Geometrical resonance analysis of chaos suppression in the bichromatically driven van der Pol oscillator

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Elimination of chaotic behavior in the harmonically driven van der Pol oscillator by means of a comparatively weak additional forcing was studied through geometrical resonance analysis. We considered commensurate and incommensurate cases together with the effect of the phase difference between the forcings. The analysis provided parameter-space regions for regularization that were corroborated by numerical experiments, including instances with clearly large chaos-inducing forcing. A reinterpretation of a classical result, due to Cartwright and Littlewood [J. London Math. Soc. **20**, 180 (1945)], was also derived by means of geometrical resonance analysis. [S1063-651X(97)06008-X]

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I. INTRODUCTION

The periodically driven van der Pol oscillator is one of the paradigms of chaos theory [1–12]. In particular, it serves as a simple model of self-excited oscillations in such diverse fields as physics, biology, and electronics, to quote just a few. In this work we consider the problem of chaos suppression for the van der Pol oscillator subjected to two harmonic forcing terms

$$\ddot{x}+d(x^2-1)\dot{x}+x=F_c\cos(\omega t)+\alpha F_c\cos(\Omega t+\Phi), \quad (1)$$

where $d,F_c>0$, $0<\alpha\ll 1$, and time is regarded as dimensionless, the framework of the study being the recently proposed geometrical resonance (GR) analysis [13–15].

In the simple case of a one-dimensional, nonautonomous, dissipative, nonlinear oscillator

$$\dot{x}=y, \quad \dot{y}=g(x)-d(x,y)+p(x,y)F(t), \quad (2)$$

where $g(x)\equiv-\partial V/\partial x$, $V(x)$ is an arbitrary time-independent potential, $-d(x,y)$ is the dissipation term, and $p(x,y)F(t)$ is a generic temporal excitation, GR means that the period, amplitude, and *wave form* of $F(t)$ must be so as to preserve an *a priori* selected periodic response from the underlying conservative system. Note, however, that the notion of GR is not limited to periodic attractors, but is defined for *any* solution from the underlying integrable system (cf. Ref. [14]). Therefore, if $x_{\mathrm{GR}}(t)$ is a GR solution of Eq. (2), it satisfies

$$-d(x_{\mathrm{GR}},y_{\mathrm{GR}})+p(x_{\mathrm{GR}},y_{\mathrm{GR}})F_{\mathrm{GR}}(t)=0, \quad (3)$$

which is equivalent to the local energy conservation requirement $\frac{1}{2}y_{\mathrm{GR}}^2(t)+V(x_{\mathrm{GR}}(t))=\mathrm{const}$. In Ref. [13] it was conjectured that GR provides the explanation of the so-called nonfeedback control of chaos (see, e.g., Refs. [16–18]) in terms of an almost-adiabatic invariant (the *action* variable) associated with each GR solution. Also, the stability of the responses of an overdamped bistable system under a periodic forcing of rectangular shape was explained [14] in terms of GR.

In this present paper we consider the forcing $F_c\cos(\omega t)$ as a chaos-inducing modulation, for given parameters and initial conditions, and the second modulation $\alpha F_c\cos(\Omega t+\Phi)$, $0<\alpha\ll 1$, as a chaos-suppressing excitation added *a posteriori*. We then look for the parameter-space regions $\{\alpha,\Omega,\Phi\}$ in which the action variable is an almost-adiabatic invariant, i.e., the regions in which one expects regularized dynamics. Notwithstanding the adiabatic character of the action conservation, we shall consider a situation in which the chaos-inducing forcing amplitude F_c and the dissipation coefficient d are not necessarily small.

The organization of the paper is as follows. In Sec. II we first obtain the general GR temporal excitation $F_{\mathrm{GR}}(t)$ corresponding to the autonomous counterpart of system (1). Then we present the almost-adiabatic invariance approach when the two harmonic excitations replace $F_{\mathrm{GR}}(t)$. In Sec. III we discuss the problem of chaos suppression under adiabatic conservation of the action. Commensurate and incommensurate cases are studied together with the effect of the phase difference between the two excitations. Section IV illustrates the scope and accuracy of the theoretical predictions with some numerical examples for the commensurate case. Finally, Sec. V gives a summary of the results.

II. GEOMETRICAL RESONANCE ANALYSIS

To obtain the GR temporal modulation $F_{\mathrm{GR}}(t)$ associated with the autonomous part of Eq. (1), let us note that the

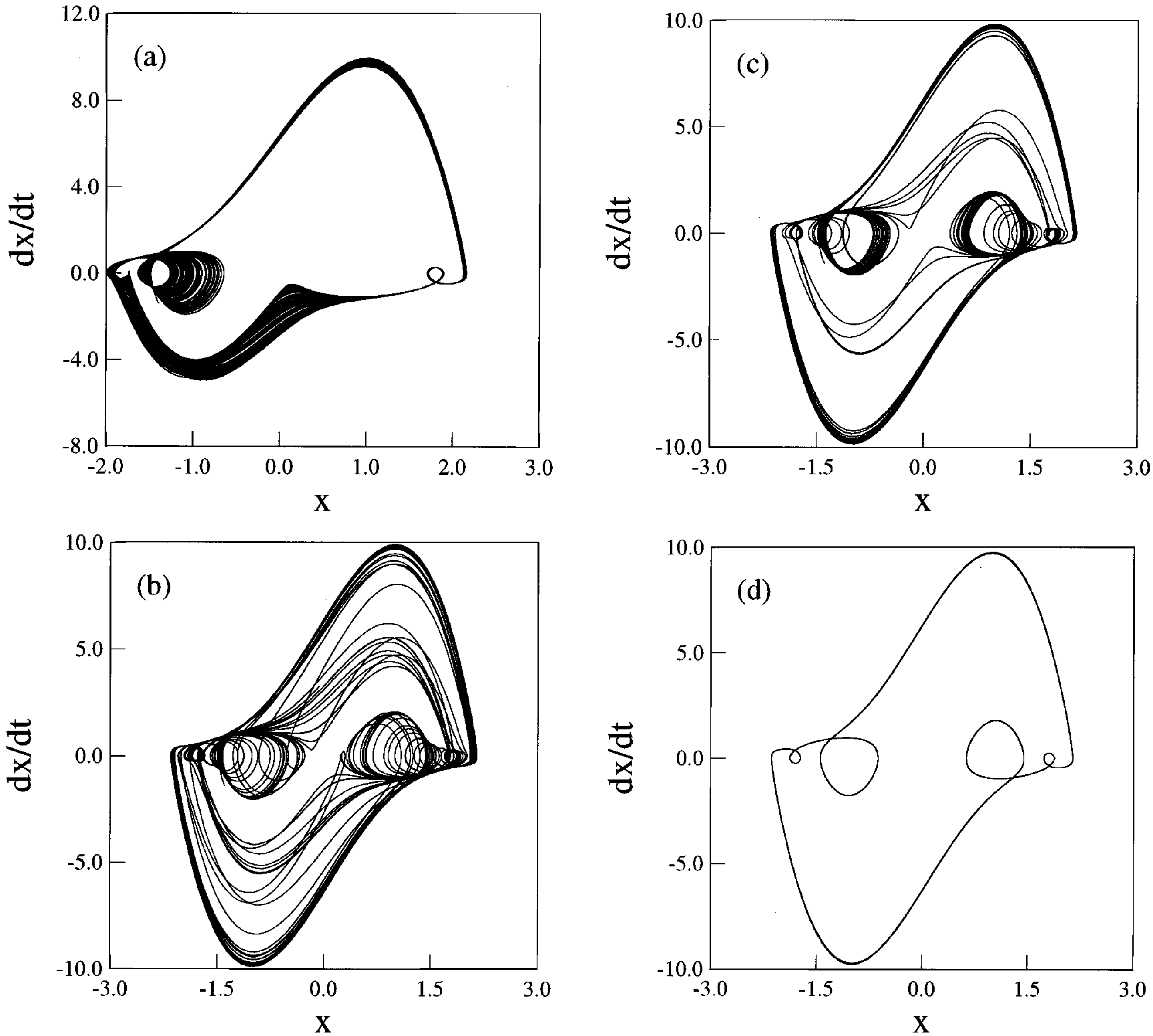

FIG. 1. Phase-space portraits for the parameters $d=5$, $F_c=5$, $\omega=2.463$, and $\Phi=2.7625$ [cf. Fig. 3(b)]: (a) $\alpha=0$, (b) $\alpha=0.04$, (c) $\alpha=0.07$, and (d) $\alpha=0.08$. The quantity x is in arbitrary current units and t is a dimensionless variable, in all the pertinent figures.

corresponding unperturbed conservative system is just the simple harmonic oscillator:

$$x_{GR}(t)=A\cos(t+\Phi'),\quad \dot{x}_{GR}(t)=-A\sin(t+\Phi'). \quad (4)$$

Therefore, from Eq. (3) one easily obtains

$$F_{GR}(t)=dA\left(1-\frac{A^2}{4}\right)\sin(t+\Phi')-\frac{dA^3}{4}\sin(3t+3\Phi'), \quad (5)$$

which depends on the initial conditions through A and Φ'. For initial conditions on a given GR solution, i.e., the corresponding action $I=\pi A^2$ remaining constant, one gets just one $F_{GR}(t)$, as expected. For $A\to 0$, $F_{GR}(t)\to dA\sin(t+\Phi')$ because of the nonlinear damping term approximates to $-d\dot{x}$ in such a limit. Notice that the above expression for $F_{GR}(t)$ is consistent with the well-known result [19] that the autonomous van der Pol oscillator has a stable limit cycle, for small d, *close* to the circle $x^2(t)+\dot{x}^2(t)=4$, i.e., $A=2$ [cf. Eqs. (4) and (5)].

Let us consider now the system (1), assuming that the dynamics is chaotic for $\alpha=0$ (for given values of d, F_c, ω, and a certain initial condition). Under such conditions we wish to find parameter values $\{\alpha,\Omega,\Phi\}$ that regularize the entire system (1). It is clear that no trio $\{\alpha,\Omega,\Phi\}$ provides an exact GR forcing [cf. Eq. (5)]. However, when dissipation and external modulation are represented by small amplitude terms, it is natural to suppose that the suitable values $\{\alpha,\Omega,\Phi\}$ for regularization will be those providing the best approximation (in the action-conservation sense) to the corresponding GR solution, i.e., those values derived from the local almost-adiabatic conservation of the action [13]

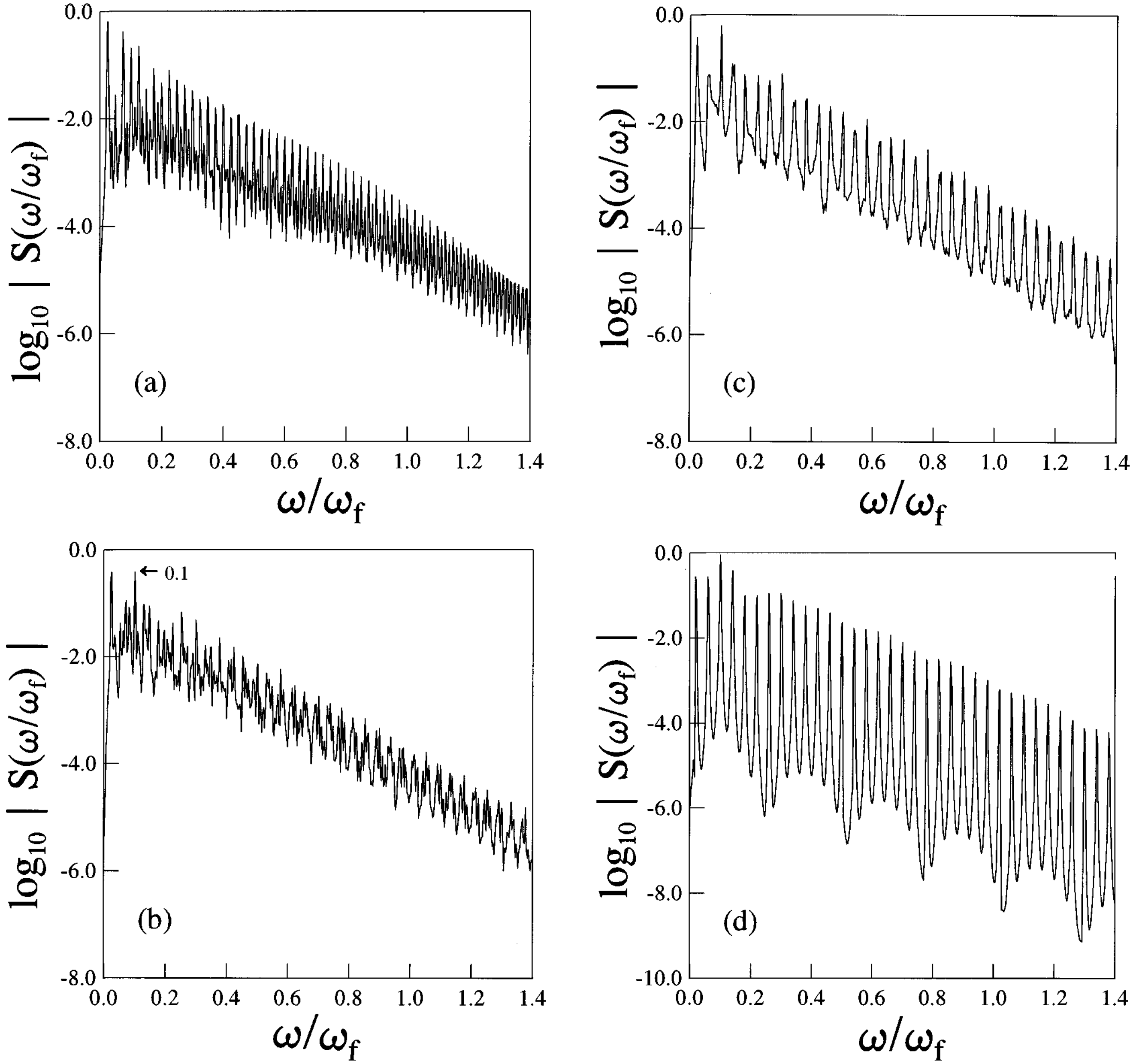

FIG. 2. Power spectra corresponding to the homonymous cases in Fig. 1; $\omega_f = 2.463$ (see the text).

$$\left\langle \frac{d}{dt}\left(\frac{E}{1/T_{\mathrm{GR}}}\right)\right\rangle_{T_{\mathrm{GR}}\equiv 2\pi} \equiv \int_0^{2\pi}\left(\frac{dE}{dt}\right)dt$$

$$\simeq \int_0^{2\pi} \dot{x}_{\mathrm{GR}}(t)\{\dot{x}_{\mathrm{GR}}(t)[1-x_{\mathrm{GR}}^2(t)] + F_c\cos(\omega t)+\alpha F_c\cos(\Omega t + \Phi)\}dt = 0, \quad (6)$$

where $x_{\mathrm{GR}}(t), \dot{x}_{\mathrm{GR}}(t)$ are given by Eq. (4). Condition (6) holds for any relationship between ω and Ω, i.e., it is valid for both commensurate and incommensurate cases. It is worth noting that condition (6) provides a chaos threshold. In other words, it represents, in general, a *necessary condition* for the suppression of chaos. This means, among other things, that one can determine the (*a priori* effective) lowest values of α for the elimination of chaos from that condition.

III. THE PROBLEM OF CHAOS SUPPRESSION

In this section we first deduce the suppressory values $\{\alpha,\Omega,\Phi\}$ for the commensurate case

$$\frac{q}{p}\,\omega = \Omega \equiv 1, \quad (7)$$

i.e., we set $2\pi/\Omega \equiv T_\Omega = T_{\mathrm{GR}} \equiv 2\pi$ and assume that there exist some integers q,p verifying $T_\Omega = (p/q)T_\omega$ ($T_\omega \equiv 2\pi/\omega$). Then, substituting Eqs. (4) and (7) into Eq. (6), the resulting integrals can be easily evaluated from standard integral tables [20]. The final result can be written

$$\alpha \simeq \frac{1}{\sin(\Phi'-\Phi)}\left\{\frac{dA}{F_c}\left(1-\frac{A^2}{4}\right)+\frac{1}{\pi}R'(p,q,\Phi')\right\}, \quad (8)$$

where

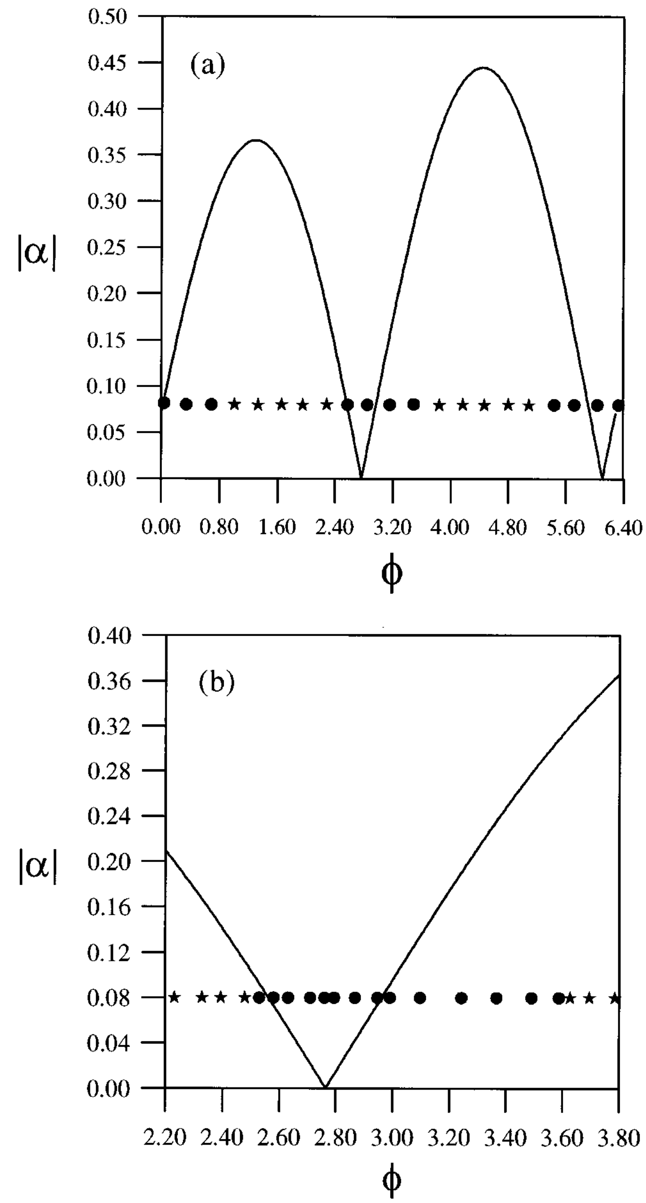

FIG. 3. (a) Theoretical boundary (solid line) for stabilization in the $|\alpha|$-Φ plane for $d=5$, $F_c=5$, $p=2463$, $q=1000$, and the same initial conditions as in Fig. 1. Asterisks (chaotic motion) and circles (periodic motions) represent numerical results for $\alpha=0.08$. (b) Enlargement of the region of the $|\alpha|$-Φ plane showing the analytical tongue with a minimum at $\Phi=2.7625$.

$$R'(p,q,\Phi')=-\pi\sin\Phi', \tag{9}$$

$$R'(p,q,\Phi')=0, \tag{10}$$

$$R'(p,q,\Phi')=\frac{-2\sin(\pi p/q)}{1-(p/q)^2}\left\{\sin\left(\frac{\pi p}{q}\right)\cos\Phi' - \frac{p}{q}\cos\left(\frac{\pi p}{q}\right)\sin\Phi'\right\} \tag{11}$$

for $p=q$, $p=nq$ $(n=2,3,4,\ldots)$, and $p\neq nq$ $(n=1,2,3,\ldots)$, respectively. As in the problem of chaos suppression one is interested first in the situation where the amplitude of the inhibitory excitation is much smaller than the amplitude of the main driving excitation, the optimal values of Φ must verify $\sin(\Phi'-\Phi)=\pm 1$ [cf. Eq. (8)], i.e.,

$$\Phi=\Phi'\mp\pi/2, \tag{12}$$

respectively. We have then instead of Eqs. (8)–(11)

$$\alpha\simeq\pm\frac{dA}{F_c}\left(1-\frac{A^2}{4}\right)+\frac{1}{\pi}R(p,q,\Phi), \tag{13}$$

$$R(p,q,\Phi)=-\pi\cos\Phi, \tag{14}$$

$$R(p,q,\Phi)=0, \tag{15}$$

$$R(p,q,\Phi)=\frac{2\sin(\pi p/q)}{1-(p/q)^2}\left\{\sin\left(\frac{\pi p}{q}\right)\sin\Phi + \frac{p}{q}\cos\left(\frac{\pi p}{q}\right)\cos\Phi\right\} \tag{16}$$

for $p=q$, $p=nq$ $(n=2,3,4,\ldots)$, and $p\neq nq$ $(n=1,2,3,\ldots)$, respectively.

We now make the following remarks. First, the phase difference Φ between the two harmonic forces plays a fundamental role in regularizing the dynamics as shown theoretically in Ref. [18(b)–18(d)]. Observe, however, that for the subharmonic case $p=nq$ $(n=2,3,4,\ldots)$ the predicted amplitude α does not depend on Φ' [i.e., or on Φ, from condition (12)]. This is a consequence of the GR solution (4) being formed by only one harmonic [just the main one: $T=2\pi$; then the corresponding integral in Eq. (6) cancels out]. Second, after substituting α from Eqs. (13)–(16) into $F_c[\cos(\omega t)+\alpha\cos(\Omega t+\Phi)]$ and taking into consideration Eq. (12), one straightforwardly obtains for the whole forcing

$$dA\left(1-\frac{A^2}{4}\right)\sin(t+\Phi')+F_c\cos(t+\Phi')\cos\Phi', \tag{17}$$

$$dA\left(1-\frac{A^2}{4}\right)\sin(t+\Phi')+F_c\cos\left(\frac{pt}{q}\right), \tag{18}$$

$$dA\left(1-\frac{A^2}{4}\right)\sin(t+\Phi')+F_c\cos\left(\frac{pt}{q}\right)+\left(\frac{2F_c}{\pi}\right)\frac{\sin(\pi p/q)}{1-(p/q)^2}\left\{\left(\frac{p}{q}\right)\cos\left(\frac{\pi p}{q}\right)\sin\Phi' - \sin\left(\frac{\pi p}{q}\right)\cos\Phi'\right\}\sin(t+\Phi') \tag{19}$$

for $p=q$, $p=nq$ $(n=2,3,4,\ldots)$, and $p\neq nq$ $(n=1,2,3,\ldots)$, respectively. The first of the two terms forming $F_{GR}(t)$ thus appears in all the approximations to the GR forcing. Moreover, for $p=3q$ and the initial condition $x(0)=0$, $\dot{x}(0)=-(4F_c/d)^{1/3}$, obtained for $A=(4F_c/d)^{1/3}$, $\Phi'=\pi/2$ [cf. Eq. (4)], the bichromatic modulation (18) *exactly* coincides with $F_{GR}(t)$, i.e., for

$$\alpha\simeq\pm 4^{1/3}\left[\left(\frac{F_c}{d}\right)^{-2/3}-2^{-2/3}\right],\quad \Phi=0,\pi, \tag{20}$$

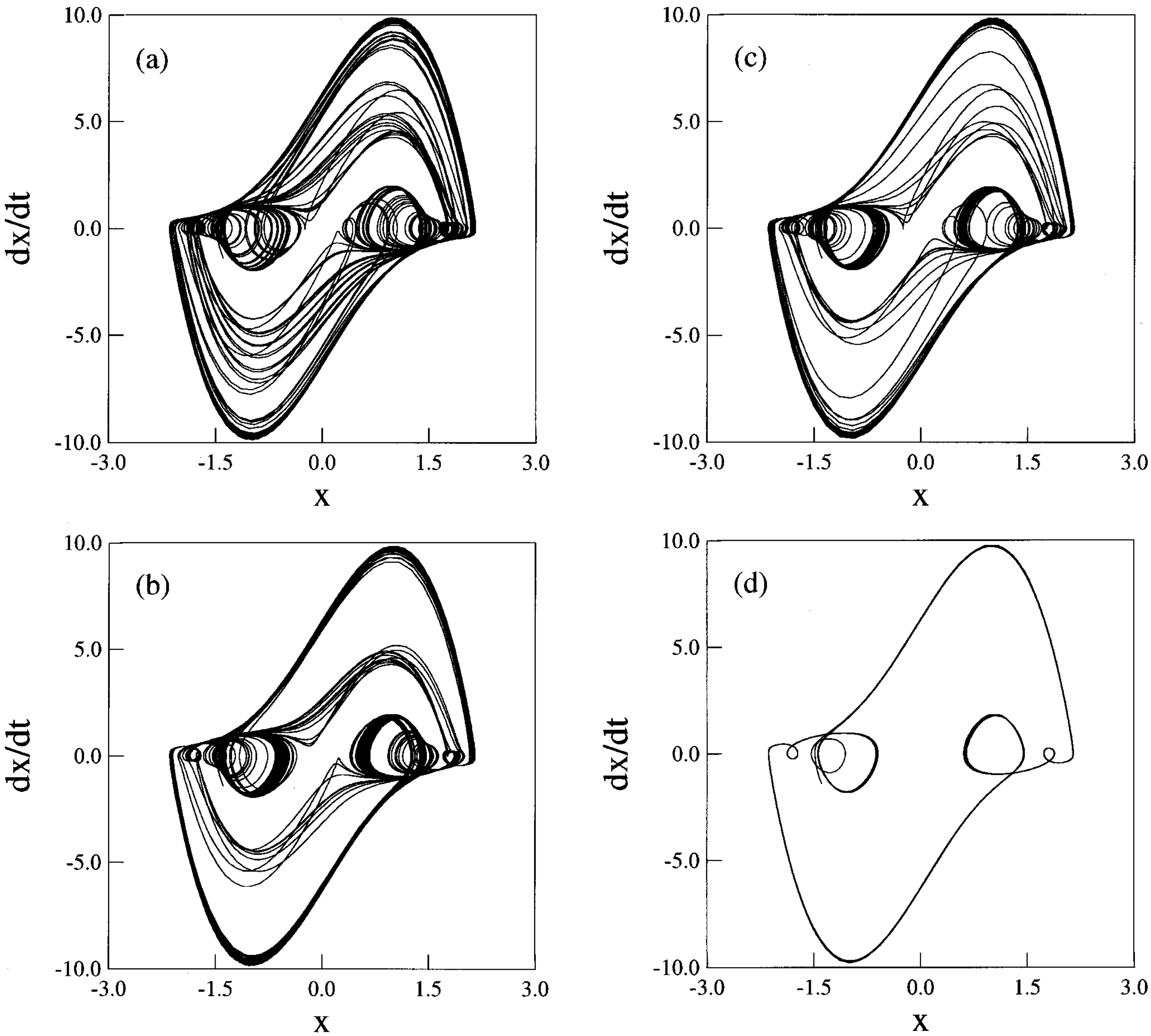

FIG. 4. Phase-space portraits for the parameters $d=5$, $F_c=5$, $\omega=2.463$, and $\alpha=0.08$: (a) $\Phi=2.4$, (b) $\Phi=2.53$, (c) $\Phi=2.63$, (d) $\Phi=2.71$, and (e) $\Phi=2.7625$ [see Fig. 1(d)].

respectively [cf. Eqs. (12), (13), and (15)]. For the particular case $F_c=2d$, Eq. (20) yields $\alpha=0$ and also $A=2$; in other words, the monochromatically ($\omega=3$) forced van der Pol oscillator has, for the initial condition $x(0)=0$, $\dot{x}(0)=-2$, an exact limit cycle given by $x_{\rm GR}^2+\dot{x}_{\rm GR}^2=4$. Thus this agrees with the aforementioned (cf. Sec. II) result concerning the autonomous van der Pol oscillator, namely, that for *small* d it has a stable limit cycle *close* to the solution $x^2+\dot{x}^2=4$. Indeed, because of the relationship $F_c=2d$, $F_c\to 0$ is equivalent to $d\to 0$. Another point concerning this particular case refers to a classical result due to Cartwright and Littlewood (CL) [2,21]: if $F_c/d>2\omega/3$ and $d>d_0(F_c,\omega)$, Eq. (1) with $\alpha=0$ has a stable periodic solution of period $2\pi/\omega$ to which all trajectories converge as $t\to+\infty$ (a *globally* stable limit cycle). Note that the result is stated without restrictions as to how large d can be, i.e., it is only required to surpass a lower threshold. However, in the above discussion for the case $\omega=3$, we deduced $d=d_{\rm GR}\equiv F_c/2$ so that the condition $F_c/d>2\omega/3$ is now written $d_{\rm GR}>d$. In other words, the result of CL requires also, for the specific case $\omega=3$, an upper threshold for d, derived from a GR analysis of the problem. Similarly, the case $d>d_{\rm GR}$, d large enough, corresponds to the situation for which CL noted the possibility of "strange" behavior [2,3,21].

Let us now consider the incommensurate case, i.e., where ω is now an irrational number. Similarly to the commensurate case, from Eqs. (6) and (12) one straightforwardly obtains

$$\alpha\simeq\pm\frac{dA}{F_c}\left(1-\frac{A^2}{4}\right)+\frac{1}{\pi}\,S(\omega,\Phi), \tag{21}$$

$$S(\omega,\Phi)=\frac{2\sin(\pi\omega)}{1-\omega^2}\,\{\sin(\pi\omega)\sin\Phi+\omega\,\cos(\pi\omega)\cos\Phi\}. \tag{22}$$

We remark that taking into account the well-known theorem that for any irrational number ω there exist arbitrarily accu-

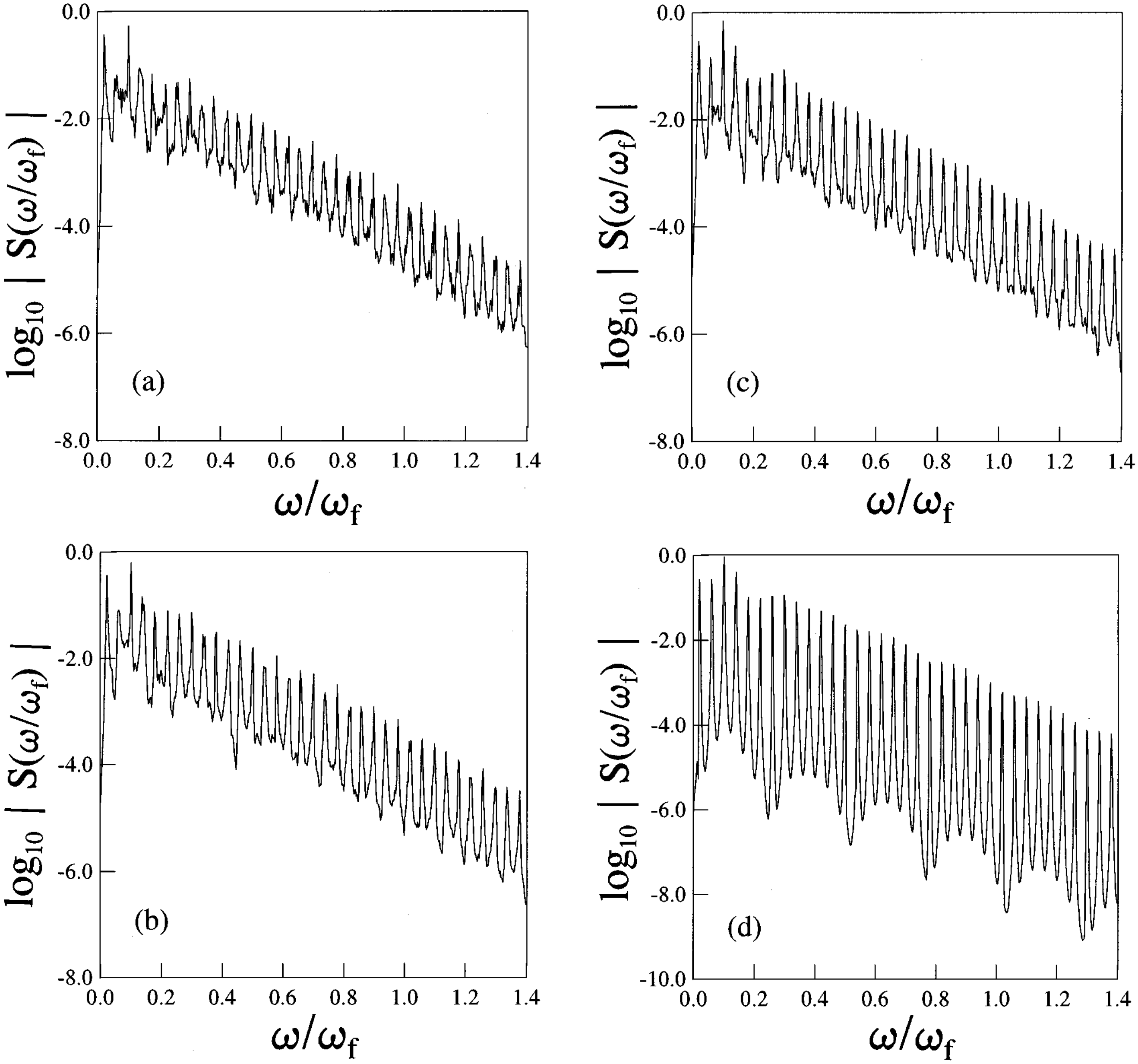

FIG. 5. Power spectra associated with the homonymous cases in Fig. 4.

rate rational approximations p/q such that $|\omega - p/q| < 1/q^2$ [22], one finds, for a given irrational frequency ω, that the functions $S(\Phi)$ and $R(\Phi)$ will be, in general, very close whenever $R(\Phi)$ corresponds to the best rational approximation p/q to ω [cf. Eq. (16)]. Consequently, the associated α values will also be very similar. The discussion of this result in the context of the routes to chaos in quasiperiodically forced systems [23] will be considered elsewhere.

IV. NUMERICAL RESULTS

We performed some numerical experiments on the driven van der Pol oscillator (1). A systematic numerical survey of its parameter space was beyond the scope of the present work. Therefore, we chose arbitrary sets of parameters in order to see the scope and accuracy of the predictions from the almost-adiabatic invariant approach. In particular, with a fixed set of parameters (d, F_c, ω) and given initial conditions for which the subsequent motion (for $\alpha=0$) is chaotic, we then let the inhibitory forcing act on the system and study the resulting orbits as the forcing parameters α, Φ are varied ($\Omega=1$). Generally, the computer simulations of the driven van der Pol oscillator (1) showed overall good agreement with the theoretical predictions, even when the damping and chaos-inducing amplitudes were clearly not small. It is worth mentioning that one cannot expect exact quantitative agreement between the two types of results due to the adiabatic character of the action conservation (see Ref. [13] for more details), which is the physical foundation of the regularization. Figure 1 shows an illustrative sequence of regularization from an initial chaotic state at $\alpha=0$ [Fig. 1(a)]. The Poincaré cross section of this chaotic attractor consists of four very thin islands, as was previously reported in Ref. [9]. As α increases the chaotic response steadily weakens [Figs. 1(b) and 1(c)] and a symmetry restoring occurs prior to complete regularization [Fig. 1(d)]. Thus, while the initial chaotic

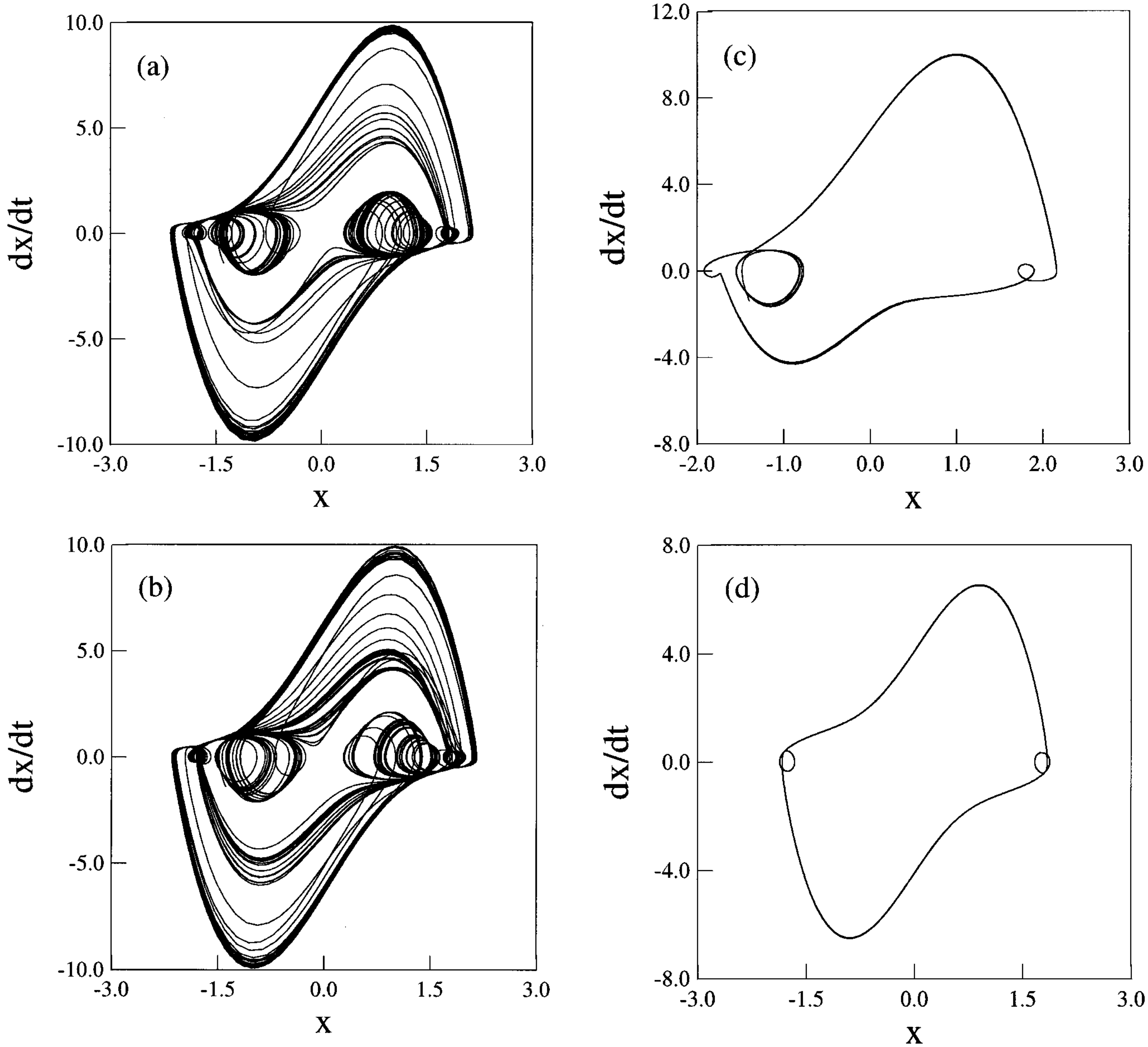

FIG. 6. Phase-space portraits for the same parameters as in Fig. 4: (a) $\Phi=3.84$, (b) $\Phi=4.47$, (c) $\Phi=5.1$, and (d) $\Phi=5.73$.

orbit arose for a fairly *large* amplitude of the chaos-inducing forcing ($F_c=5$), it was, however, eliminated by using a *relatively small* amplitude ($\alpha F_c=0.4$) of the chaos-suppressing forcing. The power spectra corresponding to the $\dot{x}$ series of the respective cases of Fig. 1 are presented in Fig. 2. We have plotted $\log_{10}|S(\omega/\omega_f)|$ versus ω/ω_f with ω_f denoting the chaos-inducing forcing frequency (dimensionless variable). Notice the gradual rise of the peak associated with the frequency $\omega=\omega_f/10$ as α increases. Figure 3 shows a plot of $|\alpha|$ versus Φ [cf. Eqs. (13) and (16)] for the case $(d,F_c,p,q)=(5,5,2463,1000)$, i.e., $\omega=2.463$, and the same initial conditions for which the motion is chaotic at $\alpha=0$ [cf. Fig. 1(a)]. Asterisks and circles represent numerical results (chaotic behavior and periodic motion, respectively) for the value $\alpha=0.08$ and the same remaining parameters, including the initial conditions. Regularization is only possible within a few ranges of phase differences that correspond to those points ($\Phi,\alpha=0.08$) that are *sufficiently* close to the theoretical curve in the $|\alpha|$-Φ plane. In particular, *all* the points ($\Phi,\alpha=0.08$) inside the theoretical tongues represent periodic motions [see Fig. 3(b)]. The numerically obtained Φ ranges for regularization are wider than those expected from the adiabatic invariance of the action due to the perturbative nature of the theoretical approach (cf. the discussion at the end of Sec. II). Figure 4 shows a regularization sequence that crosses the theoretical curve of Fig. 3, for $\alpha=0.08$ and increasing values of Φ corresponding to four of the cases depicted by asterisks or circles in Fig. 3. The respective power spectra are shown in Fig. 5. Notice the *slow* character of the regularization seen in the gradual decrease of the almost continuous background of the spectrum. Figure 6 shows another similar sequence for several values of Φ and the same remaining parameters as in Fig. 4. In this case, Φ varies in an unfavorable range for stabilization [see Fig. 3(b)]. In fact, for $\Phi=4.47$ [Fig. 6(b)], the most unfavorable value for regularization [see Fig. 3(a)], one sees that the system response is

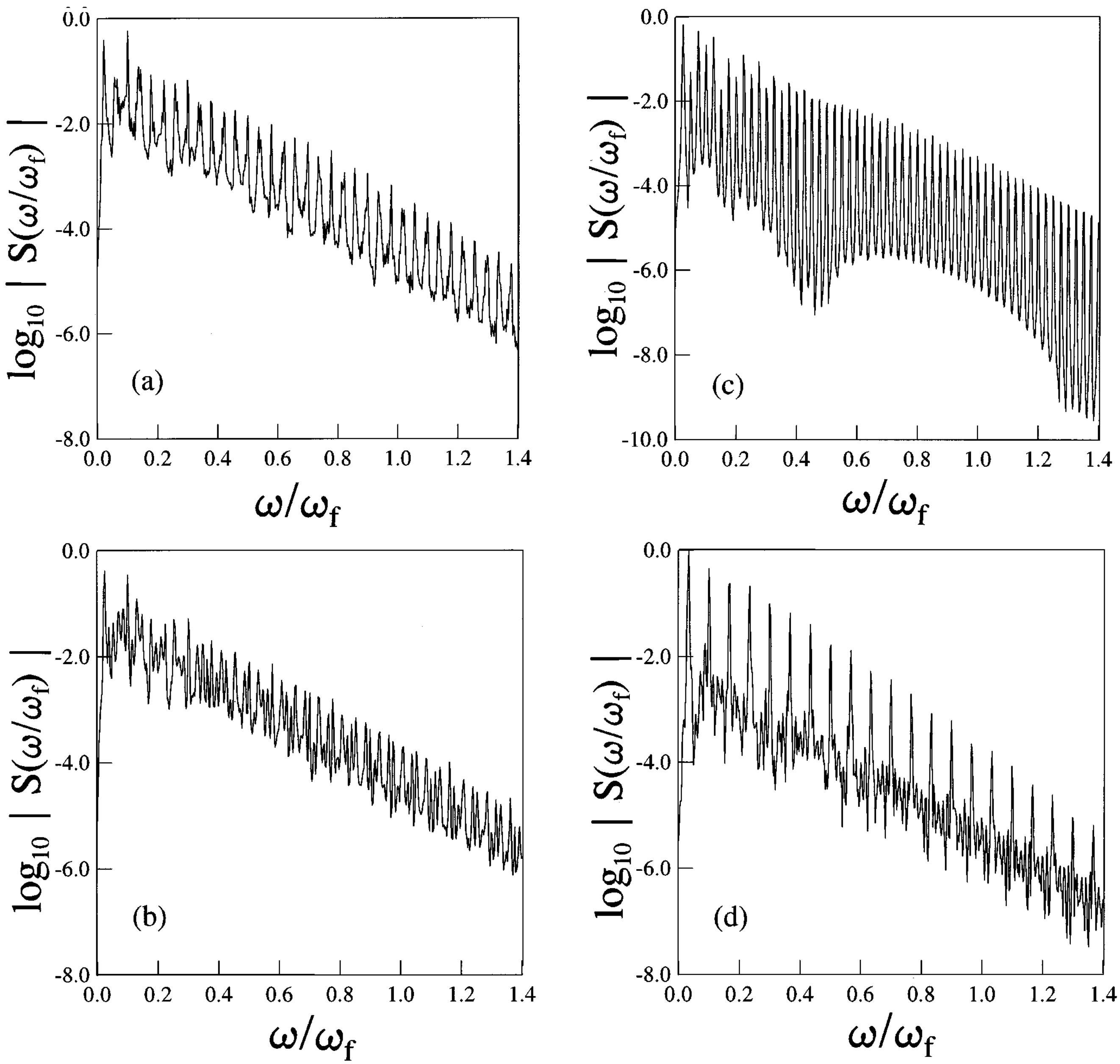

FIG. 7. Power spectra associated with the homonymous cases in Fig. 6.

the most clearly chaotic, as is also evident from the corresponding power spectrum [Fig. 7(b)]. A symmetry breaking followed by a symmetry restoring occurs for successive periodic responses as the point ($\Phi,\alpha=0.08$) approaches the theoretical curve in the $|\alpha|$-Φ plane. Therefore, it seems that a *symmetry restoring* is a hallmark of the stabilization of the dynamics under the adiabatic conservation of the action, as expected from the definition of GR.

V. SUMMARY

We have studied the inhibition of chaos in the driven van der Pol system due to a small-amplitude added forcing. Analytical estimates of the ranges of parameters for stabilization were deduced from the almost-adiabatic conservation of the action associated with each geometrical resonance solution. Computer simulations of the bichromatically driven system showed overall good agreement with the theoretical predictions, even when the dissipation term and the chaos-inducing forcing had large amplitudes. A reinterpretation of a well-known result on the monochromatically driven van der Pol oscillator, due to CL, was also derived by using geometrical resonance analysis.

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