordinate system to obtain Green's functions for the elliptic cone subject to either Dirichlet or Neumann boundary conditions. In the spheroconal coordinate system the wave equation separates into two angular Lamé equations and the spherical Bessel equation. The solution of the wave equation for a plane angular sector (PAS) is a special case of the solution of the wave equation for an elliptic cone, because, as is shown in Fig. 1, a PAS is a degenerate elliptic cone.

Since the work of Kraus and Levine other authors have studied scattering from a PAS. Radlow [3] studied the scattering of a plane wave from a quarter plane. Blume and Kirchner [4] studied the singular behavior of the field near the corner of a plane angular aperture and calculated the lowest eigenvalues for several different slot angles. Satterwhite [5] investigated the scattering of electromagnetic waves from a perfectly conducting PAS. He calculated the first few eigenvalues and eigenfunctions for the special case of a quarter plane, but did not report any results for the solutions of the scattered electric and magnetic fields. De Smedt and Van Bladel [6] also studied the singular behavior of the electric and magnetic fields near the tip of a PAS. They showed that the electric field is singular as $r^{\nu-1}$ and the magnetic field is singular as $r^{\tau-1}$, where r is distance to the tip of the sector. They calculated the lowest values for ν and τ using a variational technique. Boersma [7] used the Babinet's principle to show that the electric singularity exponent for a conducting PAS is identical to the magnetic singularity exponent for the complementary PAS. More recently, the solution of the wave equation for a PAS has been revisited by Abawi *et al.* [8]. In addition to proving theorems on the properties of the eigenvalues of the PAS, they have reported methods for calculating the eigenvalues and eigenfunctions of the wave equation subject to Dirichlet or Neumann boundary conditions on the surface of a PAS with arbitrary corner angle.

In a previous paper a formula for scattering amplitude of waves from plates of arbitrary shape in terms of a line integral around the edges of the plate was derived [9]. The use of this formula only requires the coefficient of field singularity

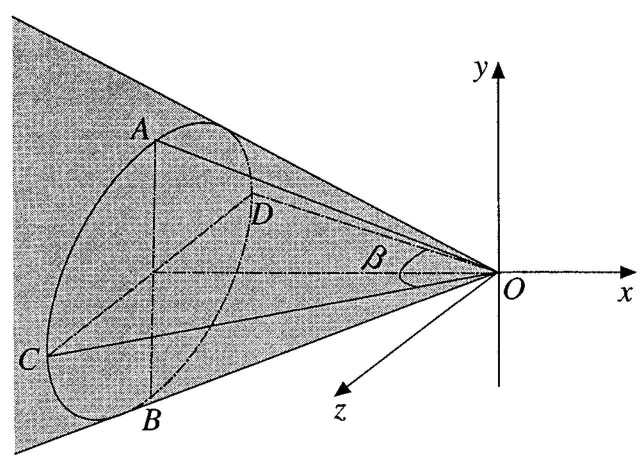

FIG. 1. This figure shows an elliptic cone with the apex at the origin, which in the spheroconal coordinate system is represented by $\vartheta=\vartheta_0$, where ϑ_0 is the angle between OA and the positive x axis. For $\vartheta_0=\pi$ the elliptic cone becomes degenerate (the elliptic base collapses to its major axis CD), resulting in the plane angular sector, COD, with corner angle β. Note that $\beta=0$ corresponds to a needle and $\beta=\pi$ corresponds to a half-plane.

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on the edge of the plate. More specifically, for Dirichlet boundary condition it requires the coefficient of the normal derivative and for Neumann boundary condition it requires the coefficient of the tangential derivative of the solution of the wave equation on the edge of the plate. If this coefficient is obtained from the exact solution of the wave equation, the scattering amplitude can be calculated exactly. In this paper, the above mentioned formula is used to formulate the scattering amplitude for a PAS, where the coefficient of the field singularity is calculated from the exact solution of the wave equation [2]. In Sec. II, the exact solution of the wave equation for a PAS is reviewed. In Sec. III this solution is used to calculate the coefficient of the field singularity on the edges of a PAS. These coefficients are then used to derive separate formulas for the scattering amplitude of waves from a PAS subject Dirichlet and Neumann boundary conditions.

II. THE SOLUTION OF THE WAVE EQUATION FOR A PLANE ANGULAR SECTOR

Kraus and Levine [2] solved the wave equation in the spheroconal coordinate system. The spheroconal coordinate system is defined by the set of equations

$$x=r\ \cos\vartheta\sqrt{1-\kappa'^2\cos^2\varphi},$$

$$y=r\ \sin\vartheta\ \sin\varphi,$$

$$z=r\ \cos\varphi\sqrt{1-\kappa^2\cos^2\vartheta}, \tag{1}$$

where

$$\kappa=\cos(\beta/2)=\cos(\varepsilon),$$

and $\kappa'=\sqrt{1-\kappa^2}$; the ranges of the variables are

$$0\leqslant\vartheta\leqslant\pi,\quad 0\leqslant\varphi\leqslant 2\pi,\quad r\geqslant 0.$$

The construction of this coordinate system is described, and its orthogonality proved, in [2]. The geometry of the coordinate system may briefly be described as follows: The coordinate r is the distance to the origin, so the surface $r=r_1$ is a sphere with its center at the origin. The coordinate surface $\vartheta=\vartheta_1$ is a semi-infinite elliptic cone whose cross section in a plane $x=$const is an ellipse centered on the x axis, with its major axis in the plane $y=0$. The surface $\varphi=\varphi_1$ is a semi-infinite elliptic half cone whose cross section in a plane z $=$const is half of an ellipse centered on the z axis with its major axis in the plane $y=0$. The geometry of the spheroconal coordinate system is shown in Fig. 2. For the case of scattering from a PAS, the scattering surface corresponds to $\vartheta=\pi$, as illustrated in Fig. 3. In this coordinate system the metric is given by

$$ds^2=h_1^2dr^2+h_2^2d\vartheta^2+h_3^2d\varphi^2,$$

where

$$h_1^2=1,$$

$$h_2^2=r^2\ \frac{\kappa^2\sin^2\vartheta+\kappa'^2\sin^2\varphi}{1-\kappa^2\ \cos^2\vartheta},$$

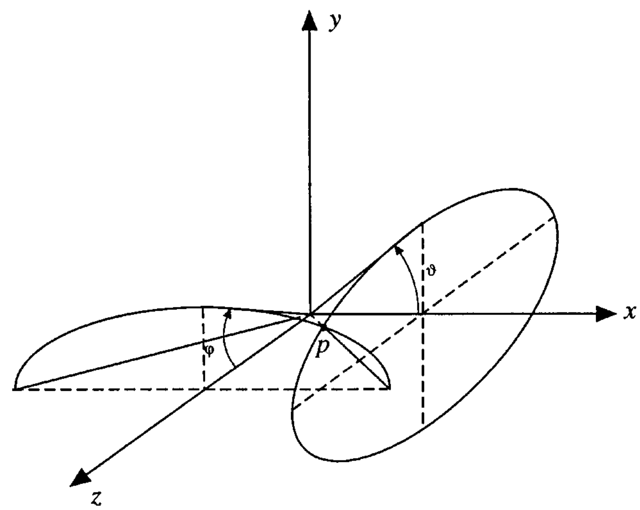

FIG. 2. The geometry of the spheroconal coordinate system. r is the distance from the origin to the point p.

$$h_3^2=r^2\ \frac{\kappa^2\sin^2\vartheta+\kappa'^2\sin^2\varphi}{1-\kappa'^2\ \cos^2\ \varphi}.$$

From the above equations we obtain the gradient

$$\nabla=\hat{\mathbf{e}}_r\frac{\partial}{\partial r}+\hat{\mathbf{e}}_\vartheta\frac{\sqrt{1-\kappa^2\cos^2\vartheta}}{r\sqrt{\kappa^2\ \sin^2\vartheta+\kappa'^2\sin^2\ \varphi}}\frac{\partial}{\partial\vartheta}+\hat{\mathbf{e}}_\varphi\frac{\sqrt{1-\kappa'^2\cos^2\varphi}}{r\sqrt{\kappa^2\ \sin^2\ \vartheta+\kappa'^2\ \sin^2\ \varphi}}\frac{\partial}{\partial\varphi}, \tag{2}$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\vartheta$, and $\hat{\mathbf{e}}_\varphi$ are unit vectors in the directions that the corresponding variables are increasing. Similarly, we find the Laplacian

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_\Omega,$$

where

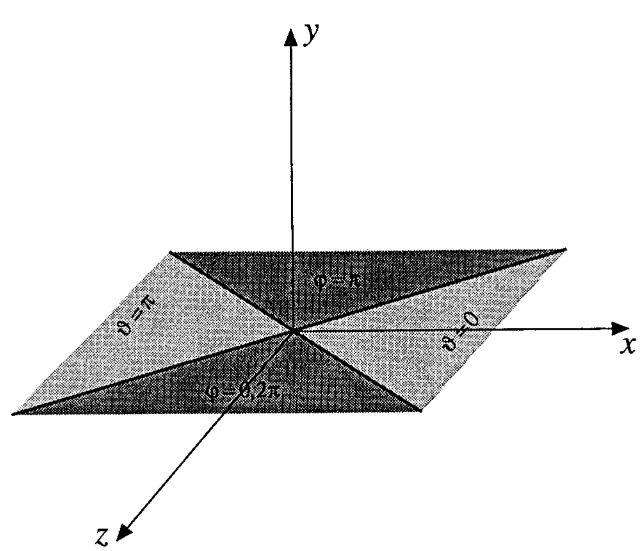

FIG. 3. The plane angular sectors $\vartheta=0$, $\vartheta=\pi$; $\varphi=0$, $\varphi=\pi$, and $\varphi=2\pi$.

$$\Delta_\Omega=\frac{1}{\kappa^2\sin^2\vartheta+\kappa'^2\sin^2\varphi}\left[\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{\partial}{\partial\vartheta}\times\left(\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{\partial}{\partial\vartheta}\right)+\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{\partial}{\partial\varphi}\left(\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{\partial}{\partial\varphi}\right)\right].\tag{3}$$

The wave equation in the spheroconal coordinate system is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\psi\right)+\frac{1}{r^2}\Delta_\Omega\psi+k^2\psi=0,$$

Kraus and Levine [2] have solved this equation by setting $\psi(r,\vartheta,\varphi)=R(r)V(\vartheta,\varphi)$. The above equation separates into a radial equation

$$\frac{d}{dr}\left(r^2\frac{d}{dr}R\right)+[k^2r^2-\nu(\nu+1)]R=0,$$

and an angular equation

$$\Delta_\Omega V+\nu(\nu+1)V=0,\tag{4}$$

where the separation constant has been written as $\nu(\nu+1)$. The solutions to the radial equation are the spherical Bessel functions

$$j_\nu(kr)=\sqrt{\frac{\pi}{2kr}}\,J_{\nu+1/2}(kr)$$

and

$$h_\nu(kr)=\sqrt{\frac{\pi}{2kr}}\,H^{(1)}_{\nu+1/2}(kr).$$

If we set $V(\vartheta,\varphi)=\Theta(\vartheta)\Phi(\varphi)$, the angular equation separates into

$$\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\left[\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\Theta(\vartheta)\right]+[\nu(\nu+1)\kappa^2\sin^2\vartheta+\mu]\Theta(\vartheta)=0\tag{5}$$

and

$$\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{d}{d\varphi}\left[\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{d}{d\varphi}\Phi(\varphi)\right]+[\nu(\nu+1)\kappa'^2\sin^2\varphi-\mu]\Phi(\varphi)=0.\tag{6}$$

Where μ is another separation constant. Equations (5) and (6) are the trigonometric form of the Lamé differential equation [10,11]. The Green's functions are found to be [2]

$$\begin{aligned}G^{(1,2)}&(r,\vartheta,\varphi;r',\vartheta',\varphi')\\&=ik\sum_{n=1}^{\infty}\frac{1}{N_{en}^{(1,2)}}j_{\nu_{en}}(kr_<)h_{\nu_{en}}(kr_>)\\&\quad\times\Theta_e^{(1,2)}(\vartheta;\kappa,\nu_{en}^{(1,2)},\mu_{en}^{(1,2)})\Theta_e^{(1,2)}(\vartheta';\kappa,\nu_{en}^{(1,2)},\mu_{en}^{(1,2)})\\&\quad\times\Phi_e^{(1,2)}(\varphi;\kappa',\nu_{en}^{(1,2)},\mu_{en}^{(1,2)})\\&\quad\times\Phi_e^{(1,2)}(\varphi';\kappa',\nu_{en}^{(1,2)},\mu_{en}^{(1,2)})\\&\quad+ik\sum_{n=1}^{\infty}\frac{1}{N_{on}^{(1,2)}}j_{\nu_{on}}(kr_<)h_{\nu_{on}}(kr_>)\\&\quad\times\Theta_o^{(1,2)}(\vartheta;\kappa,\nu_{on}^{(1,2)},\mu_{on}^{(1,2)})\Theta_o^{(1,2)}(\vartheta';\kappa,\nu_{on}^{(1,2)},\mu_{on}^{(1,2)})\\&\quad\times\Phi_o^{(1,2)}(\varphi;\kappa',\nu_{on}^{(1,2)},\mu_{on}^{(1,2)})\\&\quad\times\Phi_o^{(1,2)}(\varphi';\kappa',\nu_{on}^{(1,2)},\mu_{on}^{(1,2)}).\end{aligned}\tag{7}$$

In the above (ϑ,φ,r) represents the source point and (ϑ',φ',r') represents the observation point. Also, in the above $r_<\equiv\min(r,r')$ and $r_>\equiv\max(r,r')$ and the subscripts e and o denote even and odd solutions. The Green's functions satisfy the differential equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}G^{(1,2)}\right)+\frac{1}{r^2}\Delta_\Omega G^{(1,2)}+k^2G^{(1,2)}=-\frac{\delta(r-r')\,\delta(\vartheta-\vartheta')\,\delta(\varphi-\varphi')}{r^2\sigma},$$

subject to Dirichlet boundary condition

$$G^{(1)}(r,\vartheta,\varphi;r',\vartheta_0,\varphi')=0,$$

or Neumann boundary condition

$$G^{(2)}_\vartheta(r,\vartheta,\varphi;r',\vartheta_0,\varphi')=0.$$

In addition to the above boundary conditions the coordinate system imposes the matching conditions

$$G^{(1,2)}(r,\vartheta,\varphi;r',\vartheta',\varphi')=G^{(1,2)}(r,\vartheta,\varphi+2\pi;r',\vartheta',\varphi'),$$

$$G^{(1,2)}(r,0,\varphi;r',\vartheta',\varphi')=G^{(1,2)}(r,0,-\varphi;r',\vartheta',\varphi'),$$

$$G^{(1,2)}_\vartheta(r,0,\varphi;r',\vartheta',\varphi')=-G^{(1,2)}_\vartheta(r,0,-\varphi;r',\vartheta',\varphi').$$

In the above ϑ_0 is the boundary surface. The normalization coefficients are

$$N_{en}^{(1,2)}=\int_{-\pi}^{\pi}\int_0^{\vartheta_0}[\Theta_e^{(1,2)}(\vartheta;\kappa,\nu_{en}^{(1,2)},\mu_{en}^{(1,2)})\times\Phi_e^{(1,2)}(\varphi;\kappa',\nu_{en}^{(1,2)},\mu_{en}^{(1,2)})]^2\sigma d\vartheta\,d\varphi,$$

$$N_{on}^{(1,2)}=\int_{-\pi}^{\pi}\int_0^{\vartheta_0}[\Theta_o^{(1,2)}(\vartheta;\kappa,\nu_{on}^{(1,2)},\mu_{on}^{(1,2)})\times\Phi_o^{(1,2)}(\varphi;\kappa',\nu_{on}^{(1,2)},\mu_{on}^{(1,2)})]^2\sigma d\vartheta\,d\varphi,$$

where

$$\sigma=\frac{1}{r^2}h_1h_2h_3=\frac{\kappa^2\sin^2\vartheta+\kappa'^2\sin^2\varphi}{\sqrt{(1-\kappa^2\cos^2\vartheta)(1-\kappa'^2\cos^2\varphi)}}.$$

From the Green's functions we find the solutions of the wave equation for an incident plane wave in the direction (ϑ,φ) by taking the limit $r\rightarrow\infty$:

$$\begin{aligned}\psi_D(\vartheta,\varphi;r',\vartheta',\varphi')=&4\pi\sum_{n=1}^{\infty}\frac{1}{N_e^{(1)}}j_{\nu_{en}}(kr')(-i)^{\nu_{en}}\Theta_e^{(1)}(\vartheta;\kappa,\nu_{en}^{(1)},\mu_{en}^{(1)})\Theta_e^{(1)}(\vartheta';\kappa,\nu_{en}^{(1)},\mu_{en}^{(1)})\\&\times\Phi_e^{(1)}(\varphi;\kappa',\nu_{en}^{(1)},\mu_{en}^{(1)})\Phi_e^{(1)}(\varphi';\kappa',\nu_{en}^{(1)},\mu_{en}^{(1)})\\&+ik\sum_{n=1}^{\infty}\frac{1}{N_o^{(1)}}j_{\nu_{on}}(kr')(-i)^{\nu_{on}}\Theta_o^{(1)}(\vartheta;\kappa,\nu_{on}^{(1)},\mu_{on}^{(1)})\Theta_o^{(1)}(\vartheta';\kappa,\nu_{on}^{(1)},\mu_{on}^{(1)})\\&\times\Phi_o^{(1)}(\varphi;\kappa',\nu_{on}^{(1)},\mu_{on}^{(1)})\Phi_o^{(1)}(\varphi';\kappa',\nu_{on}^{(1)},\mu_{on}^{(1)})\end{aligned}\tag{8}$$

and

$$\begin{aligned}\psi_N(\vartheta,\varphi;r',\vartheta',\varphi')=&4\pi\sum_{n=1}^{\infty}\frac{1}{N_e^{(2)}}j_{\nu_{en}}(kr')(-i)^{\nu_{en}}\Theta_e^{(2)}(\vartheta;\kappa,\nu_{en}^{(2)},\mu_{en}^{(2)})\Theta_e^{(2)}(\vartheta';\kappa,\nu_{en}^{(2)},\mu_{en}^{(2)})\Phi_e^{(2)}(\varphi;\kappa',\nu_{en}^{(2)},\mu_{en}^{(2)})\\&\times\Phi_e^{(2)}(\varphi';\kappa',\nu_{en}^{(2)},\mu_{en}^{(2)})\\&+ik\sum_{n=1}^{\infty}\frac{1}{N_o^{(2)}}j_{\nu_{on}}(kr')(-i)^{\nu_{on}}\Theta_o^{(2)}(\vartheta;\kappa,\nu_{on}^{(2)},\mu_{on}^{(2)})\Theta_o^{(2)}(\vartheta';\kappa,\nu_{on}^{(2)},\mu_{on}^{(2)})\Phi_o^{(2)}(\varphi;\kappa',\nu_{on}^{(2)},\mu_{on}^{(2)})\\&\times\Phi_o^{(2)}(\varphi';\kappa',\nu_{on}^{(2)},\mu_{on}^{(2)}),\end{aligned}\tag{9}$$

where the eigenvalues ν and μ are determined in such a way that for the Dirichlet boundary condition

$$\psi_D(\vartheta,\varphi;r',\vartheta_0,\varphi')=0,$$

and for the Neumann boundary condition

$$\frac{\partial}{\partial\vartheta'}\psi_N(\vartheta,\varphi;r',\vartheta',\varphi')\bigg|_{\vartheta'=\vartheta_0}=0.$$

Methods for calculating the Lamé eigenvalues (ν,μ) and their corresponding eigenfunctions for both Dirichlet and Neumann boundary conditions are reported in a recent paper and the references therein [8].

III. SCATTERING AMPLITUDE FOR A PLANE ANGULAR SECTOR

The Green's functions given by Eq. (7) are the general solution of the wave equation when the boundary surface ϑ_0 is an elliptic cone. For the case of a PAS, this boundary surface collapses on its major axis resulting in the sector $\vartheta_0=\pi$. This is shown in Figs. 1 and 3. In this section we use Eqs. (8) and (9) to calculate the coefficients of the field singularity on the edges of a PAS. These coefficients are then used to calculate the scattering amplitude. Consider an incoming wave vector $\vec{\mathbf{k}}$ and a outgoing wave vector $\vec{\mathbf{q}}$:

$$\vec{\mathbf{k}}=-k_x\hat{\mathbf{x}}-k_y\hat{\mathbf{y}}-k_z\hat{\mathbf{z}},$$

$$\vec{\mathbf{q}}=q_x\hat{\mathbf{x}}+q_y\hat{\mathbf{y}}+q_z\hat{\mathbf{z}}.$$

Where for the incident angle (ϑ_k,φ_k) and the observation angle (ϑ_q,φ_q) in the spheroconal coordinate system we have

$$k_x=|k|\cos\vartheta_k\sqrt{1-\kappa'^2\cos^2\varphi_k},$$

$$k_y=|k|\sin\vartheta_k\sin\varphi_k,$$

$$k_z=|k|\cos\varphi_k\sqrt{1-\kappa^2\cos^2\vartheta_k},$$

$$q_x=|q|\cos\vartheta_q\sqrt{1-\kappa'^2\cos^2\varphi_q},$$

$$q_y=|q|\sin\vartheta_q\sin\varphi_q,$$

$$q_z=|q|\cos\varphi_q\sqrt{1-\kappa^2\cos^2\vartheta_q}.$$

A. Dirichlet boundary condition

In this case the scattering amplitude is given by [9]

$$T_D(\vec{\mathbf{k}},\vec{\mathbf{q}})=\frac{2\pi i}{|\hat{\mathbf{n}}\times\vec{\mathbf{Q}}|^2}\int C_D(\vec{\mathbf{k}},r')C_D(-\vec{\mathbf{q}},r')(\hat{\mathbf{n}}\times\vec{\mathbf{Q}})\cdot\vec{\mathbf{d}}r',$$

where

$$\vec{\mathbf{Q}}=\vec{\mathbf{k}}-\vec{\mathbf{q}}=Q_x\hat{\mathbf{x}}+Q_y\hat{\mathbf{y}}+Q_z\hat{\mathbf{z}},$$

$\hat{\mathbf{n}}$ is the outward unit normal on the surface of the PAS and $C_D(\vec{\mathbf{k}},r')$ and $C_D(-\vec{\mathbf{q}},r')$ are the coefficients of the singularity of the normal derivative of $\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\varphi')$ and $\psi_D(\vartheta_q,\varphi_q;r',\vartheta',\varphi')$ on the edges of the PAS. The above

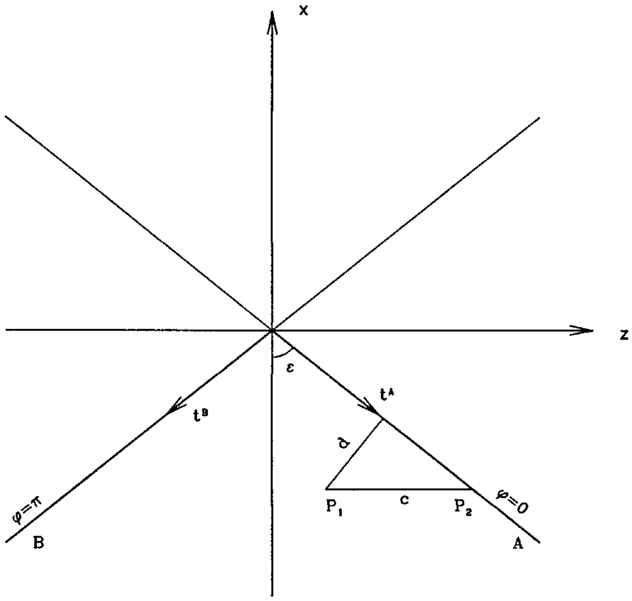

FIG. 4. The geometry of the PAS, $\vartheta_0=\pi$.

integral is to be integrated along both edges, A and B, of the PAS as illustrated in Fig. 4. Here,

$$\hat{\mathbf{n}}\times\vec{\mathbf{Q}}=\hat{\mathbf{y}}\times(Q_x\hat{\mathbf{x}}+Q_z\hat{\mathbf{z}})=Q_z\hat{\mathbf{x}}-Q_x\hat{\mathbf{z}},$$

and let us define $\hat{\mathbf{t}}^A$ and $\hat{\mathbf{t}}^B$ to be unit vectors along edges A and B:

$$\hat{\mathbf{t}}^A=-\cos\varepsilon\hat{\mathbf{x}}+\sin\varepsilon\hat{\mathbf{z}},$$

$$\hat{\mathbf{t}}^B=-\cos\epsilon\hat{\mathbf{x}}-\sin\varepsilon\hat{\mathbf{z}}.$$

With $\cos\varepsilon=\kappa$ and $\sin\varepsilon=\kappa'$, the above equations become

$$\hat{\mathbf{t}}^A=-\kappa\hat{\mathbf{x}}+\kappa'\hat{\mathbf{z}},$$

$$\hat{\mathbf{t}}^B=-\kappa\hat{\mathbf{x}}-\kappa'\hat{\mathbf{z}}.$$

Along edge A, $\vec{\mathbf{d}}r'$ can be written as

$$\vec{\mathbf{d}}r'=\hat{\mathbf{t}}^A dr'=(-\kappa\hat{\mathbf{x}}+\kappa'\hat{\mathbf{z}})dr',$$

and along edge B it can be written as

$$\vec{\mathbf{d}}r'=\hat{\mathbf{t}}^B dr'=(-\kappa\hat{\mathbf{x}}-\kappa'\hat{\mathbf{z}})dr'.$$

Thus

$$\hat{\mathbf{n}}\times\vec{\mathbf{Q}}\cdot\vec{\mathbf{d}}r'=-(\kappa Q_z+\kappa' Q_x)dr'$$

along edge A and

$$\hat{\mathbf{n}}\times\vec{\mathbf{Q}}\cdot\vec{\mathbf{d}}r'=(-\kappa Q_z+\kappa' Q_x)dr'$$

along edge B. The scattering amplitude can then be written as

$$\begin{aligned}T_D(\vec{\mathbf{k}},\vec{\mathbf{q}})=\frac{2\pi i}{|\hat{\mathbf{n}}\times\vec{\mathbf{Q}}|^2}\Bigg[&-\{\kappa Q_z+\kappa' Q_x\}\\&\times\int_0^\infty C_D^A(\vec{\mathbf{k}},r')C_D^A(-\vec{\mathbf{q}},r')dr'\\&+\{-\kappa Q_z+\kappa' Q_x\}\\&\times\int_0^\infty C_D^B(\vec{\mathbf{k}},r')C_D^B(-\vec{\mathbf{q}},r')dr'\Bigg]\end{aligned}\tag{10}$$

where the superscripts A and B refer to edges A and B. To find $C_D(\vec{\mathbf{k}},r')$ and $C_D(-\vec{\mathbf{q}},r')$ we write the normal derivative of $\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\varphi')$ as

$$\begin{aligned}\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\varphi')&=-\hat{\mathbf{e}}_\vartheta\cdot\boldsymbol{\nabla}\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\varphi')\\&=-\frac{\sqrt{1-\kappa^2\cos^2\vartheta'}}{r\sqrt{\kappa^2\sin^2\vartheta'+\kappa'^2\sin^2\varphi'}}\\&\quad\times\frac{\partial}{\partial\vartheta'}\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\varphi').\end{aligned}$$

When evaluated on the surface of the PAS, $\vartheta'=\pi$, it gives

$$\begin{aligned}&\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\varphi')|_{\vartheta'=\pi}\\&\quad=-\frac{1}{r\,\sin\varphi'}\frac{\partial}{\partial\vartheta'}\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\varphi')\Big|_{\vartheta'=\pi}\end{aligned}$$

On edges A and B ($\varphi'=0,\pi$) the normal derivative is singular. The singularity is like $1/\sqrt{d}$, where d is the perpendicular distance to the edge. Close to edge A, $\varphi'\to 0$,

$$\begin{aligned}&-\frac{1}{r'\,\sin\varphi'}\frac{\partial}{\partial\vartheta'}\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\varphi')\Big|_{\vartheta'=\pi}\\&\quad=\frac{1}{\sqrt{d^A}}C_D^A(\vec{\mathbf{k}},r'),\end{aligned}\tag{11}$$

and close to edge B, $\varphi'\to\pi$,

$$\begin{aligned}&-\frac{1}{r'\,\sin\varphi'}\frac{\partial}{\partial\vartheta'}\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\varphi')\Big|_{\vartheta'=\pi}\\&\quad=\frac{1}{\sqrt{d^B}}C_B^D(\vec{\mathbf{k}},r'),\end{aligned}\tag{12}$$

where d^A is the perpendicular distance to edge A and d^B is the perpendicular distance to edge B. To find C_D^A, we need to find a relationship between $\sin\varphi'$ and d^A. In Appendix A we find

$$d^A=r'\kappa'\sqrt{1-\kappa'^2\cos^2\varphi'}-r'\kappa\kappa'\cos\varphi'$$

and

$$d^B=r'\kappa'\sqrt{1-\kappa'^2\cos^2\varphi'}+r'\kappa\kappa'\cos\varphi'.$$

Close to edge A ($\varphi'\to 0$) we expand the above expression in powers of $\sin\varphi'$:

$$d^A = r' \frac{\kappa'}{2\kappa} \sin^2\varphi' + O(\sin\varphi')^4.$$

By substituting for d^A in Eq. (11) and taking the limit $\varphi' \to 0$, we get

$$C_D^A(\vec{\mathbf{k}},r') = -\frac{1}{\sqrt{r'}} \sqrt{\frac{\kappa'}{2\kappa}} \frac{\partial}{\partial\vartheta'} \psi_D(\vartheta_k,\varphi_k;r',\vartheta',0)\bigg|_{\vartheta'=\pi}.$$

Using Eq. (8) and noting that for Dirichlet boundary conditions the odd solutions vanish on the edges (see Appendix B), we find

$$C_D^A(\vec{\mathbf{k}},r') = \sum_{n=1}^{\infty} \chi_D(\nu_{en},\mu_{en};\vartheta_k,\varphi_k;0) \frac{j_{\nu_{en}}(r')}{\sqrt{r'}},$$

where we define

$$\begin{aligned}\chi_D(\nu_{en},\mu_{en};\alpha,\beta;\beta_0) &= -4\pi\sqrt{\frac{\kappa'}{2\kappa}}\frac{1}{N_e}(-i)^{\nu_{en}}\Theta_e(\alpha;\kappa,\nu_{en},\mu_{en})\\ &\quad\times\Phi_e(\beta;\kappa',\nu_{en},\mu_{en})\Phi_e(\beta_0;\kappa',\nu_{en},\mu_{en})\\ &\quad\times\frac{\partial}{\partial\vartheta'}\Theta_e(\vartheta';\kappa,\nu_{en},\mu_{en})|_{\vartheta'=\pi},\end{aligned}$$

and the superscript 1 has been suppressed. Similarly,

$$C_D^A(-\vec{\mathbf{q}},r') = \sum_{n=1}^{\infty} \chi_D(\nu_{en},\mu_{en};\vartheta_q,\varphi_q;0) \frac{j_{\nu_{en}}(r')}{\sqrt{r'}}.$$

Similarly close to edge B, $\varphi' \to \pi$,

$$d^B = r' \frac{\kappa'}{2\kappa} \sin^2\varphi' + O((\sin\varphi')^4).$$

By substituting for d^B in Eq. (12) and taking the limit $\varphi' \to \pi$, we find

$$\begin{aligned}C_D^B(\vec{\mathbf{k}},r') &= -\frac{1}{\sqrt{r'}}\left(\frac{\kappa'}{2\kappa}\right)^{1/2}\frac{\partial}{\partial\vartheta'}\\ &\quad\times\psi_D(\vartheta_k,\varphi_k;r',\vartheta',\pi)|_{\vartheta'=\pi},\end{aligned}$$

or we can write

$$C_D^B(\vec{\mathbf{k}},r') = \sum_{n=1}^{\infty} \chi_D(\nu_{en},\mu_{en};\vartheta_k,\varphi_k;\pi) \frac{j_{\nu_{en}}(r')}{\sqrt{r'}}$$

and

$$C_D^B(-\vec{\mathbf{q}},r') = \sum_{n=1}^{\infty} \chi_D(\nu_{en},\mu_{en};\vartheta_q,\varphi_q;\pi) \frac{j_{\nu_{en}}(r')}{\sqrt{r'}}.$$

Using these expressions, we can calculate

$$\begin{aligned}\int_0^\infty C_D^A(\vec{\mathbf{k}},r')C_D^A(-\vec{\mathbf{q}},r')dr' &= \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\chi_D(\nu_{en},\mu_{en},\vartheta_k,\varphi_k;0)\\ &\quad\times\chi_D(\nu_{en'},\mu_{en'},\vartheta_q,\varphi_q;0)\\ &\quad\times\int_0^\infty\frac{j_{\nu_{en}}(r')j_{\nu_{en'}}(r')}{r'}dr'.\end{aligned}$$

The above integral can be evaluated to give [12]

$$\begin{aligned}I(\nu,\nu') &= \int_0^\infty \frac{j_\nu(r')j_{\nu'}(r')}{r'}dr'\\ &= \frac{\cos((\pi/2)(\nu-\nu'))}{2(\nu+1/2)(\nu'+1/2)}\left\{\frac{1}{(\nu-\nu')^2-1}\right.\\ &\quad\left.-\frac{1}{(\nu+\nu'+1)^2-1}\right\}.\end{aligned} \tag{13}$$

If we define

$$\begin{aligned}\Upsilon_D^A(\vec{\mathbf{k}},\vec{\mathbf{q}}) &= \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\chi_D(\nu_{en},\mu_{en},\vartheta_k,\varphi_k;0)\\ &\quad\times\chi_D(\nu_{en'},\mu_{en'},\vartheta_q,\varphi_q;0)I(\nu_{en},\nu_{en'})\end{aligned}$$

and

$$\begin{aligned}\Upsilon_D^B(\vec{\mathbf{k}},\vec{\mathbf{q}}) &= \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\chi_D(\nu_{en},\mu_{en},\vartheta_k,\varphi_k;\pi)\\ &\quad\times\chi_D(\nu_{en'},\mu_{en'},\vartheta_q,\varphi_q;\pi)I(\nu_{en},\nu_{en'}),\end{aligned}$$

then the scattering amplitude, Eq. (10), becomes

$$\begin{aligned}T_D(\vec{\mathbf{k}},\mathbf{q}) &= -\frac{2\pi i}{|\hat{\mathbf{n}}\times\vec{\mathbf{Q}}|^2}[\kappa' Q_x\{\Upsilon_D^A(\vec{\mathbf{k}},\vec{\mathbf{q}})-\Upsilon_D^B(\vec{\mathbf{k}},\vec{\mathbf{q}})\}\\ &\quad+\kappa Q_z\{\Upsilon_D^A(\vec{\mathbf{k}},\vec{\mathbf{q}})+\Upsilon_D^B(\vec{\mathbf{k}},\vec{\mathbf{q}})\}].\end{aligned} \tag{14}$$

B. Neumann boundary condition

In this case the scattering amplitude is given by [9]

$$T_N(\vec{\mathbf{k}},\vec{\mathbf{q}}) = \frac{-2\pi i}{|\hat{\mathbf{n}}\times\vec{\mathbf{Q}}|^2}\int C_N(\vec{\mathbf{k}},r')C_N(-\vec{\mathbf{q}},r')(\hat{\mathbf{n}}\times\vec{\mathbf{Q}})\cdot\vec{\mathbf{d}}r',$$

which as in the Dirichlet case can be written as

$$\begin{aligned}T_N(\vec{\mathbf{k}},\vec{\mathbf{q}}) &= -\frac{2\pi i}{|\hat{\mathbf{n}}\times\vec{\mathbf{Q}}|^2}\left[-\{\kappa Q_z+\kappa' Q_x\}\int_0^\infty C_N^A(\vec{\mathbf{k}},r')\right.\\ &\quad\times C_N^A(-\vec{\mathbf{q}},r')dr'+\{-\kappa Q_z+\kappa' Q_x\}\\ &\quad\left.\times\int_0^\infty C_N^B(\vec{\mathbf{k}},r')C_N^B(-\vec{\mathbf{q}},r')dr'\right].\end{aligned} \tag{15}$$

In the above formula $C_N(\vec{\mathbf{k}},r')$ and $C_N(-\vec{\mathbf{q}},r')$ are the coefficients of the $1/\sqrt{d}$ singularity of the tangential gradients of $\psi_N(\vartheta_k,\varphi_k;r',\vartheta',\varphi')$ and $\psi_N(\vartheta_q,\varphi_q;r',\vartheta',\varphi')$ normal to the edge. From Eq. (2), the component of the gradient on the plane of the angular sector, $\vartheta'=\pi$, normal to edge is given by

$$\hat{\mathbf{e}}_\varphi|_{\vartheta'=\pi}\cdot\boldsymbol{\nabla}\psi_N(\vartheta_k,\varphi_k;r',\pi,\varphi')$$
$$=\frac{\sqrt{1-\kappa'^2\cos^2\varphi'}}{r'\kappa'\sin\varphi'}\frac{\partial}{\partial\varphi'}\psi_N(\vartheta_k,\varphi_k;r',\pi,\varphi').$$

On the edges ($\varphi'=0,\pi$), the above quantity is singular; close to edge A, $\varphi'\to 0$, we have

$$\frac{\kappa}{r'\kappa'\sin\varphi'}\frac{\partial}{\partial\varphi'}\psi_N(\vartheta_k,\varphi_k;r',\pi,\varphi')|_{\varphi'\to 0}$$
$$=\frac{1}{\sqrt{d^A}}C_N^A(\vec{\mathbf{k}},r').$$

Substituting for d^A, we find

$$C_N^A(\vec{\mathbf{k}},\vec{\mathbf{q}},r')=\frac{1}{\sqrt{r'}}\sqrt{\frac{\kappa}{2\kappa'}}\frac{\partial}{\partial\varphi'}\psi_N(\vartheta_k,\varphi_k;r',\pi,\varphi')|_{\varphi'=0}.$$

Using Eq. (9) and noting that for Neumann boundary conditions the derivative of the even solutions vanish on the edges, we find (see Appendix B)

$$C_N^A(\vec{\mathbf{k}},r')=\sum_{n=1}^{\infty}\chi_N(\nu_{on},\mu_{on};\vartheta_k,\varphi_k;0)\frac{j_{\nu_{on}}(r')}{\sqrt{r'}},$$

where we define

$$\chi_N(\nu_{on},\mu_{on};\alpha,\beta;\beta_0)$$
$$=-4\pi\sqrt{\frac{\kappa}{2\kappa'}}\frac{1}{N_o}(-i)^{\nu_{on}}\Theta_o(\alpha;\kappa,\nu_{on},\mu_{on})$$
$$\times\Phi_o(\beta;\kappa',\nu_{on},\mu_{on})\Theta_o(\pi;\kappa',\nu_{on},\mu_{on})$$
$$\times\frac{\partial}{\partial\varphi'}\Phi_o(\varphi';\kappa',\nu_{on},\mu_{on})|_{\varphi'=\beta_0},$$

and the superscript 2 has been suppressed. Similarly,

$$C_N^A(-\vec{\mathbf{q}},r')=\sum_{n=1}^{\infty}\chi_N(\nu_{on},\mu_{on};\vartheta_q,\varphi_q;0)\frac{j_{\nu_{on}}(r')}{\sqrt{r'}},$$

$$C_N^B(\vec{\mathbf{k}},r')=\sum_{n=1}^{\infty}\chi_N(\nu_{on},\mu_{on};\vartheta_k,\varphi_k;\pi)\frac{j_{\nu_{on}}(r')}{\sqrt{r'}},$$

and

$$C_D^B(-\vec{\mathbf{q}},r')=\sum_{n=1}^{\infty}\chi_N(\nu_{on},\mu_{on};\vartheta_q,\varphi_q;\pi)\frac{j_{\nu_{on}}(r')}{\sqrt{r'}}.$$

Using these expressions, we can calculate

$$\int_0^\infty C_N^A(\vec{\mathbf{k}},r')C_N^A(-\vec{\mathbf{q}},r')dr'=\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\chi_N(\nu_{on},\mu_{on},\vartheta_k,\varphi_k;0)\chi_N(\nu_{on'},\mu_{on'},\vartheta_q,\varphi_q;0)\int_0^\infty\frac{j_{\nu_{on}}(r')j_{\nu_{on'}}(r')}{r'}dr'.$$

By defining

$$\Upsilon_N^A(\vec{\mathbf{k}},\vec{\mathbf{q}})=\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\chi_N(\nu_{on},\mu_{on},\vartheta_k,\varphi_k;0)\chi_N(\nu_{on'},\mu_{on'},\vartheta_q,\varphi_q;0)I(\nu_{on},\nu_{on'}),$$

and

$$\Upsilon_N^B(\vec{\mathbf{k}},\vec{\mathbf{q}})=\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\chi_N(\nu_{on},\mu_{on},\vartheta_k,\varphi_k;\pi)\chi_N(\nu_{on'},\mu_{on'},\vartheta_q,\varphi_q;\pi)I(\nu_{on},\nu_{on'})$$

and using Eq. (15), we can write the scattering amplitude as

$$T_N(\vec{\mathbf{k}},\vec{\mathbf{q}})=\frac{2\pi i}{|\hat{\mathbf{n}}\times\vec{\mathbf{Q}}|^2}[\kappa' Q_x\{\Upsilon_N^A(\vec{\mathbf{k}},\vec{\mathbf{q}})-\Upsilon_N^B(\vec{\mathbf{k}},\vec{\mathbf{q}})\}+\kappa Q_z\{\Upsilon_N^A(\vec{\mathbf{k}},\vec{\mathbf{q}})+\Upsilon_N^B(\vec{\mathbf{k}},\vec{\mathbf{q}})\}]. \tag{16}$$

In the expressions for $\Upsilon_N^A(\vec{\mathbf{k}},\vec{\mathbf{q}})$ and $\Upsilon_N^B(\vec{\mathbf{k}},\vec{\mathbf{q}})$, $I(\nu_{on},\nu'_{on'})$ is given by Eq. (13). Formulas given by Eq. (14) and Eq. (16) may be used to calculate the cross section of a plate with sharp corners. The problem of scattering of waves from plates with sharp corners cannot solved analytically. However, there are numerical methods, such as the method of moments [13], that can be employed for this purpose. At high frequencies (when the wavelength of the incident field is much smaller than the length scale of the plate) these numerical methods become computationally intensive. At this regime (high frequencies) it is reasonable to assume that each corner of the plate is a PAS. Formulas (14) or (16) could be used to calculate the scattering amplitude for each corner. The scattering cross section for the plate would then be the magnitude squared of the coherent sum of the scattering amplitude for each corner.

APPENDIX A: CALCULATION OF d^A AND d^B

Referring to Fig. 4, let $P_1(z',x')$ be a point on the surface of the PAS. Its distance to the point $P_2(z_1',x')$ on the edge is

$$c=z_1'-z'.$$

According to Eq. (1), x' and z' on the surface of the PAS, $\vartheta=\pi$, are

$$x'=-r'\sqrt{1-\kappa'^2\cos^2\varphi'},$$

$$z'=r'\kappa'\cos\varphi'.$$

On edge $A(\varphi=0)$, they become

$$x'=-r'\kappa,$$

$$z'=r'\kappa'.$$

From this we obtain the equation for edge A, $x'=-\kappa/\kappa' z'$, or $z_1'=-\kappa'/\kappa x'$.
We thus write

$$c=-\frac{\kappa'}{\kappa}x'-z',$$

and noting that $d^A=c\cos\varepsilon=c\kappa$, we find

$$d^A=-\kappa' x'-\kappa z'$$

or

$$d^A=r'\kappa'\sqrt{1-\kappa'^2\cos^2\varphi'}-r'\kappa\kappa'\cos\varphi',$$

and we similarly find

$$d^B=r'\kappa'\sqrt{1-\kappa'^2\cos^2\varphi'}+r'\kappa\kappa'\cos\varphi'.$$

APPENDIX B: BOUNDARY CONDITIONS FOR A PLANE ANGULAR SECTOR

Referring to Fig. 3, we take the boundary surface to be the sector $\vartheta=\pi$. The coordinate-imposed boundary condition on $\Phi(\varphi)$ is that it must be periodic with period 2π: $\Phi(\varphi+2\pi)=\Phi(\varphi)$, in order to ensure that it is single valued. If $\Phi(\varphi)$ is even i.e., $\partial\Phi(\varphi)/\partial\varphi|_{\varphi=0}\equiv\Phi_e'(0)=0$, we can write

$$\Phi_e(\varphi+2\pi)=\Phi_e(\varphi)=\Phi_e(-\varphi)$$

$$\text{or}\quad \Phi_e'(\varphi+2\pi)=-\Phi_e'(-\varphi).$$

This implies

$$\Phi_e'(\pi)=0.$$

On the other hand, if $\Phi(\varphi)$ is odd, $\Phi_o(0)=0$ and

$$\Phi_o(\varphi+2\pi)=\Phi_0(\varphi)=-\Phi_o(-\varphi),$$

which implies

$$\Phi_o(\pi)=0.$$

Thus for the even and odd periodic cases we must respectively have

$$\Phi_e'(0)=\Phi_e'(\pi)=0$$

and

$$\Phi_o(0)=\Phi_o(\pi)=0.$$

The boundary conditions on $\Theta(\vartheta)$ can be any of the following:

1. The even Dirichlet boundary condition

In this case $\Theta(\vartheta)$ is even $[\Theta_e'(0)=0]$ and it satisfies the Dirichlet boundary condition on the boundary surface $[\Theta_e(\pi)=0]$. It has been shown by Kraus and Levine [2] that the factors $\Theta(\vartheta)$ and $\Phi(\varphi)$ of the eigenfunction $V(\vartheta,\varphi)$ can only be both even or both odd. Since $\Theta(\vartheta)$ has been chosen to be even, $\Phi(\varphi)$ must also be even resulting in the following boundary conditions

$$\begin{aligned}\Theta_e'(0)&=0, \quad \Theta_e(\pi)=0,\\ \Phi_e'(0)&=0, \quad \Phi_e'(\pi)=0.\end{aligned} \tag{B1}$$

2. The odd Neumann boundary condition

In this case $\Theta(\vartheta)$ is odd $[\Theta_o(0)=0]$ and it satisfies the Neumann boundary condition on the boundary surface $\Theta_o'(\pi)=0$. Then $\Phi(\varphi)$ must also be odd, resulting in the following boundary conditions:

$$\begin{aligned}\Theta_o(0)&=0, \quad \Theta_o'(\pi)=0\\ \Phi_o(0)&=0, \quad \Phi_o(\pi)=0.\end{aligned} \tag{B2}$$

3. The odd Dirichlet and the even Neumann boundary conditions

By using the above arguments, for the odd Dirichlet case we have

$$\Theta_o(0)=0, \quad \Theta_o(\pi)=0$$
$$\Phi_o(0)=0, \quad \Phi_o(\pi)=0.$$

By writing

$$\Theta_o(\vartheta)=\Theta(\vartheta)-\Theta(-\vartheta),$$

and imposing the boundary condition $\Theta_o(\pi)=0$, we find

$$\Theta(-\pi)=\Theta(\pi). \tag{B3}$$

Similarly for the even Neumann boundary condition we have

$$\Theta_e'(0)=0, \quad \Theta_e'(\pi)=0,$$
$$\Phi_e'(0)=0, \quad \Phi_e'(\pi)=0.$$

In this case

$$\Theta_e(\vartheta)=\Theta(\vartheta)+\Theta(-\vartheta)$$

$$\text{or} \quad \Theta_e'(\vartheta)=\Theta'(\vartheta)-\Theta'(-\vartheta).$$

At the boundary surface the left hand side of the second equation in the above vanishes, resulting in

$$\Theta'(\pi)=\Theta'(-\pi). \tag{B4}$$

For the odd Dirichlet and the even Neumann boundary conditions both $\Phi(\varphi)$ and $\Theta(\vartheta)$ are periodic with period 2π, which results in integer eigenvalues.

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