Large-scale simulations of the Zhang sandpile model

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We consider the non-Abelian sandpile model introduced by Y.-C. Zhang [Phys. Rev. Lett. **63**, 470 (1989)] on a two-dimensional square lattice. The static and dynamical properties of the model are investigated and compared to the Abelian sandpile model of Bak, Tang, and Wiesenfeld [Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988)]. A detailed analysis that takes the finite-size effects into account yields that the exponents of the avalanche probability distribution are the same as in the Abelian model.
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I. INTRODUCTION

The idea that an externally driven physical system with many degrees of freedom can be self-organized critical was introduced several years ago by Bak, Tang, and Wiesenfeld and realized theoretically using a stochastic cellular automaton [1]. The original Bak-Tang-Wiesenfeld (BTW) model belongs to the Abelian sandpile models [2]. Here the sequence of relaxation processes is described by operators that satisfy a commutative algebra. This property allows the analytical calculation of some features of the system in the steady state [2–5]. A continuous version of this model was introduced by Zhang to study the propagation of activated energies [6]. In contrast to the BTW model, the Zhang model is a non-Abelian model, i.e., the steady-state configurations depend on the sequence in which unstable sites are toppled (see [7,8] and references therein). Despite the different microscopic dynamics both models are expected to belong to the same universality class (see, for instance, [9]). Up to now nobody has proved this assumption by direct measurements of the avalanche exponents on large lattice sizes, which reduces the finite-size effects sufficiently. We consider the Zhang model on lattice sizes that are significantly larger than those sizes used in previous investigations [6,10–12]. The energy distribution $p(E)$, which characterizes the static properties of the model, is concentrated around z distinct peaks, where z is the number of nearest neighbors. We show that the peaks are located at multiples of $(z+1)/z^2$ and the height of the peaks grow with increasing system size. Numerical simulations of the two-dimensional square and honeycomb lattices confirm this result. We also investigated the avalanche distributions on lattice sizes up to $L=2048$. A finite-size analysis of the exponents of the avalanche distributions yields values that correspond to those of the BTW model.

II. MODEL

We consider a two-dimensional square lattice of linear size L. A continuous value $E_{i,j}\geqslant 0$ representing the energy is associated with each lattice site (i,j). The boundary sites are fixed to zero $[E(\text{boundary})=0]$ for all times. A configuration $\{E_{i,j}\}$ is stable if $E_{i,j}<E_c$ for all lattice sites (i,j). For the sake of simplicity we choose in all simulations $E_c=1$. A quantum of energy δ is added to a randomly chosen lattice site (i,j), i.e.,

$$E_{i,j}\rightarrow E_{i,j}+\delta. \tag{1}$$

In the case that due to this perturbation a site exceeds the critical value E_c, an activation event will occur and the critical site relaxes to zero and the energy is added to the next neighbors, i.e.,

$$E_{i,j}\rightarrow 0, \tag{2}$$

$$E_{i,j,NN}\rightarrow E_{i,j,NN}+\frac{E_{i,j}}{z}, \tag{3}$$

where z denotes the number of next neighbors. In that way the transferred energy may activate the neighboring sites and thus an avalanche of relaxation events may take place. Energy may leave the system only at the boundary.

In our simulations we use various values of the input energies out of the interval $\delta\in\,]0,E_c]$. In the case of $\delta\rightarrow 0$ all lattice sites grow parallel. In order to implement this different perturbation process one has to find the site with the largest energy E_{max} and then increment all sites by E_c-E_{max}. In this case the Zhang model is identical to the conservative limit of the ‘‘spring block’’ model of Christensen and Olami [13].

The concept of self-organized critical systems refers to driven systems that organize themselves into a steady state. We consider the average energy

$$\langle E(t)\rangle=\frac{1}{L^2}\sum_{i,j}E_{i,j}(t) \tag{4}$$

to check if the system has reached the steady state. Starting with an empty lattice, we consider the growth of the pile. In the beginning all sites are subcritical, i.e., $E_{i,j}\ll E_c$, and no toppling event occurs. Here no relaxation process takes place (nonavalanche regime) and the average energy increases linearly in time (see Fig. 1). With further perturbations the average energy is still growing until one site reaches the critical

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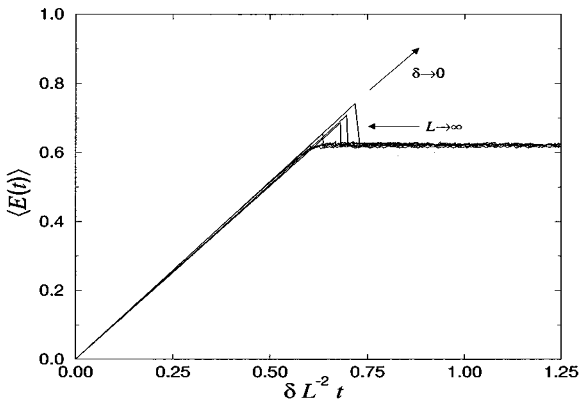

FIG. 1. Average energy $\langle E(t)\rangle$ as a function of the rescaled time $\tau=\delta L^{-2}t$ for $L\leqslant 512$ and various values of δ.

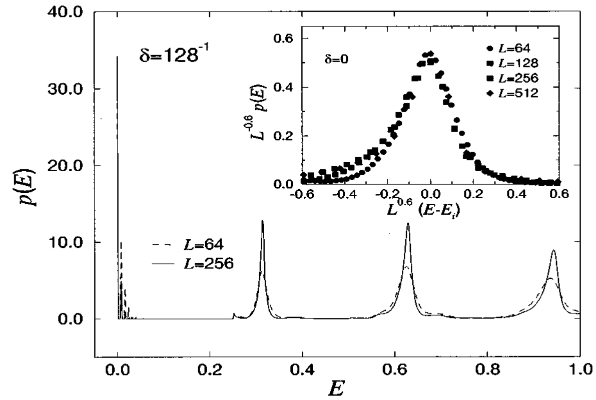

FIG. 2. Probability distribution $p(E)$ for different system sizes. The inset displays the scaling plot of the third maximum of $p(E)$.

value E_c. Now the behavior of the system changes and toppling processes occur (avalanche regime). After a certain time the average energy reaches a constant value $\langle E\rangle$, which characterizes the steady state.

In Fig. 1 the average energy is plotted as a function of the rescaled time $\tau=\delta L^{-2}t$. One can see a data collapse of all curves corresponding to different values of L and δ. Deviations from the collapse occur only at the point $\tau\approx 0.63$, where the behavior changes from the nonavalanche regime to the avalanche regime, characterized by a constant average energy. The avalanche regime occurs when the fluctuations of the energies are greater than the difference of the critical energy E_c from the average energy $\langle E(t)\rangle$, i.e., when

$$\sqrt{\langle E^2(t)\rangle-\langle E(t)\rangle^2}\geqslant E_c-\langle E(t)\rangle. \tag{5}$$

Decreasing δ reduces the fluctuations and the critical time tends to $\tau_c=1$. Larger system sizes result in a decreasing critical time.

We consider the system for lattice sizes $L\in\{64,128,256,512,1024,2048\}$ (in the case of $\delta=0$ the maximum lattice size is $L=1024$). Starting with an empty lattice the system will be equilibrated after $L^2\delta^{-1}$ perturbations. In order to provide a sufficient statistics all measurements are averaged over at least 10^6 nonzero avalanches.

III. ENERGY DISTRIBUTION

We measured the energy distribution $p(E)$ in the steady state for $\delta\in\{0,128^{-1},8^{-1},1\}$ and various system sizes L. In Fig. 2 the distribution $p(E)$ is plotted for different system sizes. The distribution is concentrated around four distinct peaks. It was assumed in previous works [6,11] that the finite spreads of the peaks are caused by ''intrinsic dynamical fluctuations.'' As one can see from Fig. 2, the peaks grow and the spreads of the peaks decrease with increasing system size L.

We found that the maximum $p_{max}(E)$ of each peak scales with the system size as

$$p_{max}(E)\sim L^y, \tag{6}$$

with $y\approx 0.6$. Since the distribution $p(E)$ is normalized, we assume that the peaks scale in the horizontal direction as L^{-y}. The location of the maxima of the distribution $p(E)$ depends slightly on the system size L. In order to produce a scaling plot this drift has to be taken into account. In the inset of Fig. 2 we plot $L^{-y}p(E)$ as a function of $L^y(E-E_{max}(L))$ and get a satisfying data collapse. The peaks of the energy distribution $p(E)$ grow to infinity and the spread of each peak vanishes for $L\rightarrow\infty$. In the case of an infinite system the energy distribution $p(E)$ in the steady state is given by

$$p(E)=\sum_{i=0}^{3} f_i\delta(E-E_i), \tag{7}$$

where f_i denotes the statistical weight and E_i denotes the location of the δ peaks.

One can calculate the discrete values of the energies E_i in the following way [14]. Suppose that the energies are already discretized with the allowed values

$$E\in\{0,E_0,2E_0,3E_0,\ldots,nE_0,\ldots\}. \tag{8}$$

Then a maximum value of n exists with

$$n_{max}E_0\leqslant E_c<(n_{max}+1)E_0. \tag{9}$$

The critical energy $E=(n_{max}+1)E_0$ relaxes and E/z should be equal to E_0, i.e.,

$$\frac{(n_{max}+1)E_0}{z}=E_0. \tag{10}$$

In this way the number of peaks equals the lattice coordination number $n_{max}+1=z$. Based on his numerical investigations of different lattice types, Díaz-Guilera has already proposed this relation [12].

Starting with a stable configuration, one perturbs the system until one site becomes critical, i.e., one adds $\Delta E=E_c-n_{max}E_0$ on each lattice site (this is correct for $\delta\rightarrow 0$). The energy of a given site is now $E=nE_0+\Delta E$. The critical site relaxes and E_c/z is added to the z next neighbors of this site. Arguing that the new energy is the next allowed energy value $E=(n+1)E_0$, one gets the relation

$$E_0=\frac{E_c}{n_{max}+1}\frac{z+1}{z}=E_c\frac{z+1}{z^2}. \tag{11}$$

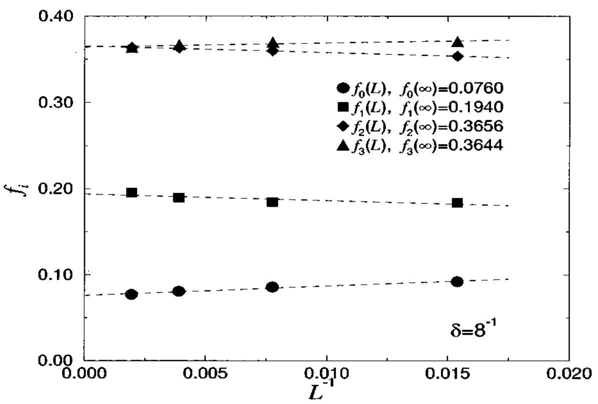

FIG. 3. Statistical weights f_i as a function of the inverse system size L^{-1} for $\delta=8^{-1}$. The values $f_i(\infty)$ are obtained by an extrapolation to the vertical axis.

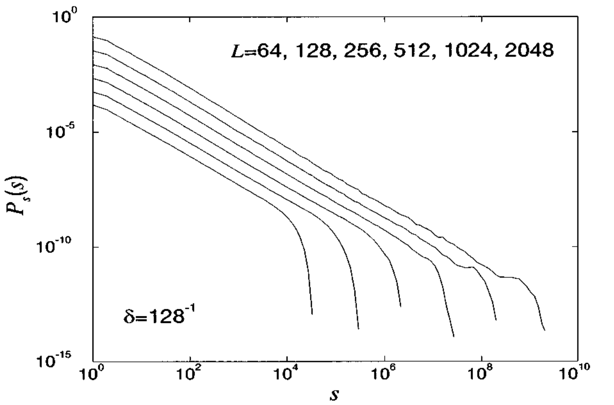

FIG. 4. Probability distribution $P_s(s)$ for different system sizes for $\delta=128^{-1}$. The curves for $L<2048$ are shifted in the downward direction.

Note that the discretization of the energies is independent of the dimension of the system. The relevant term is the lattice coordination number z. This is in contrast to the conclusions drawn from previous investigations, which are based only on simulations of square lattices in different dimensions d where the coordination number is given by $z=2d$ [10].

We compare Eq. (11) with the results obtained from computer simulations. In $d=2$ we found that $E_0\approx 0.3149$ for $\delta=128^{-1}$, $E_0\approx 0.3153$ for $\delta=0$, $E_0\approx 0.3145$ for $\delta=8^{-1}$, and $E_0\approx 0.3140$ for $\delta=1$, which are in good agreement with Eq. (11). We also measured the energy distribution of a honeycomb lattice in two dimensions $(z=3)$ and found for $\delta=128^{-1}$ the value $E_0\approx 0.443$, which corresponds very well to the exact value $E_0=0.\overline{4}$. Pietronero *et al.* have investigated the $d=3$ Zhang model on a square lattice and found six peaks in the energy distribution [11]. We measured the average distance between two peaks from Fig. 2 of [11] and determined in this way $E_0=0.190$, which agrees with the $E_0=0.19\overline{4}$ obtained from Eq. (11).

Furthermore, we determined the statistical weights f_i of the energy distribution [Eq. (7)]. We divided the interval $[0,E_c]$ in four parts and measured in each part the area $f_i(L)$ under the curve $p(E)$ for various system sizes L. The statistical weights f_i are given by an extrapolation to $L\to\infty$ (see Fig. 3) and the obtained values are listed in Table I. Analogous to the locations of the peaks, the statistical weights do not depend on the input energy δ. On the other hand one can see that the values differ from those of the BTW model, which are known exactly [4].

Pietronero *et al.* [15] introduced a renormalization-group approach for sandpile models where the density of the critical sites determines the fixed point of the renormalization transformation. Here the density of the critical sites corresponds to the statistical weight f_3 in the sense that any perturbation of a coarse-grained particle (E_0) leads to a relaxation event. Following our results, both models are characterized by different fixed points and thus one might expect that both models belong to different universality classes. But one has to emphasize that this renormalization-group approach and its improvement by Ivashkevich [8] neglects fluctuations at the steady state. Due to this ‘‘mean-field-type approximation’’ [16] we think that the different critical densities of the Zhang and the BTW model cannot lead to an answer of the universality question.

TABLE I. Statistical weights of the energy distribution.

f_i	$\delta=0$	$\delta=128^{-1}$	$\delta=8^{-1}$	$\delta=1$	BTW model
f_0	0.077	0.076	0.076	0.077	0.074
f_1	0.197	0.196	0.194	0.195	0.174
f_2	0.365	0.362	0.366	0.364	0.306
f_3	0.362	0.366	0.364	0.364	0.446

IV. AVALANCHE DISTRIBUTIONS

In this section we examine the probability distribution of an avalanche of size s, area s_d, duration t, and radius r, where s denotes the total number of toppled sites and s_d is the number of distinct sites that correspond to the area of an avalanche. The duration t of an avalanche is equal to the number of update sweeps needed until all sites are stable again. The linear size of an avalanche r is measured via the radius of gyration of the avalanche cluster. In the critical steady state the corresponding probability distributions should obey a power-law behavior characterized by exponents τ_s, τ_d, τ_t, and τ_r according to

$$P_s(s)\sim s^{-\tau_s}, \tag{12}$$

$$P_d(s_d)\sim s_d^{-\tau_d}, \tag{13}$$

$$P_t(t)\sim t^{-\tau_t}, \tag{14}$$

$$P_r(r)\sim r^{-\tau_r}. \tag{15}$$

The distribution $P_s(s)$ is plotted in Fig. 4 for $\delta=128^{-1}$ and various system sizes L. All curves fit in the middle region to a straight line and the corresponding exponents are determined via regression of this region. First we investigate whether the exponents depend on the input energy δ and second we examine how the finite system size affects the results. Figure 5 shows the exponent τ_s for $L=256$ and for various values of δ. In the limit $\delta\ll E_c=1$ the exponents are independent of the input energy. This behavior changes

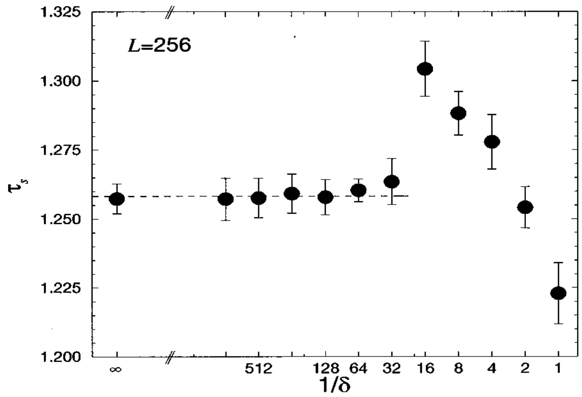

FIG. 5. Values of the exponent τ_s as a function of the input energy δ for a fixed system size L. Note that the values of the exponent are independent of δ in the limit $\delta \ll E_c$.

abruptly for $\delta \geqslant 32^{-1}$, where the exponent displays a complex δ dependence. In the following we focus our attention on the limit $\delta \ll E_c$.

The exponents τ_s corresponding to different values L and δ are plotted in Fig. 6. Significant differences between the values of the exponents $\tau_s(L,\delta=0)$ and $\tau_s(L,\delta=128^{-1})$ are caused by the system size only and not by the input energy. Both exponents tend to $\tau_s \approx 1.28$ with increasing L. In order to determine the exact value of the exponent τ_s we assume that its system size dependence is given by

$$\tau_s(L) = \tau_s + \frac{\text{const}}{L^x}. \tag{16}$$

We tried several values of x and got the best results for $x=1$, i.e., the finite-size effects are of the relative magnitude of the boundary ($\sim L^{-1}$). In the inset of Fig. 6 the exponents $\tau_s(L)$ are plotted as a function of the inverse system size. The exponent τ_s is given by an extrapolation to $L \to \infty$, which yields $\tau_s = 1.282 \pm 0.01$.

The exponents of the avalanche probability distribution of the area and radius are characterized by the same finite-size corrections. The exponents corresponding to different system sizes are plotted in Fig. 7. Except for the deviation for $L=64$ in the case of the exponent τ_d, both exponents depend on the inverse system size [corresponding to Eq. (16)]. From the extrapolation to the infinite system size we obtain the values $\tau_d = 1.338 \pm 0.015$ and $\tau_r = 1.682 \pm 0.018$, respectively. The finite-size dependence explains why lower values of the exponents were reported in previous works based on numerical simulations of one system size only (see, for instance, [10]).

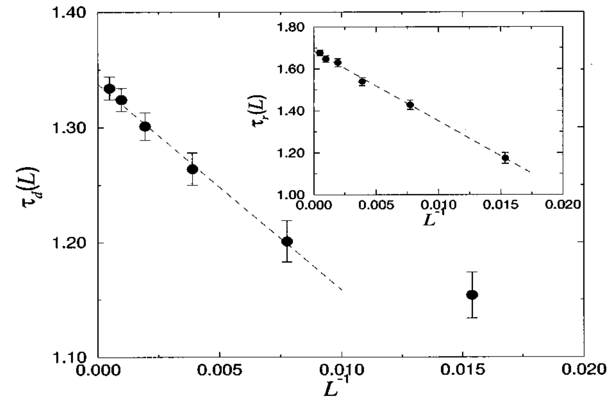

FIG. 7. Values of the exponent τ_d and τ_r as a function of the inverse system size L^{-1} for $\delta = 128^{-1}$. The dashed lines correspond to the extrapolation according to Eq. (16).

The finite-size analysis described above fails in the case of the duration exponent τ_t. Here the probability distribution exhibits a finite curvature that makes it impossible to determine the exponent via regression (see Fig. 8). Using a momentum-space analysis of the corresponding Langevin equations, Díaz-Guilera showed that the dynamical exponent of the BTW and Zhang's model is given by $z=(d+2)/3$ [9]. This result allows one to determine the exponent τ_t because the exponents z, τ_t, and τ_r have to fulfill the scaling relation (see, for instance, [17])

$$z = \frac{\tau_r - 1}{\tau_t - 1}. \tag{17}$$

Using the above value of τ_r and $z=\frac{4}{3}$ for the two-dimensional model, we obtain the value $\tau_t = 1.512 \pm 0.014$.

Recently, it has been shown numerically that the exponents of the BTW model are consistent with the values

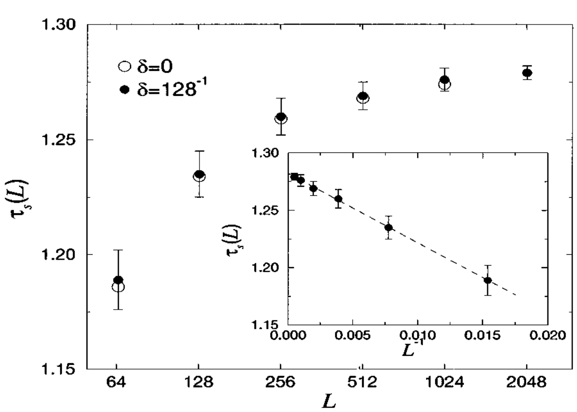

FIG. 6. System size dependence of the exponent τ_s for $\delta=0$ and $\delta=128^{-1}$. The inset displays the determination of τ_∞ according to Eq. (16) (dashed line).

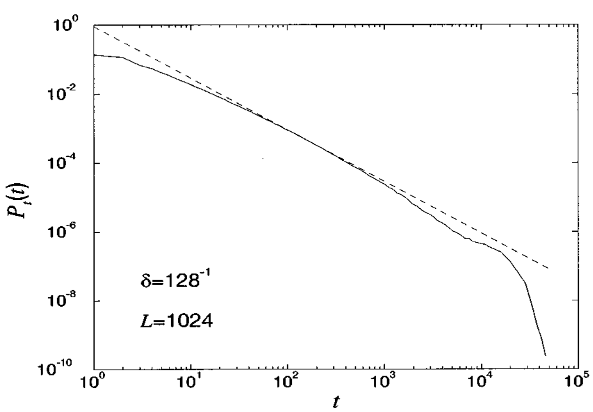

FIG. 8. Probability distribution $P_t(t)$ for a fixed system size. The dashed line corresponds to a power law with the exponent of the BTW model $\tau_t = \frac{3}{2}$ (see [17]).

$\tau_t = \frac{3}{2}$, $\tau_d = \frac{4}{3}$, and $\tau_r = \frac{5}{3}$ [17]. Because of the lack of a scaling relation, the exact value of τ_s is still unknown, but the authors estimate the value $\tau_s = 1.293 \pm 0.009$. These values are in agreement with our results, strongly suggesting that both models are characterized by the same exponents.

Note that we determined the exponents of the Zhang model for the limit $\delta \ll E_c$ only. The measurements for a fixed system size and larger values of the input energy δ yield different values of the exponents (see Fig. 5). But this does not mean that the exponents of the infinite system size depend on δ. It is also possible that the finite-size behavior [Eq. (16)] changes outside the limit $\delta \ll E_c$. Further work has to be done to examine how the finite system size affects the values of the avalanche exponents for $\delta \approx E_c$.

V. CONCLUSION

We have studied numerically the static and dynamical properties of the non-Abelian Zhang model on large system sizes. The steady-state energy distribution is concentrated around z distinct peaks that are located at multiples of $(z+1)/z^2$, where z denotes the lattice coordination number. The statistical weights of the peaks are independent of the input energy δ but differ from those of the BTW model. A finite-size analysis of the avalanche probability distributions in the limit $\delta \ll E_c$ yields exponents that are in agreement with the values of the exponents of the BTW model. Both models belongs to the same universality class, i.e., both models displays the same large-scale behavior, characterized by the avalanche exponents.

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