Tracking of synchronized chaotic systems with applications to communications

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(Received 12 December 1996)

In this paper, we examine the problem of the self-synchronizing chaotic receiver, extracting the parameters of the transmitter system from the available transmitted signals and thereby achieving synchronization. Ascertaining the parameter equality will be an important issue in any practical implementation of a chaotic communication system. We define a cost function whose iterative minimization allows the receiver to extract the parameters. The method is illustrated for the Lorenz system and two coupled Lorenz systems. Last, we suggest *chaotic multiplexing* as a possible application of the method. [S1063-651X(97)01707-8]

PACS number(s): 05.45.+b, 43.72.+q

The self-synchronization property possessed by a class of chaotic systems [1,2] has been of great interest recently, especially due to its possible application to communications [3]. The synchronization requires that the parameters characterizing the transmitter dynamical system be closely matched in the receiver system. By synchronization, we mean that the receiver follows the transmitter exactly, termed identical synchronization (IS) in [11]. Unless explicitly stated, we will refer to IS as synchronization. In this paper, we illustrate a simple method that enables the receiver, initially ignorant of the transmitter parameters, to extract transmitter parameters to achieve synchronization. The transmitter dynamical system, with the exception of the parameters to be estimated, is assumed to be known to the receiver. The signals available at the receiver are the driving or transmitted signals that enable it to synchronize with the chaotic signals evolving at the transmitter if the parameters of the transmitter and the receiver match.

Let the transmitter be given by

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\mathbf{a_t}),$$

where $\mathbf{x},\mathbf{f}(\mathbf{x})\in\mathbb{R}^n$ or $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{f}(\mathbf{x})=(f_1(\mathbf{x}),f_2(\mathbf{x}),\ldots,f_n(\mathbf{x}))$, and $\mathbf{f}$ is such that a synchronizing receiver is possible. The vector $\mathbf{a_t}\in\mathbb{R}^p$ is the parameter vector characterizing the system. Without loss of generality, let the m transmitted signals be $x_1,\ldots,x_m$. The full dimensional receiver is given by

$$\dot{\mathbf{y}}=\mathbf{g}(x_1,\ldots,x_m,\mathbf{y},\mathbf{a_r}),$$

where $\mathbf{y},\mathbf{g}\in\mathbb{R}^n$ and the receiver parameter vector is $\mathbf{a_r}\in\mathbb{R}^p$. It is not necessary that the function $\mathbf{g}$ be a replica of $\mathbf{f}$. This may enhance synchronism for particular choices of $\mathbf{g}$ [4]. In the case of parameter match, i.e., $\mathbf{a_t}=\mathbf{a_r}$, the synchronization is ensured, i.e., $||\mathbf{x}-\mathbf{y}||\to 0$ as $t\to\infty$. This rate of synchronization may be quite fast for some systems, such as the Lorenz system, where the rate is exponential [5].

The problem that we now address is: can the receiver determine the transmitter parameters $\mathbf{a_t}$ from the available transmitted signals $(x_1,\ldots,x_m)$? Carroll and Pecora [6] and Kozlov *et al.* [7] address this problem for a single parameter estimation by defining a *control law*, a nonlinear control system based on the error signals and of the form

$$\dot{\mathbf{a}}_\mathbf{r}=\mathbf{h}(x_1-y_1,\ldots,x_m-y_m,\mathbf{a_r}).$$

Finding a function $\mathbf{h}$ is specific to the transmitter-receiver system. We introduce a cost function whose iterative minimization accomplishes the same task, though with considerable simplicity. The cost function T is defined based on the error signals measurable at the receiver, $e_i(t)=x_i(t)-y_i(t)$, $(i=1,\ldots,m)$, as

$$T(\mathbf{a_r},t,t_2)=\sum_{i=1}^{m}\int_t^{t+t_2} e_i{}^2(t)dt. \tag{1}$$

The transmitted signals are stored over the interval $[t-t_1,t+t_2]$ and the receiver is driven using this stored data, and the error signals are used for the evaluation of the cost function. The transmitter parameters are assumed to be slowly varying so that they can be assumed to be constant over the interval $[t-t_1,t+t_2]$. Brown *et al.* [8] define a measure of the synchronization error based on $||\mathbf{x}-\mathbf{y}||$ and report its monotonic variation for small parameter difference, but such a function is not measurable at the receiver due to nonavailability of all transmitter state variables. In the case of parameter match, cost function (referred to hereafter as T) is very close to zero due to synchronization; otherwise, it is a positive quantity. One can thus iteratively minimize T over the receiver parameters $\mathbf{a_r}$, and T becoming very close to zero (less than some ϵ) is a simple test of convergence. It should be noted, however, that, depending on the nature of the parameters $\mathbf{a_r}$ to be tracked, it is not necessary that $\mathbf{a_r}=\mathbf{a_t}$ be the only point where T becomes zero. This is because minimization of T ensures synchronization with the transmitted signals only. Such a case is shown by the following example. The Lorenz transmitter system is given by

$$\dot{x}_1=\sigma_t(x_2-x_1);\quad \dot{x}_2=r_tx_1-x_2-x_1x_3;$$
$$\dot{x}_3=x_1x_2-b_tx_3 \tag{2}$$

and the receiver by

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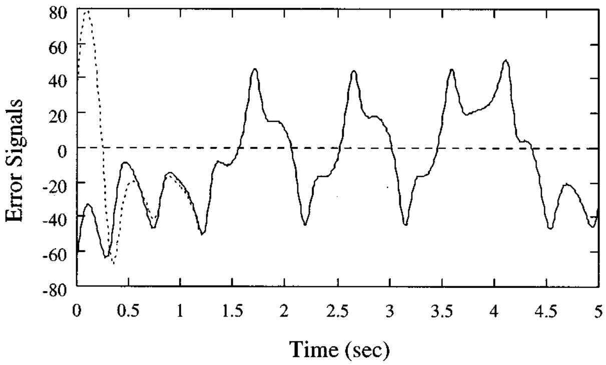

FIG. 1. Two error signals for different receiver initial conditions and unmatched parameters.

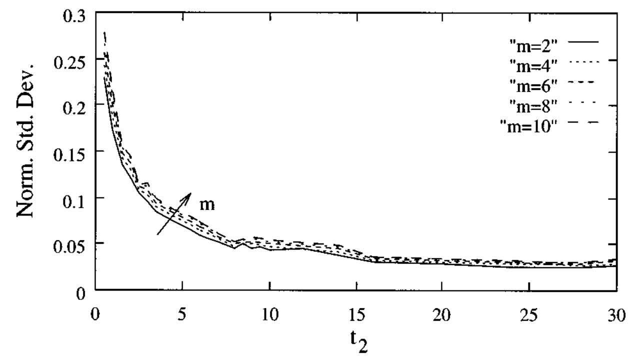

FIG. 2. The normalized standard deviation of the cost function as a function of t_2.

$$\dot{y}_1=\sigma_r(y_2-y_1);\quad \dot{y}_2=r_rx_1-y_2-a_1x_1y_3;$$

$$\dot{y}_3=a_2x_1y_2-b_ry_3. \tag{3}$$

In this case, T is a function of the error signal $e_1(t)=x_1(t)-y_1(t)$. Let $(\sigma_r,r_r,b_r)=(\sigma_t,r_t,b_t)$ and let (a_1,a_2) be the two receiver parameters modified until T becomes zero. It can easily be checked that for $a_2=1/a_1$, where $a_1\neq 0$, a change of variables of y_3 to a_1y_3 gives back the fully matched synchronizing receiver, and hence T attains its global minima of zero at an infinite number of points $(a_1,1/a_1)$, although the parameters match only when $a_1=a_2=1$. In such cases, $\mathbf{a_t}$ remains indeterminate, though a minimization of T will ensure synchronization with the transmitted signals, which may be sufficient for applications such as chaotic signal masking or chaotic digital communication [9].

We now comment on the various times, t,t_1,t_2, involved in definition (1). The selection of time t is arbitrary. However, it is assumed that the transmitter system has come out of its transient due to its arbitrary initial conditions and resides on its attractor in the interval $[t-t_1,t+t_2]$, a legitimate assumption under which synchronization may occur. It is desired that T be independent of the receiver state variable values at time $(t-t_1)$ or the initial conditions, which are set arbitrarily when T is evaluated. In case of matching parameters, all error signals $e_i(t)$, due to difference in the transmitter and receiver state variables at time $(t-t_1)$, die down due to synchronization. Though the error signals will not become identically zero for mismatched parameters, they may, however, become independent of the receiver initial conditions if time t_1 is chosen to be large. This is essentially due to the phenomenon of generalized synchronization (GS) proposed by Rulkov *et al.* [10], where, instead of following the transmitter exactly, the receiver follows its fixed image, i.e., $\mathbf{y}=\mathbf{H}(\mathbf{x})$, where $\mathbf{H}$ is nonlinear in general. The necessary and sufficient conditions for GS, given by Kocarev and Parlitz [11], are that the conditional Lyapunov exponents are all negative, or alternately that a Lyapunov function can be defined for the mismatched receiver, the latter approach being the more direct way. Thus, if these conditions are satisfied for the receiver parameters, the error signals, and hence T, become very nearly independent of the receiver initial conditions at time $(t-t_1)$ due to synchronization to a fixed image of the transmitter.

The time t_2 is chosen such that the synchronization with the transmitted signals in the interval $[t,t+t_2]$ (due to minimization of T) ensures synchronization for all time, and hence for accurate parameter recovery. We observe the normalized standard deviation of T (i.e., the ratio of standard deviation with its mean) as a function of t_2, and t_2 is chosen such that this is small. The various readings for calculating the standard deviation at a given t_2 are taken by observing T at varying t. Such a choice of t_2 works quite well. We comment here that we are currently investigating whether one can choose a t_2 at a large standard deviation (possibly a lower value than the above choice) and still recover the parameters accurately. About the nature of iterative algorithm, since the derivatives of the cost function are not directly available, one needs to choose a method where only the function evaluations are required. We used the downhill simplex method in multidimensions [12].

We now illustrate the method for the Lorenz transmitter-receiver system [see Eqs. (2) and (3)], where it is assumed that $a_1,a_2=1$ and (σ_t,r_t,b_t) are the three parameters to be estimated. First, we show that T attains its global minima of zero only in the case of parameter match, rather than an infinite number of points, as in the case of estimation of a_1,a_2. From the minimization of T, $x_1=y_1$, and hence $\dot{x}_1=\dot{y}_1$; thus, $y_2=(\sigma_r-\sigma_t)x_1/\sigma_r+\sigma_tx_2/\sigma_r$. Differentiating both sides, we get $[r_r-\sigma_tr_t/\sigma_r+(\sigma_r-\sigma_t)(\sigma_t-1)/\sigma_r+\sigma_tx_3/\sigma_r-y_3]x_1=(\sigma_r-\sigma_t)\sigma_tx_2/\sigma_r$. Now it is possible to

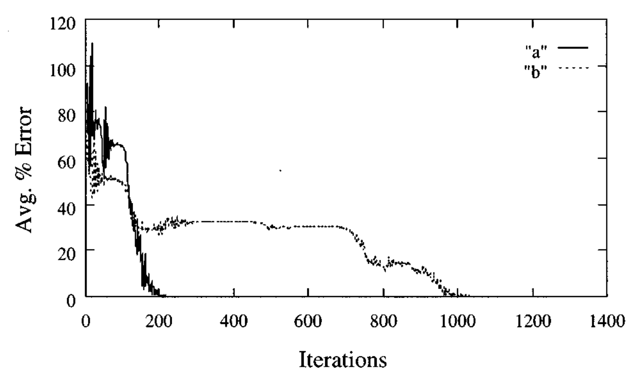

FIG. 3. The average percentage parameter mismatch as a function of iterations for (a) the Lorenz system and (b) two coupled Lorenz systems.

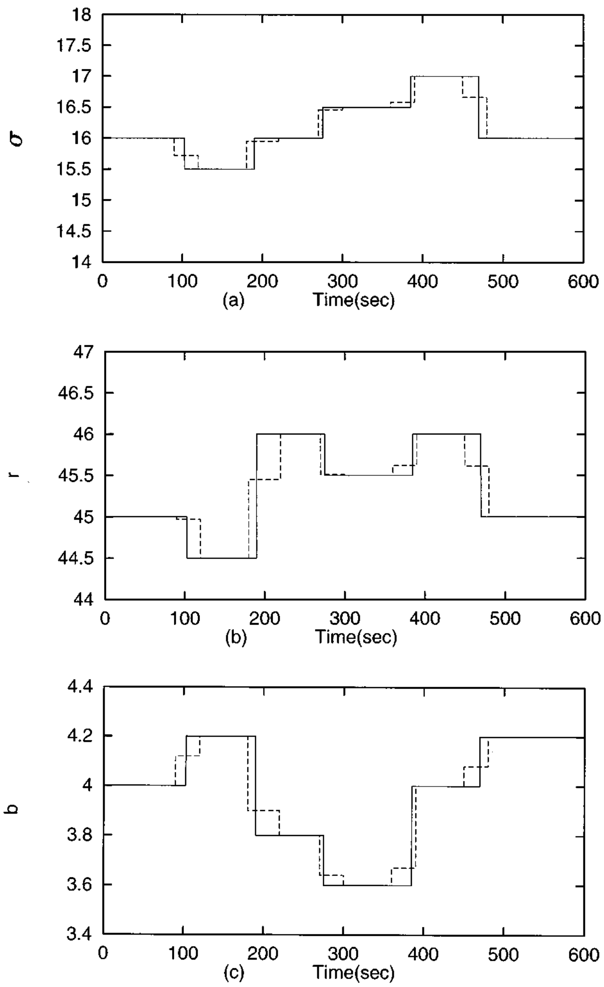

FIG. 4. Communication by the estimation of (a) σ; (b) r; and (c) b. The solid and the dashed lines indicate the transmitter and receiver parameters, respectively.

choose a point on the chaotic attractor such that $x_1=0$ and $x_2\neq 0$, where the right-hand side$=0$, implying that $\sigma_r=\sigma_t$. Hence, the left-hand side$=0$ for all x_1, so $y_3=x_3+r_r-r_t$. By differentiating, we get $(b_r-b_t)x_3=b_r(r_t-r_r)$. For this to hold for all x_3, $(r_r,b_r)=(r_t,b_t)$.

Let $(\sigma_t,r_t,b_t)=(16.0,45.6,4.0)$ and the initial choice of the receiver parameters be $(\sigma_r,r_r,b_r)=(29.0,80.6,7.8)$. Figure 1 shows the error signal $e_1(t)$ for two largely different receiver initial conditions. The transmitter state variable values at time $(t-t_1)$ are fixed at $(x_1,x_2,x_3)=(-12.1,5.3,59.8)$ and the two receiver initial conditions are $(y_1,y_2,y_3)=(50,50,50),(-50,-50,-50)$. Let $\mathbf{y}'$ and $\mathbf{y}''$ be the two receiver trajectories emanating from different initial conditions at time $(t-t_1)$. By an analysis similar to the one in the appendix of [5], it can be shown that these two trajectories approach each other at least exponentially and that $|y_1'-y_1''|$ decreases at least as max $\{O(e^{-\sigma_r t}),O(e^{-b_r t}),O(e^{-t})\}$. This gives a good measure of t_1. We chose $t_1=5$ sec for all iterations. The conditions for GS are not satisfied if σ_r or b_r is negative and the receiver trajectories are unbounded, making T very large. In systems for which the expressions for the decay of error signals are not explicitly available, the choice of t_1 would be based on the largest conditional Lyapunov exponent for the particular choice of receiver parameters. To choose t_2, we observed the normalized standard deviation of T for various receiver parameters. As an example, in Fig. 2, the receiver parameters are varied as $\sigma_r=16.0+1.3m$, $r_r=45.6+3.5m$, and $b_r=4.0+0.38m$. The receiver parameters chosen at the start of the minimization routine correspond to $m=10$. It can be seen that there is not much dependence on m, and t_2 was chosen to be 25 sec for all iterations.

A pertinent question in the minimization will be to check whether T has local minima where any minimization algorithm may get stuck. For the above chosen values of t_1,t_2, we observed T, by numerical experiments, as a function of parameter difference defined as $(\Delta\sigma,\Delta r,\Delta b)=(\sigma_r,r_r,b_r)-(\sigma_t,r_t,b_t)$. With some curve fitting, subsequent models were found to be close to the observed T when only a single parameter was varied. The transmitter parameters are fixed at $(\sigma_t,r_t,b_t)=(16.0,45.6,4.0)$,

$$T(\Delta\sigma,0,0)=\begin{cases}14.8(\Delta\sigma-2.46)+36.41\exp(-\Delta\sigma/2.46), & \Delta\sigma>0\\ -0.55(\Delta\sigma)^3, & \Delta\sigma\leqslant 0,\end{cases}$$

$$T(0,\Delta r,0)=1.93(\Delta r)^2,$$

$$T(0,0,\Delta b)=213.75(\Delta b)^2.$$

The variation of the cost function is monotonic as the mismatch increases, a feature also observed when more than one parameter was varied. Figure 3 shows the plot of average percentage parameter mismatch against the iterations. The recovered parameter values at iteration 250 were (16.001,45.599,3.999). The transmitter parameters have clearly been recovered. Next, we tried to test the parameter recovery for the chaotic signal masking application [9], where a chaotic carrier masks the low-power and low-pass speech signal. To make the error signal approximately independent of the speech signal, it can be high-passed and then used for the evaluation of T. The recovered parameters for the same initial receiver parameters as above were (16.231,45.793,3.954). The parameter recovery for this case was not as accurate as for the simple Lorenz system without the speech signal, but it does enable an intelligible recovery of the speech signal. It was observed that the parameter estimation can be made more accurate if the minimization of the cost function is done at various time segments, instead of only one, and then taking the average value of the obtained set of parameters.

A procedure for synthesizing self-synchronizing chaotic arrays was developed in [13], where lower-dimensional systems are coupled by a linear system to build high-dimensional arrays. We applied our scheme to the system of two coupled Lorenz oscillators synthesized in [13]. There are six parameters characterizing the system (three for each oscillator), which we attempted to extract by our algorithm. The linear system coupling the two systems was assumed to be known to the receiver. The transmitter parameters were fixed at $(\sigma_1,r_1,b_1,\sigma_2,r_2,b_2)=(14.0,44.0,3.5,15.0,55.0,4.5)$ and the initial values of the receiver parameters were (30.0,70.6,6.8,25.0,50.0,3.0), respectively. The convergence was slower than the simple Lorenz system (see Fig. 3) and the recovered parameters at iteration 1200 were (13.999,43.970,3.502,14.987,55.030,4.497). Recently, Peng *et al.* [14] proposed synchronizing receivers for hyperchaotic systems with a scalar transmitted signal. The transmitted signal in the four-dimensional hyperchaotic Rössler system [14] is of the form $\sin\theta x_1 + \cos\theta x_3$. Hence, the error signal for cost function evaluation is $\sin\theta(x_1 - y_1) + \cos\theta(x_3 - y_3)$. Our limited simulations show that parameter recovery for this system seems possible when the **K-B** space, as defined in [14], is known at the receiver [14]. This point is, however, under consideration and the scope of future work.

This parameter recovery scheme has useful and interesting applications in communications. First, this can be used as a safeguard against any possible parameter drift with time. Second, if the parameters to be tracked are such that they can be uniquely determined at the receiver (i.e., synchronization with transmitted signals is sufficient to guarantee parameter match, as is the case for σ, r, b of the Lorenz system), then each of such parameters can be used for information transmission. The transmitter parameters can be varied slowly so that the receiver can accurately recover them. This can be referred to as *chaotic multiplexing* of various information channels, where each parameter corresponds to a channel. Figure 4 illustrates the idea for the Lorenz system where (σ_t, r_t, b_t) are the parameters to be estimated, and hence are the three information channels. Note that the parameter recovery may not be accurate at the sudden changes, since the transmitter parameters are not constant over $[t - t_1, t + t_2]$. This can also be inferred at the receiver, since it will not be possible to minimize T to zero. In any practical implementation of this scheme, the computational speed at which the minimization of the cost function can be done will be an important issue.

In conclusion, we have shown the receiver extracting the transmitter parameters and achieving synchronization by an iterative minimization of a defined cost function. This extraction of parameters can be used for multiplexing of various information signals.

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