Slowly driven sandpile formation with granular mixtures

D. A. Head* and G. J. Rodgers†
Department of Physics, Brunel University, Uxbridge, Middlesex UB8 3PH, United Kingdom
(Received 3 March 1997)

We discuss a one-dimensional sandpile model with N different particle types and an infinitesimal driving rate. The parameters for the model are the N^2 critical slopes for one type of particle on top of another. The model is trivial when $N=1$, but for $N=2$ we observe four broad classes of sandpile structures in different regions of the parameter space. We describe and explain the behavior of each of these classes, giving quantitative analysis wherever possible. The behavior of sandpiles with $N>2$ essentially consists of combinations of these four classes. We investigate the model's robustness and highlight the key areas that any experiment designed to reproduce these results should focus on. [S1063-651X(97)11408-8]

PACS number(s): 46.10.+z, 05.40.+j, 64.75.+g

I. INTRODUCTION

Granular materials display a variety of unusual behavior not normally associated with either solids or liquids [1]. One such example is the segregation that occurs when a mixture of different sized granules is repeatedly shaken, in which the larger particles rise to the top [2–4]. Similarly, a granular mixture placed inside a rotating cylinder segregates into alternate bands along the cylinder's axis [5,6]. Segregation in the absence of external perturbations has recently been demonstrated for a mixture poured between two vertical plates separated by a narrow gap [7]. A *sandpile* forms in which particles of different sizes tend to remain near the top or bottom. Moreover, certain mixtures also *self-stratify* into alternating layers parallel to the surface of the pile. It is this segregation in sandpiles that this paper seeks to address.

A sandpile is formed by the addition of particles that then move over the surface of the pile until finding a resting place. Modeling this process is a highly nontrivial problem even without the added complication of mixtures of particles. There are two basic approaches to modeling the surface transport of pure granular media. Firstly, Bak *et al.* [8] introduced a cellular automata model as a paradigm of their more general concept of self-organized criticality. In this model, sequentially added particles can initiate a series of locally defined topples known collectively as an *avalanche*. The system is said to be *slowly driven* as the time scale for particle injection is infinitely slower than that of the subsequent avalanche [9]. However, experimental evidence [10–12] disagrees with the model's predicted power-law distribution of avalanche sizes, except possibly in the limit of overdamped particle motion [13]. A second approach treats the sandpile as a continuum with a fluid *rolling layer* interacting with the static bulk of the pile. Analysis of the field equations of this model appears to give a greater correspondence with the experiments [14].

Variations of both discrete and continuous approaches have been used to try to explain the behavior observed in the sandpile formed by pouring a mixture between two plates. A discrete model, in which the particles are added in groups and also move down the slope in groups, exhibits self-stratification but gives no insights into the mechanism responsible for the formation of the layers. However, extending the continuum rate equations to incorporate mixtures demonstrates the existence of a *kink*, which backpropagates up the slope forming two layers at once, giving a possible explanation of the experimental results [15].

In this paper, and in contrast to the experiments [7] and their subsequent analysis [15], we consider a slowly driven system in which particles are added individually rather than being poured. The entire pile is stable between particle additions and there is no formation of a rolling layer. The model we have chosen to study is a one-dimensional cellular automata with mixed particle types, based on the model in [8], which has no avalanching in one dimension. For binary mixtures we observe four broad classes of sandpile structures as the relative strengths of the interactions between the particles are varied. Two of these classes correspond to the self-segregation and self-stratification observed in the rolling layer case.

The algorithm for the model is described in Sec. II. The four classes of behavior are described and their evolution explained in Sec. III, and the results for mixtures with more than two particle types are also given. In Sec. IV we discuss how experiments to observe these classes might proceed.

II. THE MODEL

The sandpile profile is described by the set of heights h_i, $i\geq 1$, where all the h_i are initially set to zero. An infinite wall at $i=0$ serves as a lower bound for i, whereas there is no upper bound on the values of i. At each time step, a particle is chosen from one of N possible types, where each particle has the same dimensions and is chosen with equal probability. A particle of type $\alpha\in[1,N]$ is then added to the top of site 1 and subsequently slides to the first site $i\geq 1$ that obeys $z_i<z_{\alpha\beta}$, where $z_i=h_i-h_{i+1}$ and β is the type of particle currently on the surface at site i. The particle is added to the top of site i, $h_i\rightarrow h_i+1$, and another particle can now be added to the system. An example of this process is given in Fig. 1.

The N^2 parameters $z_{\alpha\beta}$ correspond to the maximum slope

*Electronic address: David.Head@brunel.ac.uk
†Electronic address: G.J.Rodgers@brunel.ac.uk

 © 1997 The American Physical Society

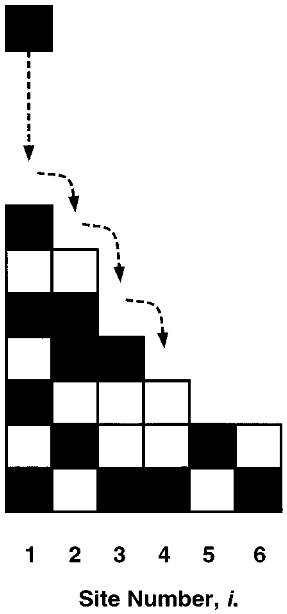

FIG. 1. Example of sandpile evolution for $N=2$, $z_{11}=1$, $z_{12}=2$, $z_{21}=3$, and $z_{22}=4$. Particles of type 1 are black, and those of type 2 are white. For the sandpile shown above, a particle of type 1 added to site 1 will eventually come to rest on site 4.

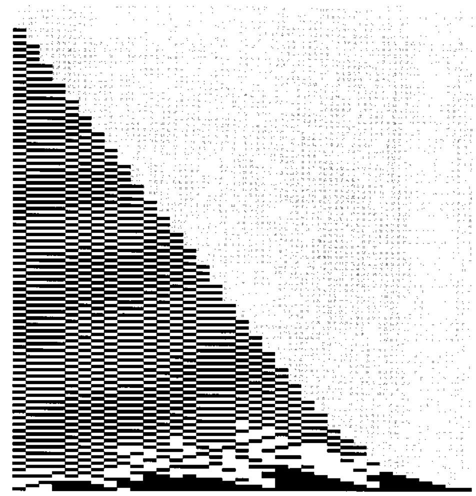

FIG. 2. Example of a class I sandpile with the parameters $z_{11}=1$, $z_{12}=5$, $z_{21}=7$, and $z_{22}=3$, which exhibits *periodic mixing*. Particles of type 1 are shown as black, particles of type 2 are shown as white. The pile given here is small, just 35 sites wide at its base, to show the mixing clearly. In this and all subsequent cases, simulations have been extended up to $O(10^5)$ particles without any observed deviation from the characteristic behavior. Note that we have rescaled the y axis to give a roughly square picture.

on which a particle of type α can remain on top of a particle of type β without sliding off. A pure pile of just type α particles has uniform slope $z_{\alpha\alpha}$, so $\tan^{-1}(z_{\alpha\alpha})$ can be identified as the *angle of repose*. The $z_{\alpha\beta}$ for $\alpha\neq\beta$ are microscopically defined quantities with no obvious macroscopic counterparts. Generally $z_{\alpha\beta}\neq z_{\beta\alpha}$, since the critical slope will depend upon which type of particle is moving on top of the pile, for instance, types α and β may have different densities. Due to the complex nature of granular surface-to-surface interactions it is unclear how much of this parameter space corresponds to physical reality. The labels for each particle type are just dummy variables, so without loss of generality we choose to fix $z_{11}\leqslant z_{22}\leqslant\cdots\leqslant z_{NN}$.

III. RESULTS

For $N=1$ the system reduces to the original model, which is trivial in one dimension [8]. For $N=2$ and $z_{11}<z_{22}$ there are four classes of solution, which we label I–IV. Each class can be identified according to the *domain stability* of each particle type, which is defined as follows. A compact region of sites with surface composition of particle type α is stable if it can reach a uniform slope $z_{\alpha\alpha}$ such that particles of type $\beta\neq\alpha$ will slide through the region—that is, if $z_{\beta\alpha}<z_{\alpha\alpha}$. Unstable regions can only form in instances where there is no incoming flux of particles of the other type, which may occur towards the right-hand side of the sandpile. We now describe each of the four classes in turn.

Class I : $z_{21}>z_{11}$ and $z_{12}>z_{22}$. Neither particle type can form stable domains and particles will usually come to rest on particles of the other type. We call this *periodic mixing*. An example of such a sandpile is given in Fig. 2. This behavior can be demonstrated by the following single-site analysis. A site with an uppermost particle of type α and slope z_i is represented by (α,z_i). Then the transition amplitude for a particle of type β coming to rest on this site, $(\alpha,z_i)\rightarrow(\beta,z_i+1)$, is given by the particle addition operator $P_{\alpha\beta}(z_i)$ defined by

$$P_{\alpha\beta}(z_i)=\theta(z_{\alpha\beta}-z_i), \tag{1}$$

where $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x\leqslant 0$. Low values for z_i are transitory, but for large z_i added particles slide through, possibly coming to rest on site $i+1$ and reducing z_i by 1. Hence the bulk properties of the sandpile will be characterised by the action of P in the region of large z_i. An example is given in Fig. 3 for the case $z_{11}<z_{22}<z_{12}<z_{21}$, which clearly shows periodicity for z_i near the maximum

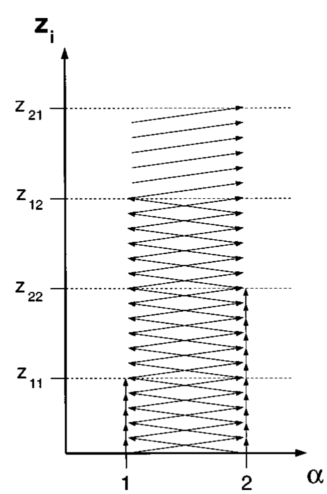

FIG. 3. Phase portrait of the particle addition operator $P_{\alpha\beta}(z_i)$ in the case $z_{11}<z_{22}<z_{12}<z_{21}$, where each arrow corresponds to $P_{\alpha\beta}(z_i)=1$, that is, an allowed particle addition. α denotes the type of particle on top and z_i is the slope.

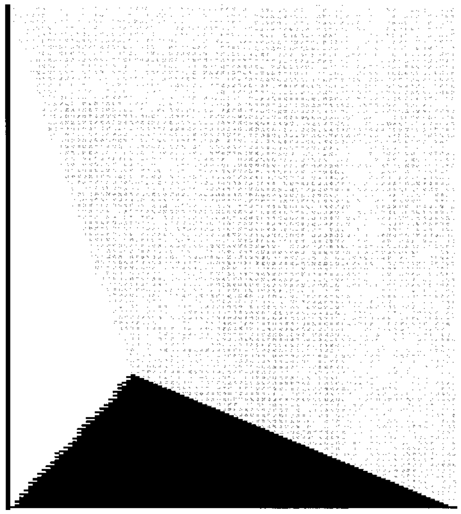

FIG. 4. Example of a class II sandpile with $z_{11}=1$, $z_{12}=3$, $z_{21}=5$, and $z_{22}=7$, demonstrating *discrete self-segregation*. Particles of type 1 are black, those of type 2 are white. The base of the pile is 100 sites wide.

slope, which in this case is $\approx z_{12}$. Note that for $z_{22}>z_{21}$ we do not have strict periodicity as type 2 particles are occasionally added consecutively.

Class II : $z_{21}>z_{11}$ *and* $z_{12}<z_{22}$. A stable domain of type 2 particles with uniform slope z_{22} builds up to the left of the sandpile, and type 1 particles slide through to the right forming a domain with uniform slope z_{11}. The result is *discrete self-segregation*, as the example in Fig. 4 demonstrates. The boundary between the two domains moves to the right as particles of type 2 come to rest on sites with slope z_{11}, so if $z_{12}>z_{11}$ some periodic mixing may occur. If $z_{21}>z_{12}$, the mixing remains confined to a narrow layer at the boundary, but if $z_{21}<z_{12}$ it expands to the bulk of the pile, giving a sandpile similar in appearance to the case of the periodic mixing described previously. The self-segregation and boundary behavior can also be seen from the phase portraits of the particle addition operator $P_{\alpha\beta}(z_i)$. We justify referring to the case $z_{11}<z_{21}<z_{12}<z_{22}$ as discrete self-segregation by the observation that the region of periodic mixing collapses to a much narrower layer under most small alterations in the dynamical rules. We will return to this point later when we discuss the robustness of the model.

It is possible to construct a global solution of this class of sandpiles. Define L_B and L to be position of the boundary and the right-hand edge of the sandpile, respectively, so that $h_L>0$, $h_{L+1}=0$, all sites $1\leqslant i\leqslant L_B$ have a type 2 particle on top and all sites $L_B<i\leqslant L$ have type 1 particles on top. A type 2 particle added to $i=L_B$ will reduce the slope of site L_B-1 by 1, so the next type 2 particle added will stop at L_B-1, then L_B-2, L_B-3, and so on. Similarly, type 1 particles will be added to L, $L-1$, $L-2,\ldots,L_B+1$, in that order. L will move to the right by one step when h_L increases from 0 to z_{11}, so in the continuum limit

$$\frac{dL}{dt}=\frac{1}{2z_{11}}\left(\frac{1}{L-L_B}\right), \tag{2}$$

where the time scale has been normalized to one particle addition of either type per unit time. The slope at L_B increases with each type 2 particle that stops on L_B and decreases with each type 1 particle that stops on L_B+1, and since L_B increases by one whenever the slope increases from z_{11} to z_{22} we get

$$\frac{dL_B}{dt}=\frac{1}{2(z_{22}-z_{11})}\left(\frac{1}{L_B}-\frac{1}{L-L_B}\right). \tag{3}$$

A steadily evolving sandpile corresponds to a constant ratio L_B/L, so

$$\frac{d}{dt}\left(\frac{L_B}{L}\right)=0. \tag{4}$$

Using this together with Eqs. (2) and (3), we find

$$\frac{L_B}{L}=\frac{\sqrt{z_{22}/z_{11}}-1}{z_{22}/z_{11}-1}, \tag{5}$$

giving the average slope of the entire pile as

$$\left(1-\frac{L_B}{L}\right)z_{11}+\frac{L_B}{L}z_{22}=\sqrt{z_{11}z_{22}}, \tag{6}$$

in agreement with simulations [16].

Class III : $z_{21}<z_{11}$ *and* $z_{12}>z_{22}$. It might be expected that self-segregation will also occur here, this time with type 1 particles to the left of the boundary. However, since $z_{11}<z_{22}$ the type 2 particles to the right of the boundary form steeper slopes than the type 1 particles to the left, and so now the boundary moves up the slope rather than down the slope, creating a double layer of 2's on top of 1's. Once the boundary reaches the left-hand wall, a thin layer of 1's quickly covers the surface of the whole sandpile, the boundary returns to the bottom, and starts propagating upwards once more. This moving interface corresponds to the kink described in the rolling layer case [7,15], and, indeed, the region of parameter space in which it occurs is the same. However, there are three significant differences between the nature of the self-stratification formed by this slowly driven process and that formed by rolling layers. (i) The slope of the layers varies between z_{11} and z_{22}, as opposed to the uniform slope of z_{22} observed in the rolling layer case. (ii) The layers are narrower, typically just one particle wide. (iii) The rate at which the interface moves is no longer a constant but varies according to statistical fluctuations in the types of incoming particles. It is even possible for the interface to stop moving altogether, resulting in a vertical build up of particles that is reminiscent of a miniature self-segregated sandpile. However, this state is unstable and the interface will eventually start moving again, either up the slope to continue the layering process or quickly down the slope to the bottom of the pile. Thus, layers can now start and stop in the bulk of the pile. An example of a self-stratified sandpile is given in Fig. 5.

Class IV : $z_{21}<z_{11}$ *and* $z_{12}<z_{22}$. In this final class, alternating stable domains of 1's and 2's form parallel vertical bands. An example of this *vertical stratification* is given in Fig. 6. The phase portrait of $P_{\alpha\beta}(z_i)$ for this class demonstrates the separation of the particle types, but further analysis requires some knowledge of the global solution. Once the

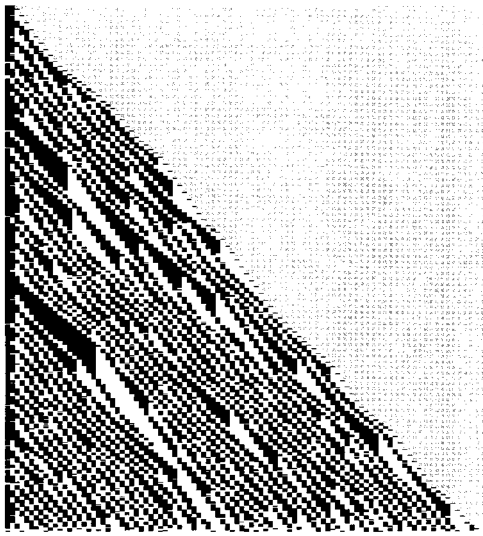

FIG. 5. Example of a class III sandpile with $z_{11}=5$, $z_{12}=15$, $z_{21}=1$, and $z_{22}=10$, demonstrating *self-stratification*. Type 1 particles are black and type 2 particles are white. The base of the pile is 100 sites wide.

slope of the site at the right-hand edge of any domain decreases by one, a single layer of particles of similar type backpropagates to the left-hand edge, when the adjacent domain will undergo a similar process. Thus the bulk of the sandpile builds up a layer at a time in this piecewise fashion, from the bottom of the pile to the top. The process that initially generates each domain depends upon the interaction between the bulk of the sandpile and the qualitatively different *end region* at the far right-hand side. In the end region, a layer of 2's of thickness $\approx z_{11}-z_{21}+1$ backpropagates to the first domain of type 2 particles. The domain then broadens to the right, and the process is repeated. Alternatively, if $z_{12}>z_{21}$ the back-propagating layer of 2's can be stopped prematurely by an incoming flux of 1's, resulting in the formation of a new stable domain of type 1 particles. Since the

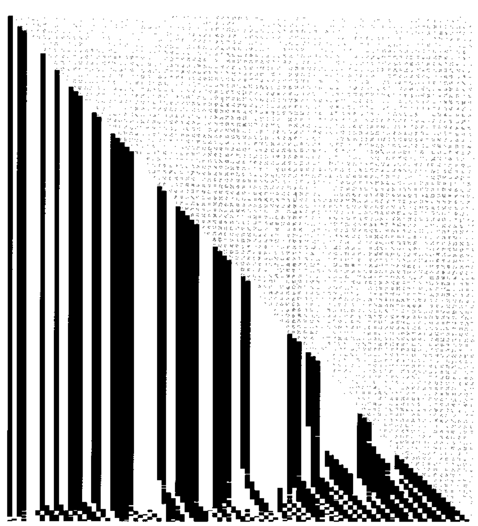

FIG. 6. Example of a class IV sandpile with $z_{11}=5$, $z_{12}=3$, $z_{21}=1$, and $z_{22}=7$, demonstrating *vertical stratification*. Type 1 particles are white, type 2 particles are white and the base of the pile is 100 sites wide.

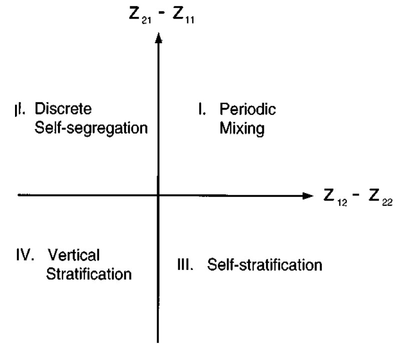

FIG. 7. Schematic diagram of $z_{21}-z_{11}$ vs $z_{12}-z_{22}$ for $N=2$ and $z_{11}<z_{22}$, showing where each class of sandpile solution applies.

particles that flow into the end region are precisely those that did *not* stop in the bulk of the pile, the number of domains of each different type tends to remain equal, although the domains themselves get broader as the end region expands. If $z_{12}<z_{21}$ the formation of new type 1 domains can still occur when the sandpile is small due to statistical fluctuations, but this rarely occurs for larger piles, which take on a self-segregated appearance.

A diagram of the parameter space showing the regions in which each of these classes occurs is given in Fig. 7. The borders between these regions correspond to $z_{21}=z_{11}$ or $z_{12}=z_{22}$, when domain stability is not well defined. In these cases the sandpile behavior is either indeterminate between the two classes in question or just reduces to random mixing. For $z_{11}=z_{22}$ periodic mixing and vertical stratification are unaffected but there is no longer any distinction between discrete self-segregation and self-stratification. Instead, these two classes are replaced with a hybrid class that exhibits self-segregation with a broad, layered boundary.

Each class is said to be *robust* if its existence is insensitive to the exact choice of dynamical rules. It is possible to vary the volumes of each particle type added or to introduce an open or closed right-hand boundary condition without any significant alteration in the resultant sandpile. Similarly, initially adding the particles over a range of sites does not affect the sandpile to the right of the range, and introducing annealed disorder to the $z_{\alpha\beta}$ just increases the noise. More significant is the effect of averaging the $z_{\alpha\beta}$ over adjacent pairs of sites, corresponding perhaps to the nestling of the upper particle in between the two lower ones, which destroys vertical stratification and instead gives random mixing or discrete self-segregation. Crudely modeling inertia by allowing the moving particle to stochastically drift a short distance further than normal distorts vertical stratification and replaces layering and periodic mixing with random mixing. We conclude that the model is robust except when we include displacements in the particle's horizontal motion.

For $N>2$ the parameter space for $z_{\alpha\beta}$ becomes too large to explore systematically. The situation improves somewhat if only domain stability is considered, but this still leaves $2^{N(N-1)}$ possible combinations, so we have limited ourselves to a brief survey of all these cases for $N=3$ and a representative sample for $N=4$. The resultant sandpiles are essentially just combinations of the four classes identified for $N=2$, with a significant feature being that periodic mixing can now occur with periodicity $\leqslant N$. Any particle separation and the allowed orders of periodic mixing can be predicted in each case by extending the particle addition operator $P_{\alpha\beta}(z_i)$ to include $\alpha,\beta \in [1,N]$. As before, there is no generic way of constructing global solutions but in many cases known solutions for $N-1$ particle systems can be used instead. This is possible when particle types α and β would by themselves periodically mix, when to a good approximation they can be replaced by a single particle type α' that alternates between the two. A domain is stable to α' only if it is stable to both α and β, a domain of type α' particles is stable to another type only if both α and β are, and $z_{\alpha'\alpha'} = \min(z_{\alpha\beta}, z_{\beta\alpha})$. This reduced system usually exhibits the correct qualitative structure of the original but significantly underestimates the amount of noise.

IV. DISCUSSION

The discrete model studied here is perhaps the simplest conceivable model describing slowly driven sandpile formation with granular mixtures. Nonetheless it exhibits a wide variety of nontrivial behavior in one dimension, and we can only suppose that it will continue to do so in higher dimensions. The behavior of sandpiles for binary mixtures falls into one of four classes, two of which have known counterparts in the rolling layer model [7,15]. They are also much more susceptible to statistical fluctuations in the order of particle types added.

We have recently initiated a series of experiments in an attempt to reproduce these classes with real granular materials [17]. On first inspection, there may appear to be little hope that such a simple model could describe real sandpile mixtures. For instance, the possibility of particles bouncing or dislodging surface material has not been addressed. However, our numerical investigations into the robustness of the model leads us to suppose that agreement might be possible in the limit of overdamped particle motion. It is also important to realize the limit of infinitely slow driving as closely as possible, to minimize the probability of a rolling layer forming, because even a thin rolling layer would displace surface material along the length of the slope. This would interfere with the formation of both vertical stratification and periodic mixing.

In summary, sandpile formation by granular mixtures exhibits a greater diversity of behavior in the slowly driven limit than in the rolling layer case, at least numerically. If toppling were included in this model [18], more than one particle would be able to move simultaneously in the form of an avalanche. Similarly, allowing moving particles to dislodge surface material in some manner might allow for something akin to a rolling layer to form. It would be interesting to see if the system diversity was reduced in either of these two cases.

ACKNOWLEDGMENT

We would like to thank Peter Hobson for useful discussions.

[1] H. M. Jaeger and S. R. Nagel, Science **255**, 1523 (1992).
[2] A. Rosato, K. J. Strandburg, F. Prinz, and R. H. Swendsen, Phys. Rev. Lett. **58**, 1038 (1987).
[3] J. B. Knight, H. M. Jaeger, and S. R. Nagel, Phys. Rev. Lett. **70**, 3728 (1993).
[4] W. Cooke, S. Warr, J. M. Huntley, and R. C. Ball, Phys. Rev. E **53**, 2812 (1996).
[5] K. M. Hill and J. Kakalios, Phys. Rev. E **49**, R3610 (1994).
[6] O. Zik, D. Levine, S. G. Lipson, S. Shtrikman, and J. Stavans, Phys. Rev. Lett. **73**, 644 (1994).
[7] H. A. Makse, S. Havlin, P. R. King, and H. E. Stanley, Nature **386**, 379 (1997).
[8] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
[9] G. Grinstein, in *Scale Invariance, Interfaces and Non-Equilibrium Dynamics*, Vol. 344 of *NATO Advanced Study Institute, Series B: Physics*, edited by A.McKane *et al.* (Plenum, New York, 1995).
[10] H. M. Jaeger, C. Liu, and S. R. Nagel, Phys. Rev. Lett. **62**, 40 (1989).
[11] G. A. Held, D. H. Solina II, D. T. Keane, W. J. Haag, P. M. Horn, and G. Grinstein, Phys. Rev. Lett. **65**, 1120 (1990).
[12] P. Evesque and J. Rajchenbach, Phys. Rev. Lett. **62**, 44 (1989).
[13] V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang, and P. Meakin, Nature (London) **379**, 49 (1996).
[14] J.-P. Bouchaud, M. E. Cates, J. R. Prakash, and S. F. Edwards, Phys. Rev. Lett. **74**, 1982 (1995).
[15] H. A. Makse, P. Cizeau, and H. E. Stanley, Phys. Rev. Lett. **78**, 3298 (1997).
[16] This result can also be derived by a geometric argument if it is assumed that both domains sweep out equal areas in time, corresponding to the equal rates of particle addition.
[17] D. A. Head, P. R. Hobson, and G. J. Rodgers (unpublished).
[18] L. P. Kadanoff, S. R. Nagel, L. Wu, and S. Zhou, Phys. Rev. A **39**, 6524 (1989).