Critical dynamics of nonperiodic Ising chains

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The critical dynamics of the nonperiodic ferromagnetic Ising chains with two different coupling constants ($J_1>J_2>0$) arranged in nonperiodic sequences are studied by trace map method. For Glauber dynamics, it is found that the dynamical critical exponent $z=1+J_1/J_2$ for the Fibonacci, general Fibonacci (e.g., silver-mean, copper-mean), and period-doubling ferromagnetic Ising chains. The applicability of the trace map method and the origin of the nonuniversality are briefly discussed. [S1063-651X(97)05007-1]

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I. INTRODUCTION

In the last two decades, there has been much interest in studying the universality of the dynamical critical exponent z for the one-dimensional Ising model within Glauber dynamics [1–15]. It is well known that $z=2$ for the infinite uniform ferromagnetic chain [2] and $z=1$ for the finite ferromagnetic chain with open boundary conditions [3]. Moreover, the exponent of the infinite ferromagnetic chain can vary with a transition rate parameter in the master equation [4]. Droz *et al.* [5] found that the alternating-bond ferromagnetic Ising chain does not belong to the universality class of the uniform ferromagnetic chain but has a nonuniversal value $z=1+J_1/J_2$, where J_1 and J_2 are the alternating coupling constants with $J_1>J_2>0$. Recently, Achiam and Southern [9,10] pointed out that this nonuniversality is a result of two different contributions. One is due to long ranged fluctuations near the critical point and the other is due to short ranged phenomena. For more inhomogeneous ferromagnetic chains, such as a random two coupling constant ferromagnetic chain, it is also known that $z=1+J_1/J_2$ by the movement of domain-wall argument [5,11,12]. Furthermore, it is found that $z=1+J_M/J_m$ for a periodic Ising chain with a basic unit cell $\{J_1,\ldots,J_n\}$ containing an arbitrary set of n ferromagnetic coupling constants, where J_M and J_m are the maximum and minimum of $\{J_1,\ldots,J_n\}$, respectively [11,13].

On the other hand, there are many kinds of nonperiodic one-dimensional systems [16–18]. These systems lack the translational invariance but are self-similar by construction, and can be loosely regarded as the intermediate between the periodic and random systems. Therefore it is natural to examine if $z=1+J_1/J_2$ also holds for two coupling constant nonperiodic Ising chains. Recently, some authors [14,15] have studied the dynamics of the two coupling constant Fibonacci Ising chain and obtained $z=1+J_1/J_2$ by renormalization-group method.

It has been shown that the trace map method is one of the most effective techniques for studying deterministic nonperiodic systems [19,20]. In this paper we use this method to study the dynamics of two coupling constant nonperiodic Ising chains. The paper is organized as follows. In Sec. II we present the trace map method and illustrate its use by reproducing the known results for the uniform and alternating-bond ferromagnetic Ising chains. We study the dynamics of general Fibonacci and period-doubling Ising chains in Secs. III and IV. Some conclusions and discussions are given in Sec. V.

II. TRACE MAP METHOD

The one-dimensional (1D) nonperiodic ferromagnetic Ising model is defined by the following Hamiltonian [15,21]:

$$\beta H=-\sum_i K_i S_{i-1}S_i, \tag{1}$$

where $\beta=1/k_BT$, $S_i=\pm1$, and $\{K_i=J_i/k_BT\}$ are the reduced ferromagnetic couplings. The coupling constants J_i between the nearest-neighbor spins take two values J_1 and J_2 ($J_1>J_2>0$) arranged in a nonperiodic sequence. The static properties of this model can be easily obtained [21]. The critical temperature is $T_c=0$. The correlation length near T_c depends only on the weak interaction J_2, i.e.,

$$\xi\sim\exp[2K_2]. \tag{2}$$

The time evolution of the system is described by a Markov process with Glauber dynamics. Therefore the probability $P(S_1,S_2,\ldots,S_N;t)$ of finding the system in the configuration $\{S_i\}$ at time t obeys the master equation

$$\frac{d}{dt}P(S_1,\ldots,S_N;t)=-\sum_{i=1}^N \omega_i(S_i)P(S_1,\ldots,S_N;t)+\sum_{i=1}^N \omega_i(-S_i)\times P(S_1,\ldots,-S_i,\ldots,S_N;t). \tag{3}$$

Here $\omega_i(S_i)$ is the transition probability per unit time that the ith spin flips from the value S_i to $-S_i$ while all others are unaffected. These transition rates satisfy the detailed balance condition. In the present situation, they take the Glauber form

$$\omega_i(S_i)=\frac{1}{2}\Gamma(1-S_iC_i^+S_{i-1}-S_iC_i^-S_{i+1}), \tag{4}$$

with

$$C_i^{\pm}=\frac{1}{2}[\tanh(K_i+K_{i+1})\pm\tanh(K_i-K_{i+1})] \tag{5}$$

and Γ a positive constant defining the time scale of the evolution of the system. In this paper we take $\Gamma=1$ without loss of generality.

The equation of motion for local magnetization

$$q_i(t)=\langle S_i\rangle_t=\mathrm{Tr}_{\{S\}}[S_iP(S_1,\ldots,S_N;t)] \tag{6}$$

is

$$\frac{d}{dt}q_i(t)=-2\langle S_i\omega_i(S_i)\rangle_t$$
$$=-q_i(t)+C_i^{+}q_{i-1}(t)+C_i^{-}q_{i+1}(t). \tag{7}$$

The Laplace transform of Eq. (7) yields [14,15]

$$-\lambda Q_i=-Q_i+C_i^{+}Q_{i-1}+C_i^{-}Q_{i+1}, \tag{8}$$

where $-\lambda$ is the variable conjugate to t, and $Q_i(-\lambda)$ is the Laplace transform of $q_i(t)$. The relaxation of $\{q_i(t)\}$ towards the equilibrium is governed by a set of relaxation times $\{\tau_n=1/\lambda_n\}$, where $\{\lambda_n\}$ are the solutions of Eq. (8). The asymptotic long-time behavior of $\{q_i(t)\}$ is given by the smallest value λ_s of $\{\lambda_n\}$ [13,15].

Equation (8) can be written in transfer matrix form

$$\begin{pmatrix} Q_{i+1}\\ Q_i\end{pmatrix}=\begin{pmatrix}\frac{1-\lambda}{C_i^{-}} & -\frac{C_i^{+}}{C_i^{-}}\\ 1 & 0\end{pmatrix}\begin{pmatrix} Q_i\\ Q_{i-1}\end{pmatrix}=\mathbf{M}_i\begin{pmatrix} Q_i\\ Q_{i-1}\end{pmatrix}. \tag{9}$$

Then

$$\begin{pmatrix} Q_{N+1}\\ Q_N\end{pmatrix}=\mathbf{M}^{(N)}\begin{pmatrix} Q_1\\ Q_0\end{pmatrix}, \tag{10}$$

where $\mathbf{M}^{(N)}=\mathbf{M}_N\cdots\mathbf{M}_1$. For a finite N spin chain, the eigenvalues $r_{1,2}$ of $\mathbf{M}^{(N)}$ are

$$r_{1,2}=\frac{1}{2}\{\mathrm{Tr}\mathbf{M}^{(N)}\pm[(\mathrm{Tr}\mathbf{M}^{(N)})^2-4\det\mathbf{M}^{(N)}]^{1/2}\}.$$

It is commonly required that the Q_N of a periodic system with a period of N should not diverge, thus λ satisfies

$$\frac{1}{2}|\mathrm{Tr}\mathbf{M}^{(N)}|\leqslant(\det\mathbf{M}^{(N)})^{1/2}, \tag{11}$$

which for unimodular $\mathbf{M}^{(N)}$ (i.e., $\det\mathbf{M}^{(N)}=1$) becomes

$$\frac{1}{2}|\mathrm{Tr}\mathbf{M}^{(N)}|\leqslant 1. \tag{12}$$

To illustrate the use of the trace map method, let us calculate the dynamical critical exponents for the uniform and alternating-bond ferromagnetic Ising chains.

(1) Uniform ferromagnetic Ising chain: For the uniform ferromagnetic Ising chain, $J_1=J_2=J>0$ and $C_i^{+}=C_i^{-}=C=\frac{1}{2}\tanh(2K)$. And $\mathbf{M}^{(N)}=\mathbf{M}_0^N$ with

$$\mathbf{M}_0=\begin{pmatrix}\frac{1-\lambda}{C} & -1\\ 1 & 0\end{pmatrix}, \tag{13}$$

a unimodular matrix. From the well-known result for the powers of 2×2 unimodular matrix

$$\mathbf{M}_0^m=U_{m-1}(x_0)\mathbf{M}_0-U_{m-2}(x_0)\mathbf{I} \tag{14}$$

for $m\geqslant 1$ with $x_0=\frac{1}{2}\mathrm{Tr}\mathbf{M}_0$, we obtain the trace of $\mathbf{M}^{(N)}$ as a function of x_0:

$$x_N(x_0)\equiv\frac{1}{2}\mathrm{Tr}\mathbf{M}^{(N)}=x_0U_{N-1}(x_0)-U_{N-2}(x_0). \tag{15}$$

Here $U_m(x)$ is the mth Chebyshev polynomial of the second kind. It satisfies the recursion relation

$$U_m(x)=2xU_{m-1}(x)-U_{m-2}(x)\quad \text{for } m\geqslant 1, \tag{16}$$

with $U_{-1}(x)=0$ and $U_0(x)=1$.

Because $x_0=(1-\lambda)/2C$, λ_s corresponds to the largest x_0 satisfying the condition $|x_N(x_0)|\leqslant 1$. From Eqs. (15) and (16), we can see that $x_N(x_0)$ is an N order polynomial of x_0 and the coefficient of x_0^N is positive. Hence the largest x_0 satisfying the condition (12) is the largest root of the equation $x_N(x_0)=1$. It can be easily checked, by noting the equation $U_N(1)=N+1$, that $x_0=1$ is the root of the equation $x_N(x_0)=1$.

We can further prove that $x_0=1$ is actually the largest root. First we note that $x_N(x_0)-x_{N-1}(x_0)=(x_0-1)\{U_{N-1}(x_0)+U_{N-2}(x_0)\}$. If $U_N(x_0)>0$ when $x_0>1$ for every N, then $x_N>x_{N-1}>\cdots>x_1=x_0>1$. This implies that there is no root for the equation $x_N(x_0)=1$ when $x_0>1$, thus $x_0=1$ is the largest root. By mathematical induction it can be proved that indeed that is the case. For $N=1$, $U_1(x)=2x>U_0(x)=1>0$ for $x>1$. Now assume that $U_{N-1}(x)>U_{N-2}(x)>0$ for $x>1$, then $U_N(x)=2xU_{N-1}(x)-U_{N-2}(x)>2U_{N-1}(x)-U_{N-2}(x)>U_{N-1}(x)>0$. That is, $U_N(x)>U_{N-1}(x)>\cdots>U_0(x)>0$ for $x>1$. Therefore $U_N(x)>0$ when $x>1$ for every N.

Thus λ_s is the root of the equation $x_0=1$, i.e., $1-\lambda_s=2C$ or $\lambda_s=1-2C=1-\tanh(2K)\to\exp[-4K]$ as $K=J/k_BT\to\infty$. According to the defintion for the dynamical critical exponent z,

$$\tau_s=1/\lambda_s\sim\xi^z, \tag{17}$$

we obtain $z=2$, which is the well-known result [2].

(2) Alternating-bond ferromagnetic Ising chain: We consider an alternating-bond ferromagnetic Ising chain with $J_{2i-1}=J_1>0$ and $J_{2i}=J_2>0$. Then $\mathbf{M}^{(N)}=\mathbf{M}_1^{N/2}$ with

$$\mathbf{M}_1=\begin{pmatrix}\frac{1-\lambda}{C_1} & -\frac{C_2}{C_1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1-\lambda}{C_2} & -\frac{C_1}{C_2}\\ 1 & 0\end{pmatrix}$$
$$=\begin{pmatrix}\frac{(1-\lambda)^2}{C_1C_2}-\frac{C_2}{C_1} & -\frac{1-\lambda}{C_2}\\ \frac{1-\lambda}{C_2} & -\frac{C_1}{C_2}\end{pmatrix}. \tag{18}$$

Here

$$C_{1,2}=\frac{1}{2}[\tanh(K_1+K_2)\pm\tanh(K_1-K_2)] \tag{19}$$

and $\mathbf{M}_1$ is also a unimodular matrix. Similarly, λ_s is the smallest root of the equation $\frac{1}{2}\mathrm{Tr}\mathbf{M}_1=1$, or

$$\frac{(1-\lambda)^2}{C_1C_2}-\frac{C_1}{C_2}-\frac{C_2}{C_1}=2. \tag{20}$$

We obtain $\lambda_s=1-C_1-C_2=1-\tanh(K_1+K_2)\rightarrow\exp[-2(K_1+K_2)]$ as $T\rightarrow 0$. From Eqs. (2) and (17), we obtain the dynamical critical exponent $z=1+J_1/J_2$, which is the same as the result obtained by Droz *et al.* [5].

III. QUASIPERIODIC CHAIN

For the quasiperiodic ferromagnetic Ising chain, the coupling constants J_i of the nearest neighbor take two values J_1 and J_2 $(J_1>J_2>0)$ arranged in a general Fibonacci sequence. The general Fibonacci sequence S_∞ is constructed recursively as $S_{l+1}=\{S_l^m S_{l-1}^n\}$ for $l\geqslant 1$, with $S_0=\{J_2\}$ and $S_1=\{J_1\}$, where S_l^m denotes the string of m S_l. Here, m and n are integers. Alternatively, the general Fibonacci sequence can be generated from a seed (e.g., J_1), by following the substitution rule: $J_1\rightarrow J_1^m J_2^n$, $J_2\rightarrow J_1$, where J_1^m represents a string of m J_1's. Due to the construction rule for S_l, the total number of symbols in the sequence S_l follows the recursion relation $F_{l+1}=mF_l+nF_{l-1}$ for $l\geqslant 1$, with $F_0=F_1=1$. In the limit $l\rightarrow\infty$, F_{l-1}/F_l approaches the value $\sigma=(1/2n)[(m^2+4n)^{1/2}-m]$, which is the positive root of the equation $n\sigma^2+m\sigma-1=0$.

(1) Fibonacci ferromagnetic Ising chain: The Fibonacci ferromagnetic Ising chain is a quasiperiodic Ising chain with $m=n=1$, for which $\sigma=\frac{1}{2}(\sqrt{5}-1)$. The substitution rule gives the following sequence of J_1 and J_2:

$$J_1J_2J_1J_1J_2J_1J_2J_1J_1J_2J_1\cdots. \tag{21}$$

The sequence C_i^+ is

$$C_1C_2C_3C_1C_2C_1C_2C_3C_1C_2\cdots, \tag{22}$$

where the $C_{1,2}$ are given by Eq. (19) and $C_3=\frac{1}{2}\tanh(2K_1)$. This sequence can be obtained from sequence (21) by the substitution rule: $J_1\rightarrow C_1C_2$, $J_2\rightarrow C_3$. Similarly, the sequence C_i^- is obtained from sequence (21) by the constitution rule: $J_1\rightarrow C_2C_1$, $J_2\rightarrow C_3$, yielding

$$C_2C_1C_3C_2C_1C_2C_1C_3C_2C_1\cdots. \tag{23}$$

For the lth Fibonacci sequence, there are F_l spins in the chain. Under the periodic boundary conditions, the transfer matrix $\mathbf{M}_l\equiv\mathbf{M}^{(F_l)}$ follows the recursion relation

$$\mathbf{M}_l=\mathbf{M}_{l-2}\mathbf{M}_{l-1}, \tag{24}$$

with

$$\mathbf{M}_0=\begin{pmatrix}\frac{1-\lambda}{C_3} & -1\\ 1 & 0\end{pmatrix} \tag{25}$$

and $\mathbf{M}_1$ given by Eq. (18).

The matrices $\mathbf{M}_0$ and $\mathbf{M}_1$ are both unimodular. According to the recursion relation (24), we can see that $\det\mathbf{M}_l=1$ for all $l\geqslant 0$. From Eq. (24) and the unimodularity of $\mathbf{M}_l$, we obtain the following trace map for $x_l=\frac{1}{2}\mathrm{Tr}\mathbf{M}_l$ [19]:

$$x_l=2x_{l-1}x_{l-2}-x_{l-3}\quad\text{for}\quad l\geqslant 3, \tag{26}$$

with initial conditions

$$x_0=\frac{1-\lambda}{2C_3},\quad x_1=\frac{(1-\lambda)^2}{2C_1C_2}-\frac{C_1}{2C_2}-\frac{C_2}{2C_1},$$

and

$$x_2=\frac{(1-\lambda)^3}{2C_1C_2C_3}-\frac{(1-\lambda)C_2}{2C_1C_3}-\frac{1-\lambda}{C_2}.$$

For the Fibonacci ferromagnetic Ising chain, the dynamical critical exponent is determined by the smallest λ_s satisfying the condition $|x|\leqslant 1$. From Eq. (26) one can see that x_l is an F_l order polynomial of λ. That is,

$$x_l=\sum_{n=0}^{F_l}A_n^{(l)}\lambda^n. \tag{27}$$

Under the periodic boundary conditions there are F_l bands satisfying the condition $|x_l|\leqslant 1$ for the lth Fibonacci Ising chain. The $2F_l$ edges of F_l bands are determined by the equations $x_l=1$ and $x_l=-1$. We can also see that the coefficient of λ^{F_l} of x_l is positive for F_l even and negative for F_l odd, x_l goes to positive infinite as $\lambda\rightarrow-\infty$ for every l and decreases as λ increases. Therefore λ_s is the smallest root of the equation $x_l=1$. At very low temperature there is no need for the full solution of the equation $x_l=1$. This is because the largest relaxation time $\tau_M=1/\lambda_s$ dominates the $\Sigma\tau_n$ at low T [13], where $\tau_n=1/\lambda_n$ and λ_n $(n=1,\ldots,F_l)$ are the F_l roots of the equation $x_l=1$. Therefore, at sufficiently low temperature, the relevant relaxation time is given by

$$\tau_M \approx \sum_{n=1}^{F_l} \tau_n = \sum_{n=1}^{F_l} \frac{1}{\lambda_n} = \frac{A_1^{(l)}}{A_0^{(l)}-1} \equiv \frac{h_l}{g_l-1}, \tag{28}$$

where $g_l \equiv A_0^{(l)}$ and $h_l \equiv A_1^{(l)}$. The dynamical critical exponent can be determined by the equation

$$z = \lim_{l\to\infty} \frac{d\ln(\tau_M)}{d(2K_2)}. \tag{29}$$

g_l and h_l satisfy the recursion relations

$$g_l = 2g_{l-1}g_{l-2} - g_{l-3},$$
$$h_l = 2g_{l-1}h_{l-2} + 2g_{l-2}h_{l-1} - h_{l-3}, \tag{30}$$

with initial conditions

$$g_0 = \frac{1}{2C_3}, \quad g_1 = \frac{1-C_1^2-C_2^2}{2C_1C_2}, \quad g_2 = \frac{1-C_2^2-2C_1C_3}{2C_1C_2C_3}, \tag{31}$$

and

$$h_0 = -\frac{1}{2C_3}, \quad h_1 = -\frac{1}{C_1C_2}, \quad h_2 = \frac{-3+C_2^2+2C_1C_3}{2C_1C_2C_3}. \tag{32}$$

By using Eqs. (28)–(30) and initial conditions (31) and (32), we can obtain numerically the dynamical critical exponent z for large l and small T. In Fig. 1 we show the value of z for the finite Fibonacci ferromagnetic Ising chain with $F_{11}=144$ spins as functions of $K_2 = J_2/k_BT$ (full lines) at $J_1/J_2 = 1.2$, 1.5, 2.0, and 2.5. From Fig. 1 one can see that $z = 1 + J_1/J_2$ as $K_2 \to \infty$ $(T\to 0)$, which agrees with the result obtained by renormalization-group method [14,15].

We can also study the dynamics of the Fibonacci ferromagnetic Ising chain with $J_1 < J_2$. This chain is equivalent to the chain (21) but with J_1 and J_2 interchanged. We call this chain a dual chain of chain (21). We can obtain the dynamical critical exponent by using Eqs. (28)–(32) with C_1, C_2, and C_3 substituted by C_2, C_1, and $\frac{1}{2}\tanh(2K_2)$. In Fig. 1 we also plot the similar $z \sim K_2$ curves (dashed lines) for $J_1/J_2 = 1.2$, 1.5, 2.0, and 2.5. Again one finds that $z = 1 + J_1/J_2$ as $T \to 0$.

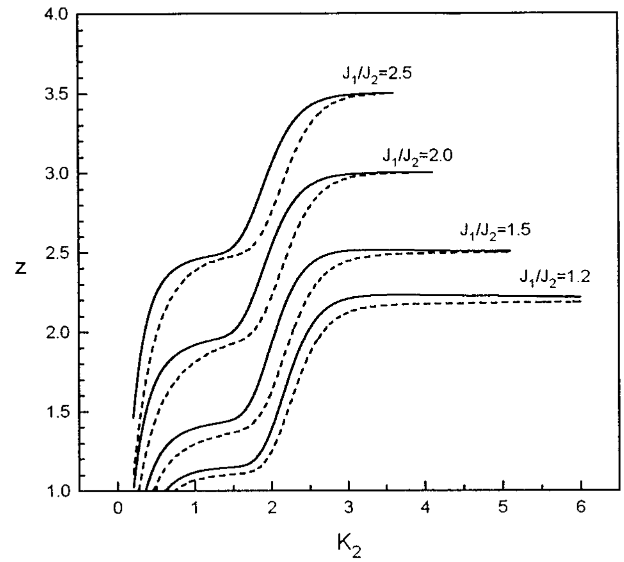

FIG. 1. The dynamical critical exponent of the Fibonacci ferromagnetic Ising chain as functions of $K_2 = J_2/k_BT$ at $J_1/J_2 = 1.2$, 1.5, 2.0, and 2.5 for the chain (21) (full lines) and its dual chain (dashed lines). The curves are obtained for the $F_{11} = 144$ spin chain under periodic boundary conditions. We can see that $z = 1 + J_1/J_2$ as $K_2 \to \infty$.

(2) Silver-mean ferromagnetic Ising chain: The silver-mean ferromagnetic Ising chain is a general quasiperiodic Ising chain with $m=2$ and $n=1$, for which $\sigma = \sqrt{2}-1$. The substitution rule gives the following sequence of J_1 and J_2:

$$J_1J_1J_2J_1J_1J_2J_1J_1J_1J_2J_1J_1J_2\cdots. \tag{33}$$

Then the sequence C_i^+ is

$$C_3C_1C_2C_3C_1C_2C_3C_3C_1C_2C_3C_1C_2\cdots. \tag{34}$$

This sequence can be obtained from sequence (33) by the substitution rule: $J_1 \to C_3C_1C_2$, $J_2 \to C_3$. Similarly, the sequence C_i^- is obtained from sequence (33) by the constitution rule: $J_1 \to C_3C_2C_1$, $J_2 \to C_3$,

$$C_3C_2C_1C_3C_2C_1C_3C_3C_2C_1C_3C_2C_1\cdots. \tag{35}$$

Under the periodic boundary conditions, the transfer matrix $\mathbf{M}_l \equiv \mathbf{M}^{(F_l)}$ of the F_l spin chain follows the recursion relation

$$\mathbf{M}_l = \mathbf{M}_{l-2}\mathbf{M}_{l-1}^2, \tag{36}$$

with $\mathbf{M}_0$ given by Eq. (25) and

$$\mathbf{M}_1 = \begin{pmatrix} \frac{1-\lambda}{C_1} & -\frac{C_2}{C_1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\lambda}{C_2} & -\frac{C_1}{C_2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\lambda}{C_3} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{(1-\lambda)^3}{C_1C_2C_3} - \frac{(1-\lambda)C_2}{C_1C_3} - \frac{1-\lambda}{C_2} & -\frac{(1-\lambda)^2}{C_1C_2} + \frac{C_2}{C_1} \\ \frac{(1-\lambda)^2}{C_2C_3} - \frac{C_1}{C_2} & -\frac{1-\lambda}{C_2} \end{pmatrix}. \tag{37}$$

From the unimodularity of matrices $\mathbf{M}_0$ and $\mathbf{M}_1$ and the recursion relation (36), we obtain $\det\mathbf{M}_l=1$ for all $l\geqslant 0$. Also, from Eq. (36) and the unimodularity of $\mathbf{M}_l$, we can obtain the following trace map for $x_l=\frac{1}{2}\mathrm{Tr}\mathbf{M}_l$ [20]:

$$x_l=4x_{l-1}^2x_{l-2}-\frac{x_{l-1}^2}{x_{l-2}}-x_{l-2}-\frac{x_{l-3}x_{l-1}}{x_{l-2}} \quad \text{for } l\geqslant 3, \tag{38}$$

with initial conditions

$$x_0=\frac{1-\lambda}{2C_3}, \quad x_1=\frac{(1-\lambda)^3}{2C_1C_2C_3}-\frac{(1-\lambda)C_2}{2C_1C_3}-\frac{1-\lambda}{C_2},$$

and

$$x_2=\frac{(1-\lambda)^7}{2C_1^2C_2^2C_3^3}-\frac{(1-\lambda)^5}{C_1^2C_3^3}-\frac{2(1-\lambda)^5}{C_1C_2^2C_3^2}-\frac{(1-\lambda)^5}{2C_1^2C_2^2C_3}+\frac{(1-\lambda)^3C_2^2}{2C_1^2C_3^3}+\frac{2(1-\lambda)^3}{C_1C_3^2}+\frac{5(1-\lambda)^3}{2C_2^2C_3}+\frac{(1-\lambda)^3}{C_1^2C_3}+\frac{(1-\lambda)^3}{C_1C_2^2}-\frac{(1-\lambda)}{C_3}-\frac{(1-\lambda)C_1}{C_2^2}-\frac{(1-\lambda)}{C_1}-\frac{(1-\lambda)C_2^2}{2C_1^2C_3}. \tag{39}$$

For the silver-mean ferromagnetic Ising chain, the dynamical critical exponent is determined by the smallest λ_s satisfying the condition $|x_l|\leqslant 1$. According to the discussion for the Fibonacci chain, we can conclude that λ_s is the smallest root of the equation $x_l=1$. This conclusion is also verified by numerical method. From map (38) and initial conditions (39), we can obtain the dynamical critical exponent z for large l and small T by solving the equation $x_l=1$ numerically. In Fig. 2 we show the z for the finite silver-mean ferromagnetic Ising chain with $F_6=99$ as functions of $K_2=1/k_BT$ at $J_1/J_2=1.2$, 1.5, 2.0, and 2.5. From Fig. 2, one sees that $z=1+J_1/J_2$ as $T\rightarrow 0$. As in the case of the Fibonacci chain, we can also study the dynamics of the dual chain of the silver-mean ferromagnetic Ising chain (33). This chain can be obtained from Eq. (33) by interchanging J_1 and J_2. And we find again $z=1+J_1/J_2$.

(3) Copper-mean ferromagnetic Ising chain: The copper-mean ferromagnetic Ising chain is a general quasiperiodic Ising chain with $m=1$ and $n=2$, for which $\sigma=\frac{1}{2}$. The substitution rule gives the following sequence of J_1 and J_2:

$$J_1J_2J_2J_1J_1J_1J_2J_2J_1J_2J_2\cdots. \tag{40}$$

Then the sequence C_i^+ is

$$C_1C_4C_2C_3C_3C_1C_4C_2C_1C_4C_2\cdots. \tag{41}$$

with $C_{1,2}$ given by Eq. (19), $C_3=\frac{1}{2}\tanh(2K_1)$, and $C_4=\frac{1}{2}\tanh(2K_2)$. This sequence can be obtained from sequence (40) by the substitution rule: $J_1\rightarrow C_1C_4C_2$, $J_2\rightarrow C_3$. Similarly, the sequence C_i^- is obtained from sequence (40) by the constitution rule: $J_1\rightarrow C_2C_4C_1$, $J_2\rightarrow C_3$, which is

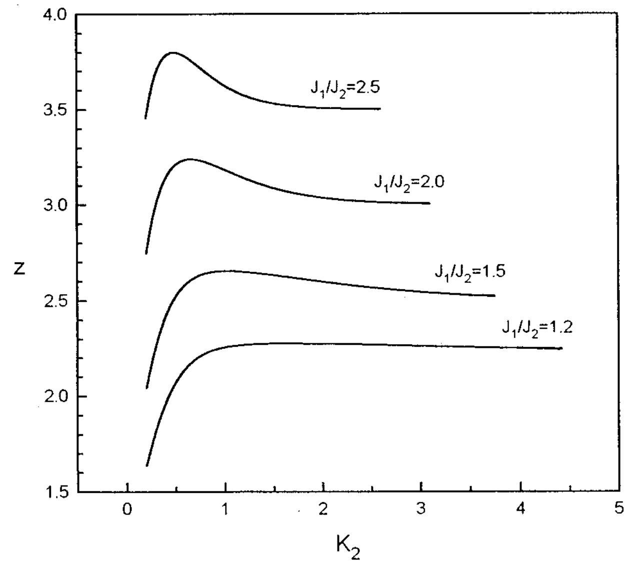

FIG. 2. The dynamical critical exponent of the silver-mean ferromagnetic Ising chain as functions of $K_2=J_2/k_BT$ at $J_1/J_2=1.2$, 1.5, 2.0, and 2.5. The curves are obtained for the $F_6=99$ spin chain under periodic boundary condition. We can see that $z=1+J_1/J_2$ as $K_2\rightarrow\infty$.

$$C_2C_4C_1C_3C_3C_2C_4C_1C_2C_4C_1\cdots. \tag{42}$$

The transfer matrix $\mathbf{M}_l\equiv\mathbf{M}^{(F_l)}$ of the F_l spin chain follows the recursion relation

$$\mathbf{M}_l=\mathbf{M}_{l-2}^2\mathbf{M}_{l-1}, \tag{43}$$

with $\mathbf{M}_0$ given by Eq. (25) and

$$\mathbf{M}_1=\begin{pmatrix}\frac{1-\lambda}{C_1} & -\frac{C_2}{C_1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1-\lambda}{C_4} & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1-\lambda}{C_2} & -\frac{C_1}{C_2}\\ 1 & 0\end{pmatrix} =\begin{pmatrix}\frac{(1-\lambda)^3}{C_1C_2C_4}-\frac{2(1-\lambda)}{C_1} & -\frac{(1-\lambda)^2}{C_2C_4}+1\\ \frac{(1-\lambda)^2}{C_2C_4}-1 & -\frac{(1-\lambda)C_1}{C_2C_4}\end{pmatrix}. \tag{44}$$

Similarly, from the unimodularity of matrices $\mathbf{M}_0$ and $\mathbf{M}_1$ and the recursion relation (43), we can also obtain $\det\mathbf{M}_l=1$ for all $l\geqslant 0$. From Eq. (43) and the unimodularity of $\mathbf{M}_l$, we can obtain the following trace map for $x_l=\frac{1}{2}\mathrm{Tr}\mathbf{M}_l$ [20]:

$$x_l=4x_{l-1}x_{l-2}^2-x_{l-1}-4x_{l-2}x_{l-3}^2+2x_{l-2} \quad \text{for } l\geqslant 3 \tag{45}$$

with initial conditions

$$x_0=\frac{1-\lambda}{2C_3}, \quad x_1=\frac{(1-\lambda)^3}{2C_1C_2C_4}-\frac{(1-\lambda)C_1}{2C_2C_4}-\frac{1-\lambda}{C_1},$$

and

$$x_2=\frac{(1-\lambda)^5}{2C_1C_2C_3^2C_4}-\frac{(1-\lambda)^3}{C_1C_3^2}-\frac{(1-\lambda)^3}{2C_1C_2C_4}-\frac{(1-\lambda)^3}{C_2C_3C_4}+\frac{1-\lambda}{C_1}+\frac{1-\lambda}{C_3}+\frac{(1-\lambda)C_1}{2C_2C_4}.$$

As for the Fibonacci ferromagnetic Ising chain, λ_s is the smallest root of the equation $x_l=\Sigma_{n=0}^{F_l}A_n^{(l)}\lambda^n=1$. The dynamical critical exponent z is determined by Eqs. (28) and (29), with the coefficients g_l and h_l derived from the recursion relations

$$g_l=4g_{l-1}g_{l-2}^2-4g_{l-2}g_{l-3}^2-g_{l-1}+2g_{l-2} \tag{46}$$

and

$$h_l=4h_{l-1}g_{l-2}^2+8g_{l-1}g_{l-2}h_{l-2}-4h_{l-2}g_{l-3}^2-8g_{l-2}g_{l-3}h_{l-3}-h_{l-1}+2h_{l-2}, \tag{47}$$

with initial conditions

$$g_0=\frac{1}{2C_3},\quad g_1=\frac{1-C_1^2-2C_2C_4}{2C_1C_2C_4},$$
$$g_2=\frac{1-2C_2C_4-2C_1C_3-C_3^2+2C_2C_3^2C_4+2C_1C_2C_3C_4+C_1^2C_3^2}{2C_1C_2C_3^2C_4}, \tag{48}$$

and

$$h_0=-\frac{1}{2C_3},\quad h_1=\frac{-3+C_1^2+2C_2C_4}{2C_1C_2C_4},$$
$$h_2=\frac{-5+6C_2C_4+6C_1C_3+3C_3^2-2C_2C_3^2C_4-2C_1C_2C_3C_4-C_1^2C_3^2}{2C_1C_2C_3^2C_4}. \tag{49}$$

Figure 3 shows the numerical result of z for the finite copper-mean ferromagnetic Ising chain with $F_8=171$ spins as functions of $K_2=1/k_BT$ at $J_1/J_2=1.2$, 1.5, 2.0, and 2.5. From this figure it is found again that $z=1+J_1/J_2$ as $T\to 0$. Similarly, we can study the dynamics of the dual chain of the copper-mean ferromagnetic Ising chain (40). This chain is obtained from Eq. (40) by interchanging J_1 and J_2. We also obtain that $z=1+J_1/J_2$.

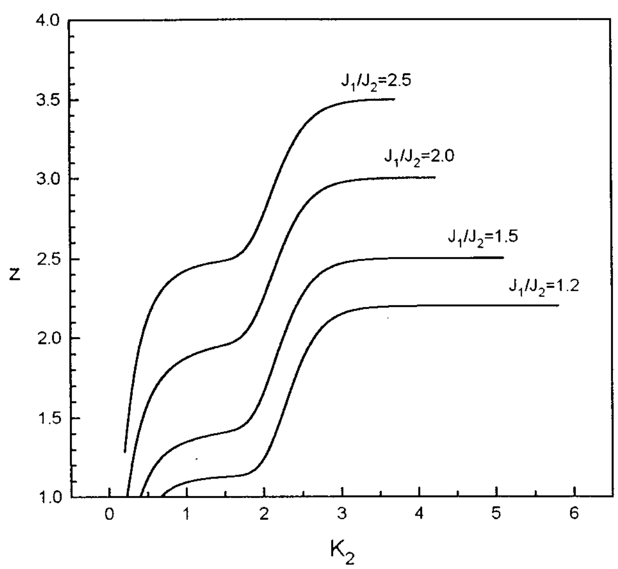

FIG. 3. The dynamical critical exponent of the copper-mean ferromagnetic Ising chain as functions of $K_2=J_2/k_BT$ at $J_1/J_2=1.2$, 1.5, 2.0, and 2.5. The curves are obtained for the $F_8=171$ spin chain under periodic boundary conditions. We can see that $z=1+J_1/J_2$ as $K_2\to\infty$.

IV. PERIOD-DOUBLING CHAIN

For the period-doubling ferromagnetic Ising chain, the nearest-neighbor coupling constants J_i take two values J_1 and J_2 ($J_1>J_2>0$) arranged in a period-doubling sequence. The period-doubling sequence is aperiodic and is different from quasiperiodic sequences. This sequence is generated from a seed (e.g., J_1), by the following substitution rule: $J_1\to J_1J_2$, $J_2\to J_1J_1$ [17],

$$J_1J_2J_1J_1J_1J_2J_1J_2J_1J_2J_1J_1J_1J_2J_1J_1\cdots. \tag{50}$$

Then the sequence C_i^+ is

$$C_1C_2C_3C_3C_1C_2C_1C_2C_1C_2C_3C_3C_1C_2\cdots, \tag{51}$$

with $C_3=\frac{1}{2}\tanh(2K_1)$. This sequence can be obtained from sequence (50) by the substitution rule: $J_1\to C_1C_2$, $J_2\to C_3$. Similarly, the sequence C_i^- is obtained from sequence (50) by the constitution rule: $J_1\to C_2C_1$, $J_2\to C_3$, which yields

$$C_2C_1C_3C_3C_2C_1C_2C_1C_2C_1C_3C_3C_2C_1\cdots. \tag{52}$$

For the lth period-doubling sequence, there are 2^l spins in the chain. Under the periodic boundary conditions, the transfer matrix $\mathbf{M}_l \equiv \mathbf{M}^{(2^l)}$ follows the same recursion relation (43) of the copper-mean chain with different initial conditions:

$$\mathbf{M}_1 = \begin{pmatrix} \frac{1-\lambda}{C_2} & -\frac{C_1}{C_2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\lambda}{C_1} & -\frac{C_2}{C_1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{(1-\lambda)^2}{C_1 C_2} - \frac{C_2}{C_1} & -\frac{1-\lambda}{C_2} \\ \frac{1-\lambda}{C_2} & -\frac{C_1}{C_2} \end{pmatrix} \tag{53}$$

and $\mathbf{M}_0$ is given by Eq. (25). $x_l = \frac{1}{2}\mathrm{Tr}\mathbf{M}_l$ satisfies the recursion relation (45) with initial conditions

$$x_0 = \frac{1-\lambda}{2C_3}, \quad x_1 = \frac{(1-\lambda)^2}{2C_1C_2} - \frac{C_2}{2C_1} - \frac{C_1}{2C_2},$$

and

$$x_2 = \frac{(1-\lambda)^4}{2C_1C_2C_3^2} - \frac{(1-\lambda)^2 C_2}{2C_1C_3^2} - \frac{(1-\lambda)^2}{2C_1C_2} - \frac{(1-\lambda)^2}{C_2C_3} + \frac{C_2}{2C_1} + \frac{C_1}{2C_2}.$$

As in the case of the copper-mean chain, we can obtain the dynamical critical exponent z by using Eqs. (46) and (47) with the initial conditions

$$g_0 = \frac{1}{2C_3}, \quad g_1 = \frac{1 - C_1^2 - C_2^2}{2C_1C_2},$$

$$g_2 = \frac{1 - C_2^2 - C_3^2 - 2C_1C_3 + C_2^2C_3^2 + C_1^2C_3^2}{2C_1C_2C_3^2}, \tag{54}$$

and

$$h_0 = -\frac{1}{2C_3}, h_1 = -\frac{1}{C_1C_2},$$

$$h_2 = \frac{-2 + C_2^2 + C_3^2 + 2C_1C_3}{C_1C_2C_3^2}. \tag{55}$$

In Fig. 4 we show z of finite period-doubling ferromagnetic Ising chain with $F_7 = 128$ spins as functions of $K_2 = 1/k_BT$ at $J_1/J_2 = 1.2$, 1.5, 2.0, and 2.5. From Fig. 4 one can see that $z = 1 + J_1/J_2$ as $T \to 0$. The study of the dual chain of the period-doubling ferromagnetic Ising chain again gives $z = 1 + J_1/J_2$.

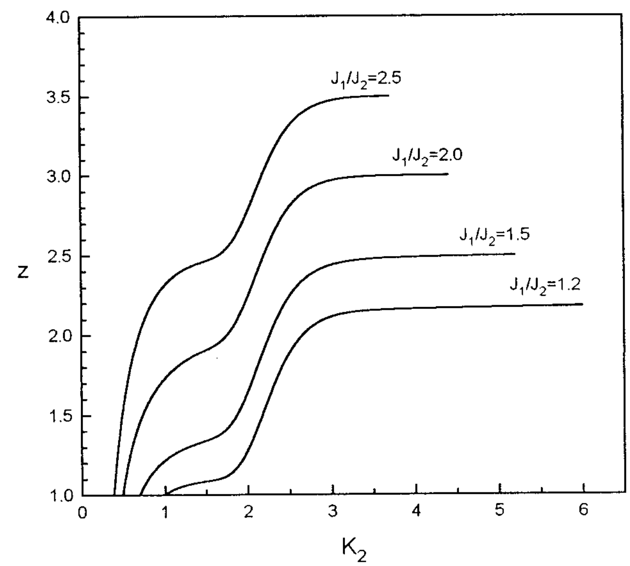

FIG. 4. The dynamical critical exponent of the period-doubling ferromagnetic Ising chain as functions of $K_2 = J_2/k_BT$ at $J_1/J_2 = 1.2$, 1.5, 2.0, and 2.5. The curves are obtained for the $F_7 = 128$ spin chain under periodic boundary conditions. We can see that $z = 1 + J_1/J_2$ as $K_2 \to \infty$.

V. CONCLUSION AND DISCUSSION

We have used the trace map method to study the dynamical critical exponent z for the general Fibonacci and period-doubling ferromagnetic Ising chains with Glauber dynamics. For the well-known uniform and alternating-bond ferromagnetic chains, we reproduced the standard results. For the Fibonacci, silver-mean, copper-mean, and period-doubling ferromagnetic Ising chains, we found that the dynamical critical exponent z is nonuniversal and is identical to that obtained for the alternating-bond ferromagnetic Ising chain. It can be verified that the domain-wall arguments, along the lines given by Droz *et al.* [5], give the same results. Our results for the Fibonacci ferromagnetic chain agree with that obtained by renormalization-group method [14,15]. Based on the distribution of inverse relaxation times (i.e., λ), which were obtained from generating function using renormalization-group method, Southern and Achiam [15] pointed out that the nonuniversal value of z is due to the width of the lowest band, which tends to zero as the temperature approaches zero. This narrowing of the band gives an additional contribution to the critical slowing down since a new time scale is introduced. From our discussion in this paper, we can obtain the width of bands by iterating numerically the trace maps and show that the width of the band tends to zero as the temperature approaches zero. Therefore, for the general Fibonacci and period-doubling ferromagnetic Ising chains with Glauber dynamics, the nonuniversality is also a result of two different contributions: the univeral long ranged fluctuations near critical point and short ranged phenomena, as first concluded by Southern and Achiam [9,10,15].

From the results of the present paper, we conclude that the trace map method is also a very useful method in studying the dynamics of the nonperiodic Ising chains. Moreover, although the nonperiodic Ising chains investigated in this paper are ferromagnetic, the trace map method is also applicable to antiferromagnetic or ferromagnetic and antiferromagnetic Ising chains, since the method used here does not rely on the sign of the coupling constants. Beside the nonperiodic sequences discussed in the present paper, the Thue-Morse sequence is one of the most popular nonperiodic sequences. But, we cannot use the trace map method to study the dynamics of the Thue-Morse Ising chain. This is because we cannot obtain the close recursion relation for the transfer matrix of the Thue-Morse chain analytically [22].

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