Wave-function method for a waveguide with centered circular diaphragm

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The scattering by a diaphragm of finite thickness with a centered circular hole in a rectangular waveguide is reinvestigated by using the wave-function method for the propagation of the electromagnetic waves in waveguides. The susceptance of the diaphragm with a centered circular aperture is readily obtained from the reflection coefficient. The transmission and the reflection coefficients of the thick circular aperture are computed. A comparison is made with the standard methods. [S1063-651X(97)03407-7]

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I. INTRODUCTION

It is well known that a small aperture that transmits radiation is a very useful microwave circuit element. Waveguide diaphragms with circular holes are frequently employed in building microwave components such as waveguide filters and an impedance matching system; waveguide-to-cavity coupling is often accomplished with a circular hole. Various approaches such as quasistatic theory, conformal mapping, variational theory, singular equation method, and mode-matching method have been applied to the scattering by a thick diaphragm with a centered circular hole in a rectangular waveguide. The study of discontinuities in waveguiding structures has always been a subject of great importance and a large number of contributions in this area can be found in the literature. In particular we mention in this respect the historical work by Schwinger and Saxon [1] and two review works by Levine [2] and Mitra and Lee [3]. Chapters of more general significance have also been devoted to this subject by Collin [4] and Harrington [5], and we mention also a more practically oriented handbook edited by Marcuvitz [6]. All these works deal essentially with analytical approximate solutions to various problems arising in waveguide physics. In the microwave field theory Lord Rayleigh [7] was the first to discuss the scattering of electromagnetic (e.m.) waves by a circular aperture and ellipsoidal obstacles. Lord Rayleigh's work provided the foundation for highly useful ''small-aperture'' and ''small-obstacle'' theories, which were revived and generalized by Bethe [8–11]. He developed a vectorial theory of diffraction of plane electromagnetic waves by a circular aperture in an infinite plane conducting screen. He showed that a wave incident on a small hole in a a metallic wall produces a field in the hole equivalent to the sum of an electric and a magnetic dipole, the polarizabilities of which are given to a good approximation by electrostatic approximations if the hole diameter is small relative to the wavelength and the hole is not near the waveguide wall. The most complete variational solutions of scattering from a diaphragm with a centered circular hole in a rectangular waveguide was given in Marcuvitz [6]. The results presented in Ref. [6] have been derived using the so-called single-mode network representation for waveguide discontinuities, which has always been considered very useful for its computational efficiency. The strongest limitation of this type of result is that it can only take into account fundamental-mode interactions. The advent of powerful digital computers allows other numerically oriented approaches. In particular for waveguide problems, the mode-matching technique provide multimode equivalent network representations for waveguide discontinuities [12].

The difficulty in the study of discontinuities consists in the lack of a powerful general analytical method, each problem requiring its own type of approximation, especially related to its own geometry. Recently, a method has been introduced [13–17], that allows one to study the transmission of electromagnetic radiation through waveguides in much the same manner as the wave tunneling through a potential barrier. This method is not restricted to radiation wavelengths much shorter than the hole size, and recently it has been applied successfully in describing the propagation of the electromagnetic waves through waveguides by analogy with the quantum tunneling phenomenon [18–20]. Apart from being free of the above-mentioned restriction, this method, which may be called the wave-function method, is also useful because of its simplicity and flexibility in dealing with various geometries encountered in the physics of the microwave guides. The wave-function method is further developed in the present paper with the aim of enlarging its applicability to the analysis of microwave circuit elements. In this paper we show that a further classical electromagnetic problem is tractable by methods inspired by quantum mechanics. In Sec. II the wave-function method is briefly outlined. In Sec. III we apply it to compute the transmission and the reflection coefficient of a circular hole centered in a diaphragm of finite thickness, transverse to the axis of a rectangular waveguide. The susceptance of a thin diaphragm with a centered circular hole is then readily obtained in Sec. IV. The results are graphically compared with the formulas and curves due to usual approaches such as the quasistatic theory, variational technique and mode-matching technique in Sec. V.

II. THE WAVE-FUNCTION METHOD

It is well known [4] that the propagation of electromagnetic waves through a waveguide proceeds by two types of transverse standing modes: TE and TM. In the former case the fields are given by the H_z component of the magnetic field along the guide axis

$$H_z = f(x,y)e^{i(k_g z-\omega_0 t)}, \tag{1}$$

where the function f satisfies the two-dimensional scalar Helmholtz equation $(\Delta_{x,y}+k_c^2)f=0$, with the boundary condition $\partial f/\partial n|_\Gamma=0$, **n** being the vector normal to the Γ contour of the cross section, the wave vector k_g along the guide axis, and k_c being related to the frequency ω_0 by

$$\omega_0^2=c^2k_g^2+\omega_c^2, \tag{2}$$

where the cutoff frequency $\omega_c=ck_c$, with c denoting the light velocity. Usually, the function $f(x,y)$ is normalized over the cross section of the waveguide, $\int|f(x,y)|^2ds=1$. For the TM modes the magnetic field H_z is replaced in Eq. (1) by the electric field E_z and the boundary condition is $f|_\Gamma=0$.

The plane wave along the z direction expressed by Eq. (1) may be reflected by, absorbed by, or transmitted through various small objects or media placed inside the waveguide, as well as variations of the cross section, which amount to connecting two or more waveguides, resonant cavities, etc. It may also be scattered by a small circular aperture centered in a metallic plate across the guide in a plane perpendicular to the guide axis, at a suitable position in the common wall between two guides, etc. In all these cases we are interested in the amplitude a_k of the plane wave, such that we are led to introduce the wave function [15,16].

$$\Psi(z,t)=\frac{1}{(2\omega_0)^{1/2}}a_k e^{i(k_g z-\omega_0 t)} \tag{3}$$

by

$$H_z=f(x,y)(8\pi)^{1/2}\omega_c\Psi(z,t) \tag{4}$$

for the TE modes, and a similar relationship for E_z in the case of TM modes. It has been shown [15] in this case that the density energy per unit length is given by

$$\omega_0|a_k|^2=2\omega_0^2|\Psi|^2 \tag{5}$$

and the energy flux is

$$S=cn\omega_0|a_k|^2 \tag{6}$$

where $n=(1-\omega_c^2/\omega_0^2)^{1/2}$ is the refractive index of the waveguide, i.e., the energy is transported with the group velocity cn. In addition, the waveguide function given by Eq. (3) satisfies a Klein-Gordon type equation in 1+1 dimensions according to Eq. (2),

$$-\frac{\partial^2\Psi}{\partial z^2}+\frac{1}{h^2c^2}(E^2-U_g^2)\Psi=0, \tag{7}$$

where E is the photon energy and $U_g=h\omega_c$ is the height of the rectangular potential barrier representing the uniform waveguide. The density and current of a ''plane wave'' of positive frequencies $\omega_0=(c^2k_g^2+\omega_c^2)^{1/2}$ can be defined that satisfy the continuity equation [15]. As one can see, the whole picture shares many essential features with quantum physics. In order to illustrate how the methods works, we apply it in the next section to computing the transmission and the reflection coefficients of a centered circular hole in a metallic diaphragm of finite thickness, transverse to the axis of a rectangular waveguide.

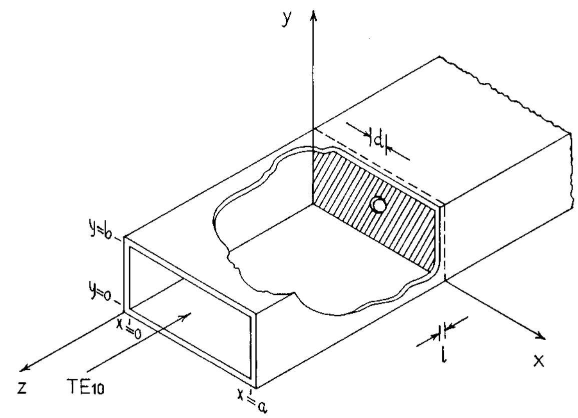

FIG. 1. Small centered circular hole in a thick metallic diaphragm transverse to the axis of a rectangular waveguide.

III. THE TRANSMISSION AND THE REFLECTION COEFFICIENTS OF A CIRCULAR HOLE CENTERED IN A DIAPHRAGM TRANSVERSE TO THE AXIS OF A RECTANGULAR WAVEGUIDE

Figure 1 illustrates a perfectly conducting diaphragm of nonzero thickness t with a centered circular hole of radius $r<b$, placed in a plane perpendicular to the axis of an ideal rectangular waveguide with sides a,b $(a>b)$, matched at both ends. Usually the range of a/λ_0 is such that only the TE_{10} mode can propagate at frequency ω_0. A wave of unit amplitude is incident on the obstacle from the region $z<0$. The TE_{10} mode is reflected from the diaphragms and some of the incident power is transmitted through the aperture. If the operating frequency is below the resonant frequency of the circular hole, the latter can be considered as a waveguide of cross section equal to the cross section of the hole and the length equal to the thickness. According to the result in Sec. II, the diaphragm with the centered circular hole can be regarded as a rectangular-circular-rectangular connection of potentials as in Fig. 2, of heights U_g and U_{ch}, respectively, where the wave numbers are given by

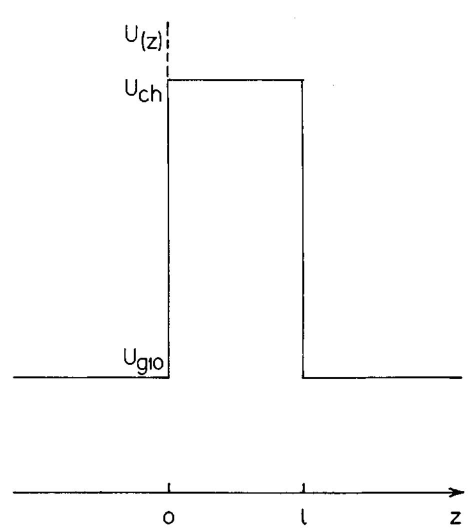

FIG. 2. Rectangular potential barrier for the physical situation from Fig. 1.

$$k_g=\frac{1}{hc}(E^2-U^2)^{1/2} \tag{8}$$

and

$$\chi_{ch}=\frac{1}{hc}(U_{ch}^2-E^2)^{1/2}. \tag{9}$$

For the three regions (see Fig. 2) the wave function is given by

$$\Psi(z)=\begin{cases} e^{ik_g z}+R_{ch}e^{-ik_g z}, & z<0\\ Ae^{\chi_{ch}l}+Be^{-i\chi_{ch}l}, & 0<z<l\\ T_{ch}e^{ik_g l}, & l<z, \end{cases} \tag{10}$$

where R_{ch} and T_{ch} are the reflection and the transmission coefficients, respectively, of the potential barrier representing the evanescent circular waveguide. They will be determined together with A and B by requiring the continuity of the wave function at $z=0$ and $z=l$. The calculations are straightforward and one obtains

$$T_{ch}=\frac{e^{ik_g l}}{\cosh\chi_{ch}l-i\,\dfrac{k_g^2-\chi_{ch}^2}{2k_g\chi_{ch}}\sinh\chi_{ch}l}, \tag{11}$$

$$R_{ch}=\frac{\cosh\chi_{ch}l-i\,\dfrac{k_g}{\chi_{ch}}\sinh\chi_{ch}l}{\cosh\chi_{ch}l-i\,\dfrac{k_g^2-\chi_{ch}^2}{2k_g\chi_{ch}}\sinh\chi_{ch}l}-1. \tag{12}$$

Since not all the photons coming from $z=-\infty$ arrive on the surface of the aperture, the equations for the transmission and the reflection coefficients depend on the geometry of the system. From the total photon flux Φ_t in the cross-section of the guide in the absence of the obstacle, a part Φ_m will be reflected by the metallic plate of the diaphragm and the difference between the two fluxes

$$\Phi_t-\Phi_m=\Phi_{ch} \tag{13}$$

represents the flux of photons arriving on the surface of the circular hole. Each flux depends on the distribution function of the photons in the transverse cross section. For the case of the TE_{10} mode the distribution function is given by $\Psi(x,y)=(2/a)^{1/2}\sin\pi x/a$. Thus one may write for the transmission and the reflection coefficients of the circular hole centered on the diaphragm

$$\mathcal{T}_{ch}=\frac{2\pi r^2}{ab}\,\frac{\cos k_g l+\eta\sin k_g l\tanh\chi_{ch}l-i(\sin k_g l-\eta\cos k_g l\tanh\chi_{ch}l)}{\cosh\chi_{ch}l(1+\eta^2\tanh^2\chi_{ch}l)}. \tag{14}$$

$$\mathcal{R}_{ch}=\frac{2\pi r^2}{ab}\,\frac{1-ik_g/\chi_{ch}\tanh\chi_{ch}l}{1-i\,\eta\tanh\chi_{ch}l}. \tag{15}$$

where

$$\eta=\frac{k_g^2-\chi_{ch}^2}{2k_g\chi_{ch}}. \tag{16}$$

The validity of the relationship

$$|T_{ch}|^2+|R_{ch}|^2=1 \tag{17}$$

is ensured.

Let us consider the case of a thin diaphragm for which

$$\frac{l}{\lambda_g}\ll 1, \tag{18}$$

with a centered circular hole whose diameter d satisfies the condition of the small aperture range

$$\frac{d}{b}\ll 1. \tag{19}$$

Both restrictions (18) and (19) are verified for a standard rectangular waveguide in the X band (with $a=2.25b$, $b=1.016\times10^{-2}$ m, and $\lambda_0\simeq3\times10^{-2}$ m) in which a diaphragm of thickness $l=10^{-4}$m, with a small circular hole of radius $r\ll\pi\times10^{-4}$ m, is placed. The wave numbers in the rectangular and cylindrical waveguides are given by the dispersion relations [4]

$$k_g^2=4\pi^2\left(\frac{1}{\lambda_0^2}-\frac{1}{\lambda_{c10}^2}\right), \tag{20}$$

$$\chi_{ch}^2=4\pi^2\left(\frac{1}{\lambda_{ch}^2}-\frac{1}{\lambda_0^2}\right), \tag{21}$$

where $\lambda_{c10}=2a$ and $\lambda_{ch}=p_{m,e}r$ are the cutoff wavelength of the propagation mode in the rectangular waveguide and the circular waveguide and $p_{m,e}$ is a numerical coefficient depending on the standing mode; the subscripts e and m refer to the "electric type" for the TM mode and the "magnetic type" for the TE mode respectively. We assume that the passage of energy proceeds by three modes excited in the cylindrical guide, TM_{01}, TE_{10}, and TE_{11} for which $p_e=2.6127$, $p_m=1.6398$, and $p_m=3.4126$ respectively [4]. Their cutoff frequencies $\omega_{c,me}=2\pi c/p_{m,e}r$ are therefore much higher than ω_0. Assuming that the modes have equal weights in carrying the e.m. energy and therefore $p\simeq2$, for the case $r\simeq\pi l$, from Eq. (21) it results that

$$\chi_{ch}=10^4\ \ \mathrm{m}^{-1} \tag{22}$$

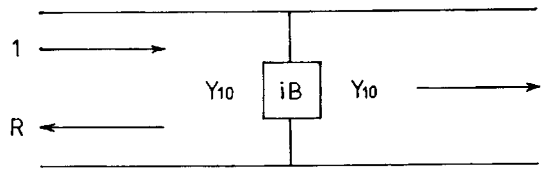

FIG. 3. Equivalent transmission-line representation of the TE_{10} mode in the rectangular waveguide with a thin diaphragm with a centered circular aperture.

and from Eq. (22)

$$k_g = 157.079 \ \mathrm{m}^{-1}. \tag{23}$$

It is straightforward now to show, using Eq. (16), that one may write

$$\eta \simeq -\frac{\chi_{ch}}{2k_g}. \tag{24}$$

In the case of the thin diaphragm $k_g l \ll 1$, $\chi_{ch} l = 1$ and then $\cos k_g l \simeq 1$, $\sin k_g l \simeq k_g l$, $\cosh \chi_{ch} l \simeq 1$, $\sinh \chi_{ch} l \simeq \chi_{ch} l$, and consequently from Eqs. (14) and (15) one obtains

$$\mathcal{T}_{ch} \simeq \frac{i}{\eta}\frac{2\pi r^2}{ab}, \tag{25}$$

$$\mathcal{R}_{ch} \simeq \frac{1}{-i\eta}\frac{2\pi r^2}{ab} - 1. \tag{26}$$

Since $\lambda_{ch} \ll \lambda_0$, from Eq. (21) it results that

$$\chi_{ch} \simeq \frac{2\pi}{pr}. \tag{27}$$

Introducing Eq. (27) into Eqs. (25) and (26) for the average value $\overline{p} = 2.5550$ corresponding to the most probable excitation modes in the circular guide, one obtains

$$\mathcal{T}_{ch} = -i5.110\frac{k_g r^3}{ab}, \tag{28}$$

$$\mathcal{R}_{ch} = i5.110\frac{k_g r^3}{ab} - 1 \tag{29}$$

for the transmission coefficient through the small circular hole and for the reflection coefficient, respectively. The reflection coefficient of an extremely small circular hole $r \to 0$ is nearly equal to -1. This is a correct result for a short-circuiting plate across the waveguide. For the same case of a centered small circular hole in a transverse thin diaphragm located in a rectangular waveguide for the dominant TE_{10} mode according to the standard e.m. field theory, one has the well-known formulas [4]

$$\mathcal{T}_{ch} = -i5.333\frac{k_g r^3}{ab}, \tag{30}$$

$$\mathcal{R}_{ch} = i5.333\frac{k_g r^3}{ab} - 1. \tag{31}$$

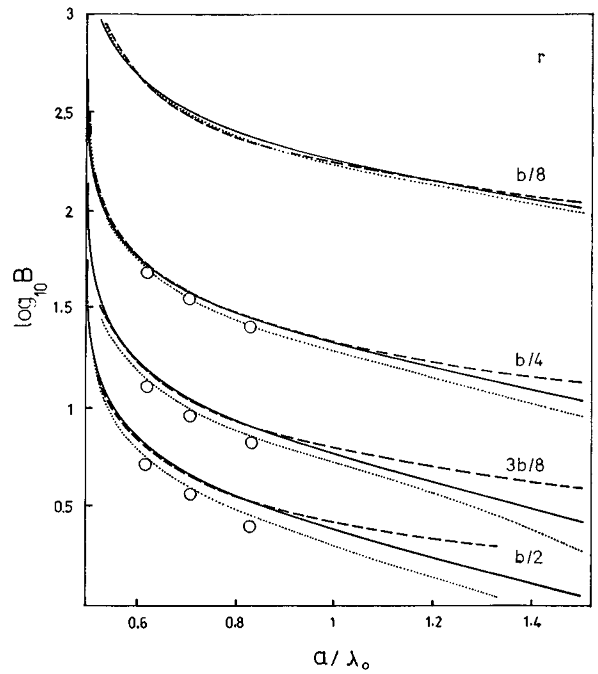

FIG. 4. Susceptance of a thin diaphragm with a centered circular hole versus a/λ_0, with radius r as a parameter. Solid lines are calculated using the present method, dashed lines are from the Bethe quasistatic theory, circled lines are from the variational theory, and dotted lines are from the mode-matching technique.

One can see that the transmission and the reflection coefficients derived in this paper practically coincide with the classical result, except for the numerical coefficient 5.110 versus 5.333. We shall use this result to obtain the susceptance of the diaphragm in the next section.

IV. THE SUSCEPTANCE OF A CIRCULAR HOLE CENTERED IN A THIN DIAPHRAGM TRANSVERSE TO THE AXIS OF A RECTANGULAR WAVEGUIDE

From the reflection coefficient one can determine the normalized load admittance of discontinuity. A thin metal diaphragm with a centered circular hole in a rectangular waveguide exhibits an equivalent circuit consisting of a single inductive shunt susceptance iB on the equivalent transmission line representation of the dominant TE_{10} mode field, located precisely at the point of discontinuity [4] as in Fig. 3.

It can be shown that

$$\mathcal{R} = \frac{-i\overline{B}}{2 + i\overline{B}}, \tag{32}$$

where $\overline{B} = B/Y_{10}$ is the normalized susceptance and Y_{10} is the wave admittance of the TE_{10} mode and hence the characteristic admittance of transmissions line. When B is large one can write

$$\mathcal{R} \simeq -1 - \frac{2}{i\overline{B}}. \tag{33}$$

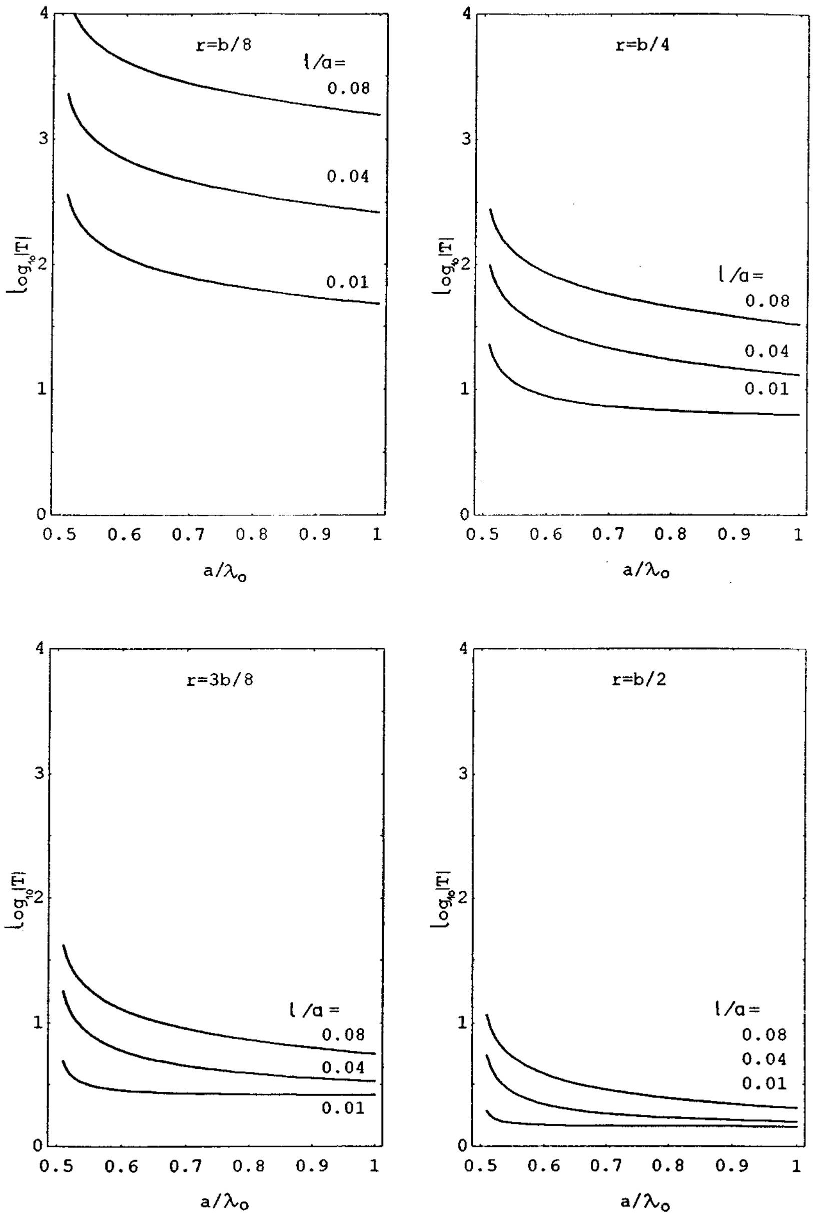

FIG. 5. Amplitude of the transmission coefficient (TE_{10} mode) given by Eq. (14) for a thick diaphragm with a centered circular hole vs frequency for aperture radii and diaphragm thickness as parameters.

Comparing Eq. (29) with Eq. (33) shows that an indefinitely thin circular small hole is equivalent to a normalized susceptance

$$\overline{B}=-\frac{ab}{pr^3k_g}=-\frac{8}{2\pi p}\frac{ab\lambda_g}{d^3}. \quad (34)$$

The minus sign indicates that the susceptance is inductive. One can see also that the susceptance is inversely proportional to the cube of radius and $\overline{B}\rightarrow\infty$ when $r\rightarrow 0$. This corresponds to a short circuit across the waveguide. On the contrary, when the waveguide is almost open, the hole occupying almost the whole cross section $\overline{B}$ will be numerically small. For the same case of the zero-thickness diaphragm with a centered small circular hole in a rectangular waveguide, according to the e.m. field theory of microwaves one has the well-known formula [6]

$$\overline{B}=-\frac{3ab}{8r^3k_g}=-\frac{3}{2\pi}\frac{ab\lambda_g}{d^3}, \quad (35)$$

originating in the quasistatic theory of Bethe. One can see that for the average value $\overline{p}=2.5550$ corresponding to the most probable excitation mode in the circular guide, the coefficient of susceptance calculated using the present method is 3.1373, which agrees well with 3 given by the e.m. field theory.

V. NUMERICAL RESULTS AND DISCUSSION

In our numerical computational that considers air-filled waveguides in the X band, we have selected the frequency range of practical interest in which only the TE_{10} mode is propagated. The normalized susceptance for the thin metallic diaphragm with a centered circular hole placed in a plane perpendicular to the waveguide axis was calculated using the wave-function method. The normalized susceptances are

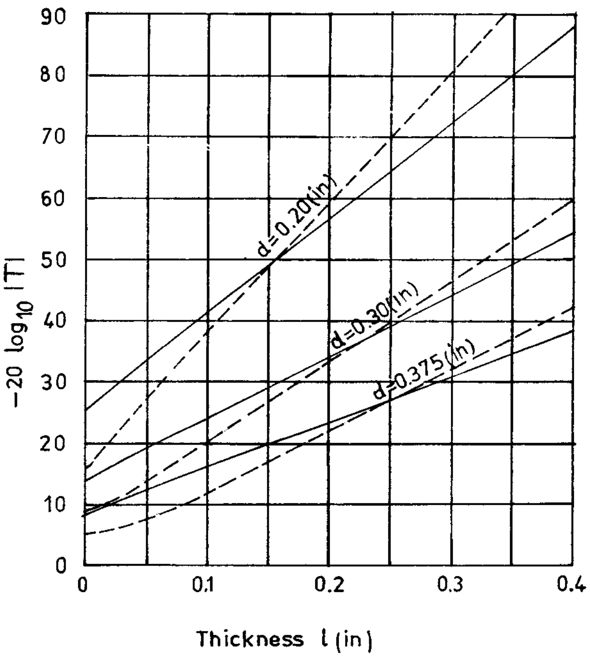

FIG. 6. Power transmission coefficient (TE_{10} mode) given by Eq. (14) of a centered circular hole in a metallic diaphragm (broken lines) and classical power transmisssion coefficient (solid line), as a function of thickness l with hole diameter d as a parameter.

plotted in Fig. 4 versus a/λ_0 with radius r as parameter. Our results (solid lines) are graphically compared with curves based on the well-known formula (35) given by the quasistatic theory of Bethe (dashed lines) [11] with the variational theory (open circles) [6] and with the mode-matching technique (dotted lines) [12]. As we can see from Fig. 4, the normalized susceptance increases as the radius of the circular decreases and diverges as $a/\lambda_0 \rightarrow 0.5$ since the TE_{10} mode's admittance Y_{10} vanishes at this point. For small circular holes $r \leqslant b/8$ the results obtained using quasistatic theory, mode-matching technique, and results obtained by our method practically coincide. Variational calculus susceptance gives a lower susceptance than the small-aperture (Bethe) theory, the mode matching technique and the present method, but for large holes $r = b/2$ it gives values that differ from ours by about 10%. This is confirmed by the results presented in Fig. 4. We note that the agreement between the standard aproaches and our solution is quite good even for large values of r/b. The dependence of the amplitude of the transmission coefficient (TE_{10} mode) given by Eq. (14) for a thick diaphragm with a centered circular hole in a rectangular guide versus a/λ_0 is plotted in Fig. 5 for four aperture radii and a series of a diaphragm thickness l/a, ranging between 0.01 and 0.08. As one can see, the effect of the thickness of the diaphragm is significant. The thinnest diaphragm $l = 0$ has the largest $|T|$. The amplitude of the transmission coefficient decreases with increasing the diaphragm thickness, the effect being greater for small circular holes $r \leqslant b/8$. In Fig. 6 the power transmission coefficient $\mathcal{T}$ given by Eq. (14) (dashed line) is compared with the classical transmission coefficient (solid line) provided by the variational calculus [6] for the X-band waveguide. One can see that the present results are in good agreement with the classical ones for $d = 0.375 - 0.30$ in. and $l = 0.15 - 0.35$ in. and for $d = 0.20$ in. and $l = 0.1 - 0.25$ in., respectively. The displacement between the values obtained using the wave-function method and those obtained by the variational calculus is below 10% for $d = 0.375$ and $d = 0.30$ in. and $t \in 0.15 - 0.35$ in. and for $d = 0.20$ in. and $t \in 0.1 - 0.25$ in., respectively.

VI. CONCLUSION

In conclusion, one may say that the diffraction by a circular hole in a metallic diaphragm transverse to the axis of the waveguide can be described in a very convenient manner by using well-known quantum-mechanical concepts. Using the model presented in this work, the thick diaphragm with a centered circular aperture in a rectangular waveguide is treated as a cascaded connection of rectangular potential barrier corresponding to rectangular-circular-rectangular guides. The availability of this simple equivalent representation allows a fast and accurate analysis. Various applications of the wave-function method, such as the analysis of evanescent mode waveguide bandpass filter and the waveguide sandwich filter, are in progress.

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