

Nonlinear conformally invariant generalization of the Poisson equation to $D > 2$ dimensions

Mordehai Milgrom

Department of Condensed-Matter Physics, Weizmann Institute, Rehovot 76100, Israel

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I propound a nonlinear generalization of the scalar-field Poisson equation of the form $[(\varphi,^i \varphi,^i)^{D/2-1} \varphi,^k]_{;k} \propto \rho$, in curved D -dimensional space. It is derivable from the Lagrangian density $L^D = L_f^D - A\rho\varphi$, with $L_f^D \propto -(\varphi,^i \varphi,^i)^{D/2}$, and ρ the distribution of sources. Specializing to Euclidean spaces, where the field equation is $\nabla \cdot (|\nabla\varphi|^{D-2} \nabla\varphi) \propto \rho$, I find that L_f^D is the only conformally invariant (CI) Lagrangian in D dimensions, containing only first derivatives of φ , beside the free Lagrangian $(\nabla\varphi)^2$, which underlies the Laplace equation. When φ is coupled to the sources in the above manner, L^D is left as the only CI Lagrangian. The symmetry is one's only recourse in solving this nonlinear theory for some nontrivial configurations. Systems comprising N point charges are special and afford further application of the symmetry. In spite of the CI, the energy function for such a system is not invariant under conformal transformations of the charges' positions. The anomalous transformation properties of the energy stem from effects of the self-energies of the charges. It follows from these that the forces \mathbf{F}_i on the charges q_i at positions \mathbf{r}_i must satisfy certain constraints beside the vanishing of the net force and net moment: for example, $\sum_i \mathbf{r}_i \cdot \mathbf{F}_i$ must equal some given function of the charges. The constraints total $(D+1)(D+2)/2$, which tallies with the dimension of the conformal group in D dimensions. Among other things I use all these to derive exact expressions for the following quantities: (1) The general two-point-charge force. (2) The full potential field for two opposite charges $\pm q$. (3) The energy function and the forces in any three-body configuration with zero total charge. (4) The few-body force for some special configurations. (5) The virial theorem for an arbitrary, bound, many-particle system relating the time-average kinetic energy to the particle charges. I also discuss briefly multiscalar theories, theories with higher derivatives, and vector- and higher-form-potential theories. [S1063-651X(97)14007-7]

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I. INTRODUCTION

It is a well-known and well-used fact that the Poisson equation, $\Delta\varphi \propto \rho$, for the potential φ produced by sources ρ , describes a conformally invariant (CI) theory in two dimensions: It is invariant under the angle-preserving coordinate transformations. In all dimensions it is linear in the field φ , and thus describes a "free" (non-self-interacting) field. Many of the special features of the $D=2$ theory stem from its linearity, but many are underpinned by the conformal invariance. The Poisson equation in $D > 2$ dimensions is not CI (while the Laplace equation is; see Sec. II B).

The Poisson equation describes many physical problems in linear media such as electrostatics, magnetostatics, steady-state diffusion, and other potential flows in the presence of sources and sinks, and, of course, Newtonian gravity. It can be generalized to

$$\nabla \cdot [\mu(|\nabla\varphi|)\nabla\varphi] \propto \rho, \tag{1.1}$$

to describe, for example, nonlinear media with a response coefficient (dielectric constant, permeability, diffusion coefficient, etc.) that is a function of the field strength. An equation of this type, with different forms of $\mu(x)$, has been studied in different contexts. For example, as an effective-action approximation to Abelianized QCD [1], as a modification of Newtonian gravity to replace the dark-matter hypothesis for galactic systems [2,3], and in the context of nonlinear composite media [4]. Some of the general properties of such theories are summarized and extended in [5].

Equation (1.1) might serve as a model for many other nonlinear phenomena, such as electrodynamics in very strong fields.

Here I point out that with the special choice of $\mu(x) \propto x^{D-2}$ the theory is a natural generalization of the (linear) two-dimensional Poisson theory. The resulting nonlinear theory is unique in certain regards. Foremost is its conformal invariance. This enables one to say much about the theory and its solutions—much beyond what is possible for the general case. The theory seems to be the only one derivable from a CI action that contains only first derivatives of the potential, with the source ρ coupled directly to the potential, i.e., with an interaction Lagrangian of the form $\rho f(\varphi)$.

In the modified dynamics discussed as an alternative to dark matter, phenomenology requires just this CI behavior in three dimensions in the limit of very small $|\nabla\varphi|$ (see [2,3]). Our results here apply then in the large-distance limit of this theory.

In material media, nonlinearities of the response coefficient appear at high values of $|\nabla\varphi|$. Our results might then apply in the short-distance limit. So, for example, our results for point charges will be valid when charges are very near each other, and those for the fields at short distances from the sources.

The present theory constitutes an instance of a highly nonlinear theory that can be solved for nontrivial configurations due to the symmetry.

I shall present two types of results: one concerns solutions for the potential field for various charge distributions obtained by conformal transformations from highly symmetric

ones; this I do in Sec. III, after discussing some general properties of the theory in Sec. II.

The other type of result concerns systems of point charges. The dynamics of these is governed by an energy function that depends on the charges and their positions. It turns out that while the theory is invariant under conformal transformations, the energy—surprisingly perhaps—is not invariant under a conformal transformation of the positions of the point charges. This can be seen already in the two-dimensional case, which is exactly solvable, where the energy of a system of charges q_i at positions \mathbf{r}_i is $E = \sum_{i \neq j} q_i q_j \ln |\mathbf{r}_i - \mathbf{r}_j|$. Under a dilatation $\mathbf{r}_i \rightarrow \lambda \mathbf{r}_i$ we have $E \rightarrow E + \ln \lambda \sum_{i \neq j} q_i q_j = E + (1/2) \ln \lambda [(\sum_i q_i)^2 - \sum_i q_i^2]$, with $a=2$. In the D -dimensional, nonlinear case we do not, in general, have a closed expression for the energy. Still, we shall see that the energy transforms under dilatations as $E \rightarrow E + K \ln \lambda$, with K a function of the charges of the same form, with a value of the power a that depends on D . The nontrivial term in the transformation law comes from the behavior of the self-energies of the charges under dilatation, including the fact that the self-energy of a charge is logarithmic in its size scale. All this is rather transparent in the linear two-dimensional case. There is also an appropriate transformation law of the energy under inversions—the other conformal transformations (and of course, the energy is invariant under translations and rotations, which do not affect the self-energies of charges). I discuss all of this in Sec. IV. Some of the applications to calculating energies and forces are discussed in Sec. V. In Sec. VI, I discuss other field actions for φ , and demonstrate the uniqueness of L^D as a CI Lagrangian. In Sec. VII, I discuss multipotential theories. In Sec. VIII I generalize briefly to nonlinear, CI extensions of Maxwellian electrodynamics in $D > 4$ dimensions. In the last section I make brief comments on possible connections with quantum field theory.

II. GENERAL PROPERTIES

Via the equation

$$\nabla \cdot [(\nabla \varphi)^2]^{D/2-1} \nabla \varphi = \alpha_D G \rho \quad (2.1)$$

a charge distribution $\rho(\mathbf{r})$ in D -dimensional, Euclidean space gives rise to a potential $\varphi(\mathbf{r})$. This field equation is derivable from the action

$$S^D = S_i^D + S_f^D \equiv - \int \rho \varphi d^D r - \frac{1}{D \alpha_D G} \int [(\nabla \varphi)^2]^{D/2} d^D r. \quad (2.2)$$

Here, S_f^D is the field action, S_i^D is the interaction action, G is a coupling constant, and $\alpha_D = 2(\pi)^{D/2} / \Gamma(D/2)$ is the D -dimensional solid angle, introduced here for convenience. For $G > 0$, like charges attract, as in gravity; for $G < 0$ they repel each other, as in electrostatics. The field equation has a unique solution in a volume V bounded by Σ when either φ or the normal component of $[(\nabla \varphi)^2]^{D/2-1} \nabla \varphi$ are dictated on Σ (see, e.g., [3,5]).

Two integral relations satisfied by solutions of the field equation were derived in [5]. The first applies for our class of theories when the total charge vanishes; it then tells us that

$$\frac{1}{\alpha_D G} \int [(\nabla \varphi)^2]^{D/2} d^D r = - \int \rho \varphi d^D r. \quad (2.3)$$

The second relation is an explicit expression for the virial integral \mathcal{V} :

$$\mathcal{V} \equiv \int \rho \mathbf{r} \cdot \nabla \varphi d^D r = (dG)^{-1} |GQ|^d \equiv \mathcal{V}_D(Q) \quad (2.4)$$

[$d \equiv D/(D-1)$], which follows by writing \mathcal{V} as a surface integral at infinity. The virial—which is shown below to control the response of the configuration's energy to rescaling—can then be written in terms of only the total charge of the system.

While I shall work in Euclidean space with its specific conformal transformations, it is useful to formulate the problem for curved space. The covariant form of the action is

$$S^D = - \int g^{1/2} \rho \varphi d^D r - \frac{1}{D \alpha_D G} \int g^{1/2} (g^{ij} \varphi_{,i} \varphi_{,j})^{D/2} d^D r, \quad (2.5)$$

giving rise to the field equation

$$[(g^{ij} \varphi_{,i} \varphi_{,j})^{D/2-1} g^{km} \varphi_{,k}]_{;m} = \alpha_D G \rho. \quad (2.6)$$

Above, g_{ij} is the metric, g^{ij} its inverse, $g = |\det(g_{ij})|$, and summation over repeated indices is understood everywhere. The density ρ is defined so as to be a coordinate scalar: the charge within a volume V is $\int_V g^{1/2} \rho d^D r$. So, for example, for a system of point charges q_i at \mathbf{r}_i ,

$$\rho(\mathbf{r}) = g^{-1/2} \sum_i q_i \delta^D(\mathbf{r} - \mathbf{r}_i). \quad (2.7)$$

Using usual derivatives instead of covariant ones Eq. (2.6) reads

$$g^{-1/2} [g^{1/2} (g^{ij} \varphi_{,i} \varphi_{,j})^{D/2-1} g^{km} \varphi_{,k}]_{,m} = \alpha_D G \rho. \quad (2.8)$$

The field stress tensor is defined as the functional derivative of the field action with respect to the metric; i.e., under a small change δg_{ij} in the metric

$$\delta S_f^D = \frac{1}{2} \int g^{1/2} \delta g_{ij} \mathcal{P}^{ij} d^D r, \quad (2.9)$$

giving

$$\mathcal{P}^{ij} = - \frac{1}{D \alpha_D G} (\varphi_{,k} \varphi_{,k})^{D/2} \left(g^{ij} - D \frac{\varphi_{,i} \varphi_{,j}}{\varphi_{,m} \varphi_{,m}} \right), \quad (2.10)$$

which has a vanishing trace. (The metric does not appear in the interaction part—because $g^{1/2} \rho$ depends only on the charges degrees of freedom—which, thus, does not contribute to \mathcal{P} .) The tracelessness results from the conformal invariance of the action S_f^D , as is well known (see below). For the Euclidean case the stress tensor becomes

$$\vec{\mathcal{P}} = -(D \alpha_D G)^{-1} |\nabla \varphi|^D (1 - D \mathbf{n} \otimes \mathbf{n}), \quad (2.11)$$

with $\mathbf{n} = \nabla \varphi / |\nabla \varphi|$.

In this flat case, the stress tensor gives the force on any volume V , bounded by the surface Σ on which $\rho=0$ as

$$\mathbf{F} \equiv - \int_V \rho \nabla \varphi d^D r = - \int_{\Sigma} \vec{\mathcal{P}} \cdot d\mathbf{s}. \quad (2.12)$$

(Compare with the expression of the force as a surface integral in [2].)

A. Conformal coordinate transformations

The crux of this paper is that the above theory is invariant under conformal coordinate transformations in D -dimensional space. These are the angle-preserving transformations $\mathbf{r} \rightarrow \mathbf{R}$, for which the metric transforms as

$$g_{ij} \rightarrow \frac{\partial r^k}{\partial R^i} g_{km} \frac{\partial r^m}{\partial R^j} = \lambda^2(\mathbf{r}) g_{ij}, \quad (2.13)$$

corresponding to local rescaling of distances. The Jacobian determinant of the transformation is thus $J = |\partial R / \partial r| = \lambda^{-D}(\mathbf{r})$.

In a flat (Euclidean) ($D > 2$) dimensional space the group of conformal coordinate transformations comprises the rigid transformations (translations, rotations, and reflections) under which the metric does not change, dilatations (rescaling) $\mathbf{r} \rightarrow \lambda^{-1} \mathbf{r}$, with a constant λ , for which $g_{ij} = \delta_{ij} \rightarrow \lambda^2 \delta_{ij}$, and inversions. Under an inversion about a sphere of radius a centered at an arbitrary point \mathbf{r}_0 a point \mathbf{r} is transformed to a point \mathbf{R} on the same ray issuing from \mathbf{r}_0 , with the geometric mean of the distances of \mathbf{r} and \mathbf{R} from \mathbf{r}_0 being a . Explicitly,

$$\mathbf{r} \rightarrow \mathbf{R} = \mathbf{r}_0 + \frac{a^2}{|\mathbf{r} - \mathbf{r}_0|^2} (\mathbf{r} - \mathbf{r}_0). \quad (2.14)$$

The Euclidean metric δ_{ij} then transforms as

$$\delta_{ij} \rightarrow J_{ik}^{-1} \delta_{km} J_{jm}^{-1} = \frac{a^4}{|\mathbf{R} - \mathbf{r}_0|^4} \delta_{ij} = \frac{|\mathbf{r} - \mathbf{r}_0|^4}{a^4} \delta_{ij}, \quad (2.15)$$

where

$$J_{ij} = \frac{\partial R_i}{\partial r_j} = \frac{a^2}{|\mathbf{r} - \mathbf{r}_0|^2} (\delta_{ij} - 2n_i n_j). \quad (2.16)$$

Here, \mathbf{n} is a unit vector in the direction of $\mathbf{r} - \mathbf{r}_0$. The matrix in brackets has eigenvalues 1 ($D-1$ degenerate) and -1 (nondegenerate). The determinant of J_{ij} , in absolute value, is thus

$$J = \frac{a^{2D}}{|\mathbf{r} - \mathbf{r}_0|^{2D}} = \frac{|\mathbf{R} - \mathbf{r}_0|^{2D}}{a^{2D}}. \quad (2.17)$$

All conformal transformations take spheres (hyperplanes included as spheres of infinite radius).

Since the scalar potential transforms as $\varphi(\mathbf{r}) \rightarrow \hat{\varphi}(\mathbf{R}) = \varphi(\mathbf{r}(\mathbf{R}))$, we have $\nabla_r \varphi \rightarrow \nabla_R \varphi[\mathbf{r}(\mathbf{R})] = (\partial r / \partial R) \nabla_r \varphi$, from which it follows that

$$(\nabla_R \hat{\varphi})^2 = \frac{a^4}{|\mathbf{R} - \mathbf{r}_0|^4} (\nabla_r \varphi)^2. \quad (2.18)$$

It is customary to use, instead of pure inversions, transformations of the form $P_{\mathbf{A}} = I_0 T(\mathbf{A}) I_0$, where $T(\mathbf{A})$ is a translation by a vector \mathbf{A} , and I_0 is the inversion at the origin about a sphere of unit length. These have certain advantages: they are connected continuously to unity, and they bring the properties of the conformal group into better relief. I prefer, however, to use pure inversions in what follows, as they are easier to handle: they are self-inverse, and have a simpler transformation Jacobian.

B. Conformal invariance of the theory

If $\mathbf{r} \rightarrow \mathbf{R}$ is a conformal coordinate transformation, our action and the field equation are invariant under replacement of $\varphi(\mathbf{r})$ by $\varphi(\mathbf{r}(\mathbf{R}))$, of $\rho(\mathbf{r})$ by $J^{-1} \rho(\mathbf{r}(\mathbf{R}))$, and of the metric $g_{ij}(\mathbf{r})$ by $g_{ij}(\mathbf{r}(\mathbf{R}))$ (and $d^D r$ by $d^D R$ in the action). This can be checked by direct substitution, but is easier to see as follows: in a general metric space, a theory is conformally invariant if its action is invariant under replacement, everywhere in the action, of g_{ij} by $\zeta^2(\mathbf{r}) g_{ij}$ (and thus of g^{ij} by $\zeta^{-2} g^{ij}$, and of g by $\zeta^{2D} g$), and of ρ by $\zeta^{-D} \rho$ [because of the factor $g^{-1/2}$ in the definition of ρ —see Eq. (2.7)] It is evident from expression (2.5) for the action, or from the field equation (2.8), that ours is indeed a conformal theory by this definition. This implies conformal invariance in the above sense, evident by applying first a conformal coordinate transformation, under which $\varphi(\mathbf{r}) \rightarrow \varphi(\mathbf{r}(\mathbf{R}))$, $\rho(\mathbf{r}) \rightarrow \rho(\mathbf{r}(\mathbf{R}))$, and $g_{ij}(\mathbf{r}) \rightarrow J^{-2D} g_{ij}(\mathbf{r}(\mathbf{R}))$ [see Eq. (2.13)]. The action, being a coordinate scalar, is invariant. Now transform the metric back by multiplying it by the conformal factor $\zeta^2 = J^{2/D}$, and ρ by ζ^{-D} . The action remains invariant by virtue of its CI. The net result is that the action is invariant under the transformation described at the head of this subsection.

It follows from this that if $\varphi(\mathbf{r})$ solves the field equation for the source $\rho(\mathbf{r})$ and metric $g_{ij}(\mathbf{r})$, then $\hat{\varphi}(\mathbf{R}) = \varphi(\mathbf{r}(\mathbf{R}))$ solves it for the source $\hat{\rho}(\mathbf{R}) = J^{-1}(\mathbf{r}) \rho(\mathbf{r}(\mathbf{R}))$, with the same metric $g_{ij}(\mathbf{r}(\mathbf{R}))$. Clearly, equipotential surfaces are transformed into equipotential surfaces. Also, field lines go to field lines, because they are perpendicular to equipotential surfaces and angles are preserved in the transformation. Charges are preserved in the transformation; i.e., the total charge in a certain volume is the same as the transformed charge in the image of that volume.

The tracelessness of the stress tensor follows by employing Eq. (2.9) with $\delta S_f^D = 0$ for $\delta g_{ij} = \epsilon(\mathbf{r}) g_{ij}$, with ϵ an arbitrary (infinitesimal) function.

The application of such conformal invariance is standard in the linear, two-dimensional case (in electrostatics, in potential-flow problems, etc.). In D dimensions such an application has special value because the symmetry is our only recourse in solving some of the problems in this strongly nonlinear theory, as I do in Secs. III–V.

The covariant Laplace (free) action, $\int g^{1/2} g^{ij} \varphi_{,i} \varphi_{,j} d^D r$, is not CI in the above sense, but can be made so by adding to the above action a term proportional to $R\varphi$, with R the scalar curvature, and taking φ to have nonzero dimension, so that it transforms as $\varphi \rightarrow \lambda^{-(D/2-1)} \varphi$ (see e.g., [6]). The Euclidean Laplace theory thus becomes CI with φ of dimension $D/2-1$ (as the term with the curvature vanishes), but then the CI of the interaction term $\int g^{1/2} \rho \varphi$ is lost. What is special about our theory, and what leads to the applications below, is

the fact that it is a CI theory in the presence of sources.

Hereafter I confine myself to the Euclidean case. In curved spaces that are conformally flat, such as maximally symmetric spaces, conformal invariance implies the existence of coordinates in which the theory takes the Euclidean form, with g_{ij} replaced by δ_{ij} everywhere.

C. Asymptotic behavior of the potential

If the sources ρ are contained within a finite volume, and sum up to a total charge $Q \neq 0$, the field becomes radial at infinity, and, applying Gauss's theorem to the field equation for a sphere of a large radius, r , we find asymptotically

$$\nabla\varphi \approx s(QG)|GQ|^{1/(D-1)}r^{-1}\mathbf{n}_r. \quad (2.19)$$

Here $s(x) = \text{sgn}(x)$, and \mathbf{n}_r is an out-pointing, radial unit vector. The potential is then logarithmic for any dimension. When $Q=0$, the asymptotic behavior of φ is determined by higher multipoles. Typically, a dipole potential dominates asymptotically, and has the form $\varphi \propto z/r^2$ (see below) with z the axis along the dipole. Outside a spherical distribution of zero total charge the field vanishes.

D. Scaling properties

The field equation enjoys a two-parameter family of scaling invariances: If $\varphi(\mathbf{r})$ solves the equation for a source $\rho(\mathbf{r})$, then, for any two constants a and b , $\hat{\varphi}(\mathbf{r}) \equiv a^{-1}|a/b|^d \varphi(b\mathbf{r})$ solves it for $\hat{\rho}(\mathbf{r}) = a\rho(b\mathbf{r})$, where $d \equiv D/(D-1)$. When $b^D = a > 0$, so that the total charge remains the same, the scaled potential is $\hat{\varphi}(\mathbf{r}) = \varphi(b\mathbf{r})$.

It follows then that the potential, the electric field, the forces, etc. scale simply with charge: if $\rho \rightarrow a\rho$, then $\varphi \rightarrow s(a)|a|^{1/(D-1)}\varphi$, and forces (which scale like $q\nabla\varphi$) $\mathbf{F} \rightarrow |a|^d \mathbf{F}$. These quantities also scale with system size.

III. EXACT SOLUTIONS FOR THE FIELD

Only few charge configurations with exact solutions are known for the general case with an arbitrary form of $\mu(x)$ in Eq. (1.1) [5]. In particular, there is a closed-form solution for any configuration with one of the D one-dimensional symmetries: plane-parallel, cylindrical, . . . , spherical: by applying Gauss's theorem we get for the present theory

$$\frac{d\varphi}{dR} \propto [q(R)]^{1/(D-1)} R^{-s/(D-1)}, \quad (3.1)$$

where R is the only coordinate on which φ depends, q is the accumulated charge, and $s=0$ for the plane-parallel case, $s=1$ for the cylindrical case, etc. For a spherical system $s=D-1$, and we have

$$\frac{d\varphi}{dr} = \frac{[Q(r)]^{1/(D-1)}}{r}, \quad (3.2)$$

where $Q(r)$ is the accumulated charge at spherical radius r (here and in the rest of the section I use $G=1$). In the plane-parallel case

$$\frac{d\varphi}{dz} = [\alpha_D \Sigma(z)/2]^{1/(D-1)}, \quad (3.3)$$

where $\Sigma(z)$ is the total surface density to the left (small- z) of z minus that to its right.

These solutions, and others, may be used to generate new ones by applying conformal transformations to the corresponding charge configuration. Some examples follow.

A. Two opposite point charges $\pm q$ at \mathbf{r}_1 and \mathbf{r}_2

Start with a point charge $q > 0$ at \mathbf{r}_1 and a spherical shell evenly charged with charge $-q$, centered at \mathbf{r}_1 , and having a very large radius (infinite in the limit). Upon inversion about a sphere of radius $a = |\mathbf{r}_1 - \mathbf{r}_2|$ centered at \mathbf{r}_2 the large spherical shell is transformed into a point charge $-q$ at \mathbf{r}_2 , and the charge q stays at \mathbf{r}_1 . The potential for the original, spherically symmetric system is $\varphi(\mathbf{r}) = q^{1/(D-1)} \ln|\mathbf{r} - \mathbf{r}_1|$ inside the spherical shell, and $\varphi=0$ outside. It transforms into

$$\varphi(\mathbf{r}) = q^{1/(D-1)} \ln \frac{|\mathbf{r} - \mathbf{r}_1|}{|\mathbf{r} - \mathbf{r}_2|} \quad (3.4)$$

(after subtraction of the constant $q^{1/(D-1)} \ln|\mathbf{r}_2 - \mathbf{r}_1|$); this applies everywhere. Interestingly, this potential is just the sum of the potentials of the two individual charges. This happens to be the case only for two opposite charges. It holds neither for two charges that are not opposite, nor for more than two charges.

B. The pure-dipole field

Asymptotically, at $r \gg \ell$, where ℓ is the dipole separation, the potential in Eq. (3.4) becomes

$$\varphi \approx -q^{1/(D-1)} \ell \frac{z}{r^2}, \quad (3.5)$$

where z is the dipole axis (positive charge to the positive- z side). This is the potential for a pure dipole of strength $q\ell^{D-1}$. It describes the field everywhere in the limit $\ell \rightarrow 0$ with $q\ell^{(D-1)}$ constant. (For $D > 2$, a standard dipole with $\ell \rightarrow 0$ and $q\ell$ finite does not contribute to the dipole field, due to self-screening effects.) The pure dipole potential is a vacuum solution of the field equation that is obtained from another vacuum solution: a constant-gradient field; the latter has the potential $\varphi \propto z$, which transforms into z/r^2 .

The dipole field has a field strength, $|\nabla\varphi| \propto r^{-2}$, that depends only on r —not on the angular coordinates. This is a well-noticed property of the dipole field in two dimensions. Here it follows directly from the transformation law (2.18) for $|\nabla\varphi|$, and the fact that the dipole field is obtained by inversion from a constant-gradient field.

For a bounded density distribution of a vanishing total charge the asymptotic behavior of the field is, generically, dominated by a dipole field Az/r^2 . I have not been able to express A as a functional of the density distribution.

C. Point charge in the presence of a grounded sphere

If, in the above example, we take the charged spherical shell to have a finite radius, we end up with a point charge q in the presence of a grounded sphere (as the potential on the original sphere vanishes). By a proper choice of the inversion radius and center, the image point charge falls inside,

or outside, the image sphere. In the first case the potential inside the sphere is that of two opposite charges, and vanishes outside; the tables are turned when the charge falls outside the sphere. The charge distribution on the grounded sphere is then easily determined.

D. Two oppositely charged spheres

More generally, starting from two oppositely charged ($\pm q$) concentric spheres (for which the potential is constant in the innermost and in the outermost regions, and is $q^{1/(D-1)} \ln r$ in between) we get the potential field of two oppositely charged, equipotential spheres of any size, either nested or detached. When the spheres are nested, the potential in the inner and outer parts is still constant; in between it is of the form (3.4). When the spheres are detached the potential is constant inside the spheres, and is of the form (3.4) outside.

Starting with two parallel hyperplanes charged with a constant surface density $\pm \Sigma$ (between which $\nabla \varphi$ is constant), and inverting about a point half-way between the planes, we obtain two oppositely charged, tangent spheres. The potential vanishes inside the spheres; outside we have an exact dipole potential. The charge distribution on each sphere—straightforwardly calculated—diverges at the origin and together the charges give a dipole of finite strength.

E. Some general comments on potential fields

Since equipotential surfaces remain so when transformed, and since spheres go to spheres, the equipotential surfaces in all the above examples are spheres (all tangent in the case of a point dipole). The field lines are all circles, being images of circles or straight lines. For example, for a finite-separation dipole the field lines are all the circles going through the two charges.

There are constraints on the field that can be deduced even when the full field cannot be calculated. As an example consider a charge distribution that lies on a circle (with $Q=0$). The field $\nabla \varphi$ at any point \mathbf{r} must be tangent to any sphere, of any dimension, containing \mathbf{r} and the circle (because the sphere can be transformed into a plane by inversion about a point on it). This provides some information on the field of any three-point charge configuration, or on that of a square quadropole.

Other vacuum solutions of the field equation can be formed by starting from the known, exact solutions of one-dimensional symmetry (uniformly charged, one-dimensional wire, two-dimensional plane, etc.). As an example take a one-dimensional wire in three dimensions with a constant line density σ . Working in cylindrical coordinates R, z we write the potential as $\varphi = (8\sigma)^{1/2} R^{1/2}$. Inverting about a point off the wire will give the field for certain ring-plus-point-charge configurations. Inverting about a point on the wire gives a configuration whose vacuum solution is $\varphi \propto R^{1/2}(R^2 + z^2)^{-1/2}$. This corresponds to a charge density $\hat{\sigma}(z) \propto z^{-2}$ (and there appears an infinite opposite charge at the origin to compensate the infinite charge of the wire).

For a general charge distribution, the field near an arbitrary point \mathbf{r} , away from charges, is conformally related to an asymptotic field: invert about a very small sphere devoid of

charges, and centered at \mathbf{r} . All the charges are transformed into the small sphere, and the \mathbf{r} goes to infinity. The asymptotic field of the new configuration is related to the field near \mathbf{r} in the original configuration. If $\nabla \varphi(\mathbf{r}) \neq \mathbf{0}$, the image asymptotic field is dominated by a dipole term. In an opposite example, look at the field near the midpoint between two equal point charges, where $\nabla \varphi = \mathbf{0}$; the inverted configuration is a quadropole, with a point charge $-2q$ flanked by two symmetric charges q ; the asymptotic field decreases faster than a dipole field. I have not been able to determine this asymptotic behavior.

IV. MANY POINT CHARGES—GENERAL CONSTRAINTS

N -point-charge configurations—comprising bodies whose extent is much smaller than their separations—afford further application of the conformal symmetry. Take then a system made of N point charges q_1, \dots, q_N at $\mathbf{r}_1, \dots, \mathbf{r}_N$, respectively. The information on the dynamics of the system is encapsulated in the energy function $E(\mathbf{r}_1, \dots, \mathbf{r}_N, q_1, \dots, q_N)$. The energy of a charge distribution may be taken as $-S$. This converges at infinity only when the total charge vanishes. We can still use it when $Q \neq 0$, provided only its changes under change of configuration are needed: only the energies of configurations with the same total charge can be compared (see [2]). Changes in E may be calculated as changes in $-S$. An infinitesimal change $\delta \rho$ in the charge distribution with no net change in the total charge ($\int \delta \rho d^D r = 0$) thus produces a change $\int \varphi \delta \rho d^D r$ in the energy ($\delta \rho$ also induces an increment of φ but the field equation implies that this does not contribute to the increment of the action). The N -point-charge energy, $E(\mathbf{r}_1, \dots, \mathbf{r}_N)$ (suppressing the q variables), is a special case giving the relative energies of the N point charges in different configurations: we are only interested in comparing E values as the charges are moved rigidly to different positions. The energy also diverges (logarithmically) near point charges, but these divergences can be subtracted as self-energies. I, in fact, treat point charges as small but finite bodies, and need not be concerned with such divergences.

The force on the i th charge is given by

$$\mathbf{F}_i = - \frac{\partial E}{\partial \mathbf{r}_i}. \quad (4.1)$$

I now derive an expression for a virial integral defined in Eq. (2.4) that involves only the positions of the charges and the net forces on them, but is oblivious to internal forces and structure (the above virial involves integration inside the charges). To this end consider the contribution \mathcal{V}_i of the i th body occupying the small volume v_i : $\mathcal{V}_i = \int_{v_i} \rho \mathbf{r} \cdot \nabla \varphi d^D r$. Write for \mathbf{r} within the body, $\mathbf{r} = \mathbf{r}_i + \mathbf{h}$, where \mathbf{r}_i is the center of charge, \mathbf{h} is small and will be taken to 0 in the limit. We can then write

$$\mathcal{V}_i = - \mathbf{r}_i \cdot \mathbf{F}_i + \int_{v_i} \rho \mathbf{h} \cdot \nabla \varphi d^D h, \quad (4.2)$$

where \mathbf{F}_i is the net force on the body. The second term in Eq. (4.2) does not necessarily vanish in the limit $\mathbf{h} \rightarrow 0$, because $\nabla \varphi$ inside the body diverges in this limit. Let $\nabla \varphi_i(\mathbf{r})$ be the

field produced by the body when it is the only one present. Write $\nabla\varphi(\mathbf{r}) = \nabla\varphi_i(\mathbf{r}) + \nabla\kappa(\mathbf{r})$, where κ is the increment in the potential due to the presence of the other bodies in the system. (Because of the nonlinearity κ is not just the field produced by all the other charges.) In the limit $\mathbf{h} \rightarrow \mathbf{0}$, $\nabla\varphi_i$ diverges like $|\mathbf{h}|^{-1}$ (from the scaling properties) but $\nabla\kappa$ remains finite. Thus, $\int_{v_i} \rho \mathbf{h} \cdot \nabla\kappa d^D h$ vanishes in the limit, and we are left with $\int_{v_i} \rho \mathbf{h} \cdot \nabla\varphi_i d^D h$. This is just the virial defined for the i th body when it is isolated. Thus, from Eq. (2.4),

$$\int_{v_i} \rho \mathbf{h} \cdot \nabla\varphi d^D h \rightarrow \mathcal{V}_D(q_i) \equiv (dG)^{-1} |Gq_i|^d. \quad (4.3)$$

Putting all the above together we finally get an expression for the reduced virial

$$\begin{aligned} \mathcal{V}_r &\equiv - \sum_i \mathbf{r}_i \cdot \mathbf{F}_i = \mathcal{V}_D(Q) - \sum_i \mathcal{V}_D(q_i) \\ &= (dG)^{-1} |G|^d \left(|Q|^d - \sum_i |q_i|^d \right). \end{aligned} \quad (4.4)$$

Note that the limit of a point charge is gotten from a finite charge distribution $\rho(\mathbf{h})$ by taking the limit $\lambda \rightarrow 0$ of $\lambda^{-D} \rho(\mathbf{h}/\lambda)$; this has enabled us to take the limit of $\int_{v_i} \rho \mathbf{h} \cdot \nabla\kappa d^D h$ to 0. Pointlike bodies of finite, higher multipoles (dipole, quadropole, etc.) cannot be described in this way (for example, for a point dipole $\int_{v_i} \rho \mathbf{h} d^D h$ remains finite in the limit). Our ensuing results are not valid for such bodies.

A. Scaling behavior of the energy function

Expression (4.4) implies then that $E_\lambda \equiv E(\lambda \mathbf{r}_1, \dots, \lambda \mathbf{r}_n)$ satisfies

$$\begin{aligned} \frac{\partial E_\lambda}{\partial \lambda} &= \sum_i \lambda^{-1} [\lambda \mathbf{r}_i \cdot \mathbf{F}_i(\lambda \mathbf{r})] \\ &= \lambda^{-1} \left[\mathcal{V}_D(Q) - \sum_i \mathcal{V}_D(q_i) \right]. \end{aligned} \quad (4.5)$$

Integrating over λ between 1 and λ we get an important homogeneity property of $E(\mathbf{r}_1, \dots, \mathbf{r}_N)$:

$$\begin{aligned} E(\lambda \mathbf{r}_1, \dots, \lambda \mathbf{r}_N) \\ = E(\mathbf{r}_1, \dots, \mathbf{r}_N) + \left[\mathcal{V}_D(Q) - \sum_i \mathcal{V}_D(q_i) \right] \ln \lambda. \end{aligned} \quad (4.6)$$

I shall now derive this transformation law of E in a different way, which illuminates better its origin, and which will be of further use below. This is based on the invariance of the theory under space dilatations—a fact that also underlies the derivation of Eq. (4.4).

Consider first the change in the energy of an arbitrary charge distribution $\rho(\mathbf{r})$ under a dilatation: $\rho(\mathbf{r}) \rightarrow \rho_\lambda(\mathbf{r}) = \lambda^{-D} \rho(\mathbf{r}/\lambda)$. In light of the scaling laws described in Sec. II D, $\varphi(\mathbf{r}) \rightarrow \varphi_\lambda(\mathbf{r}) = \varphi(\mathbf{r}/\lambda)$. Integrating $\delta E / \delta \lambda = \int \varphi_\lambda (\partial \rho_\lambda / \partial \lambda) d^D r$ between 1 and λ gives

$$E(\rho_\lambda) - E(\rho) = \ln \lambda \int \rho(\mathbf{r}) \mathbf{r} \cdot \nabla \varphi d^D r = \mathcal{V}_D(Q) \ln \lambda, \quad (4.7)$$

where Q is the total charge, and I have used Eq. (2.4) for the virial integral. This is the result that underlies all our findings below. It says that the virial integral is the single system parameter that determines the variations in energy under scaling transformations.

What is the change in E when the charges are moved from positions \mathbf{r}_i to positions $\lambda \mathbf{r}_i$? This can be achieved in two steps: first apply a space dilatation to the charge distribution. The centers of charge are then moved to the new positions. But also, the charges themselves (taken as very small but finite bodies) are dilated by the same factor; this is more than we want, as we need to move the charges *rigidly* to the new positions. After this first step we have from Eq. (4.7)

$$\tilde{E}(\lambda \mathbf{r}_1, \dots, \lambda \mathbf{r}_N) = E(\mathbf{r}_1, \dots, \mathbf{r}_N) + \mathcal{V}_D(Q) \ln \lambda, \quad (4.8)$$

where \tilde{E} is the energy of the dilated charges at their new configuration.

In the second step we dilate each charge separately by the inverse factor to bring the configuration to the desired one. The energy change in the second step can be calculated in the limit of very small size for the charges. It is then the sum of changes due to separate dilation of the individual charges. This cannot be done when the bodies are not much smaller than their separations, and, as before, does not apply if the bodies have finite, higher multipoles. Using Eq. (4.7) again for the individual charges yields the change $\Delta E = -\ln \lambda \sum_i \mathcal{V}_D(q_i)$. Putting the two together we get Eq. (4.6). The nontrivial transformation properties of the energy function under $\mathbf{r}_i \rightarrow \lambda \mathbf{r}_i$ (even when $Q=0$) thus have to do with the transformation of the self-energies of the point charges.

Invariance under translations implies that E must be a function of only differences of \mathbf{r}_i , such that $E(\mathbf{r}_i + \mathbf{a}) = E(\mathbf{r}_i)$. The derivative of this with respect to \mathbf{a} gives

$$\mathbf{F} \equiv \sum_i \mathbf{F}_i = 0. \quad (4.9)$$

Similarly, invariance under rotations implies that E depends only on scalars, and that the total moment on the system must vanish:

$$\vec{\mathcal{M}} \equiv \sum_i \mathbf{r}_i \otimes \mathbf{F}_i - \mathbf{F}_i \otimes \mathbf{r}_i = \mathcal{O}. \quad (4.10)$$

B. Behavior of the energy under inversions

What does inversion invariance tell us about how E changes under inversions, namely, when moving q_i rigidly from \mathbf{r}_i to \mathbf{R}_i according to Eq. (2.14)? We saw above that if the inversion is not to produce a new charge at the center we must start with a total charge $Q=0$; when $Q \neq 0$ we may annul it by putting a charge $-Q$ at infinity. Then, from conformal invariance, the energy (action) is conserved under an

inversion transformation of the charge distribution. As in the case of dilatations, such a transformation does not just move the charges to their new positions, it also transforms their inner structure, but how? When the charges are of very small size their shape change is determined by the first derivatives $\partial \mathbf{R} / \partial \mathbf{r}$ [Eq. (2.16) with \mathbf{r} and \mathbf{R} interchanged]. This describes a reflection about a hyperplane perpendicular to \mathbf{n} through the body's center, and a dilatation by a factor $a^2 / |\mathbf{r}_i - \mathbf{r}_0|^2 = R_{i0}^2 / a^2$, with $\mathbf{R}_{i0} = \mathbf{R}_i - \mathbf{r}_0$. As before, bringing the charges back to their original size changes the total energy, which leads to

$$E(\mathbf{R}_1, \dots, \mathbf{R}_N) = E(\mathbf{r}_1, \dots, \mathbf{r}_N) - \sum_i \mathcal{V}_D(q_i) \ln \left(\frac{R_{i0}^2}{a^2} \right). \quad (4.11)$$

Again, the fact that E is not invariant under inversion of the positions results from the effect on the self-energy of the charges.

The derivatives of Eq. (4.11) with respect to a^2 and to \mathbf{r}_0 , at fixed \mathbf{R}_i , give sum relations for the forces. The first gives Eq. (4.4) for the reduced virial, again. The second gives the vector sum relation

$$\begin{aligned} -a^2 \sum_i \mathbf{F}_i + \sum_i r_{i0}^2 \mathbf{F}_i - 2 \sum_i (\mathbf{r}_{i0} \cdot \mathbf{F}_i) \mathbf{r}_{i0} \\ + 2 \sum_i \mathcal{V}_D(q_i) \mathbf{r}_{i0} \\ = \mathbf{0}. \end{aligned} \quad (4.12)$$

Equation (4.12) holds for the forces \mathbf{F}_i to which the point charges are subject when at \mathbf{r}_i , for all values of a and \mathbf{r}_0 . Separating the dependence on \mathbf{r} and \mathbf{r}_0 this equation can be written as

$$\mathbf{I} - a^2 \mathbf{F} + \mathbf{r}_0^2 (1 - 2\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{F} + 2V \mathbf{r}_0 + 2\vec{\mathcal{M}} \mathbf{r}_0 = \mathbf{0}, \quad (4.13)$$

where $V \equiv \sum_i [\mathbf{r}_i \cdot \mathbf{F}_i - \mathcal{V}_D(q_i)]$, $\mathbf{n} = \mathbf{r}_0 / |\mathbf{r}_0|$, and

$$\mathbf{I} \equiv \sum_i r_i^2 \mathbf{F}_i - 2 \sum_i (\mathbf{r}_i \cdot \mathbf{F}_i) \mathbf{r}_i + 2 \sum_i \mathcal{V}_D(q_i) \mathbf{r}_i. \quad (4.14)$$

For Eq. (4.13) to hold for any \mathbf{r}_0 and a we must have separately $\mathbf{F} = \mathbf{0}$, $\vec{\mathcal{M}} = \vec{0}$, $V = 0$, and the new sum relation $\mathbf{I} = \mathbf{0}$. The number of such relations totals $D + D(D-1)/2 + 1 + D = (D+1)(D+2)/2$ —and tallies with the dimension of the conformal group. The first three relations were derived above from transformation properties of E under translations, rotations, and dilatations, respectively. Now we see that they all follow solely from the transformation properties under inversions. This need not be surprising: locally, all the former transformations can be obtained from combinations of inversions. Any set of points in a finite volume can be translated, rotated, or dilated by using a succession of inversions alone.

How do the forces on the charges transform under inversion? If \mathbf{F}_i and \mathbf{F}_i^* are the forces in the old and new posi-

tions, respectively, then, from Eq. (4.11), and choosing the origin at the inversion point ($\mathbf{r}_0 = 0$)

$$\mathbf{F}_i^* = - \frac{\partial E(\mathbf{R}_1, \dots, \mathbf{R}_N)}{\partial \mathbf{R}_i} = \frac{\partial \mathbf{r}_i}{\partial \mathbf{R}_i} \cdot \mathbf{F}_i + 2 \mathcal{V}_D(q_i) \frac{\mathbf{R}_i}{R_i^2}. \quad (4.15)$$

V. N -POINT CHARGE CONFIGURATIONS— SOME APPLICATIONS

Energy functions and forces for certain N -point charge configurations can be calculated with the use of the results in the previous section.

A. The two-body system

The force $F(q_1, q_2, \ell)$ between two point charges q_1 and q_2 , a distance ℓ apart (F is positive for attraction) can be obtained from expression (4.4) for the reduced virial: here $\mathbf{F}_1 = -\mathbf{F}_2 = \mathbf{F}$, and the vanishing of the moment implies that \mathbf{F} points from one charge to the other, so that $\mathcal{V}_r = F\ell$, and we can write

$$\begin{aligned} F = \frac{s(G)}{\ell} [\mathcal{V}_D(q_1 + q_2) - \mathcal{V}_D(q_1) - \mathcal{V}_D(q_2)] \\ = \frac{s(G)}{\ell} d^{-1} |G|^{d-1} (|q_1 + q_2|^d - |q_1|^d - |q_2|^d) \end{aligned} \quad (5.1)$$

[$d \equiv D/(D-1)$]. This result was derived in [7] for the three-dimensional case in a roundabout manner. [For $D=2$ expression (5.1) reduces to the standard two-dimensional linear-medium result $F = Gq_1q_2/\ell$.] For example, for two equal charges $q_1 = q_2 = q$, $F = 2s(G)\ell^{-1}d^{-1}|G|^{d-1}|q|^d(2^{d-1} - 1)$. For opposite charges, $q_1 = -q_2 = q$, $F = -2s(G)\ell^{-1}d^{-1}|G|^{d-1}|q|^d$, also to be gotten from the force-transformation law (4.15) starting with one charge at infinity, and hence $\mathbf{F}^* = \mathbf{0}$.

The energy function in the two-body case is

$$E(\mathbf{r}_1, \mathbf{r}_2, q_1, q_2) = \beta_{12} \ln |\mathbf{r}_1 - \mathbf{r}_2|, \quad (5.2)$$

where

$$\beta_{ij} \equiv \mathcal{V}_D(q_i + q_j) - \mathcal{V}_D(q_i) - \mathcal{V}_D(q_j). \quad (5.3)$$

B. The three-body system with vanishing total charge

Consider three charges q_i at \mathbf{r}_i , with $q_1 + q_2 + q_3 = 0$. Perform an inversion with one of the \mathbf{r}_i 's as center, say \mathbf{r}_3 . Then, q_3 is transformed to infinity, and $q_{1,2}$ are transformed to $\mathbf{R}_{1,2}$. The force on q_1 , say, can be calculated in the new configuration from the two-body force formula Eq. (5.1). From this the force \mathbf{F}_1 in the original three-body configuration is calculated by employing the force-transformation law Eq. (4.15) to obtain

$$\mathbf{F}_1 = \beta_{12} \frac{\mathbf{r}_2 - \mathbf{r}_1}{r_{12}^2} + \beta_{13} \frac{\mathbf{r}_3 - \mathbf{r}_1}{r_{13}^2}. \quad (5.4)$$

(This could also be derived from the above constraints on the forces $\mathbf{F} = \mathbf{0}$, $\vec{\mathcal{M}} = \vec{0}$, $V = 0$, $\mathbf{I} = \mathbf{0}$.) Interestingly, the force is the sum of the two forces that would have been exerted by

q_2 and q_3 separately, the nonlinearity notwithstanding. Integrating Eq. (5.4) over \mathbf{r}_1 we get the explicit form of the three-body energy function (for the zero-total-charge case):

$$E(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \ln[r_{12}^{\beta_{12}} r_{13}^{\beta_{13}} r_{23}^{\beta_{23}}]. \quad (5.5)$$

C. The virial theorem

I now derive the analogue of the standard virial theorem, $\langle E_k \rangle = -\langle E_p \rangle / 2$, relating the mean kinetic energy and mean potential energy of an N -body system held together by Newtonian gravity in three dimensions. Consider a bound system made of any number of point charges q_i , of masses m_i , moving under the sole influence of the φ field they produce (other forces may act inside each body to hold it together). The center-of-mass acceleration of each body $\ddot{\mathbf{r}}_i = \mathbf{F}_i / m_i$, with the \mathbf{F}_i satisfying Eq. (4.4). Now

$$\begin{aligned} \mathcal{V}_r = - \sum_i \mathbf{r}_i \cdot \mathbf{F}_i &= - \sum_i m_i \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i = - \frac{1}{2} \frac{d^2}{dt^2} \left[\sum_i m_i \mathbf{r}_i^2 \right] \\ &+ \sum_i m_i \dot{\mathbf{r}}_i^2. \end{aligned} \quad (5.6)$$

The first term vanishes in the stationary case (or its long-time average vanishes for a general bound system); the second term equals twice the kinetic energy $E_k = M \langle \mathbf{V}^2 \rangle / 2$, where M is the total mass of the system, and $\langle \mathbf{V}^2 \rangle$ is the mean square velocity. Together with Eq. (4.4) this finally gives the desired virial theorem

$$\begin{aligned} 2E_k = M \langle \mathbf{V}^2 \rangle &= \mathcal{V}_D(Q) - \sum_i \mathcal{V}_D(q_i) \\ &= (dG)^{-1} |G|^d \left(|Q|^d - \sum_i |q_i|^d \right), \end{aligned} \quad (5.7)$$

by which the mean-square velocity depends only on the charges. This is exact for a stationary system; for a general bound system the long-time average of the left-hand side has to be taken.

This relation has been used to estimate the mass of large-scale, cosmological, galaxy filaments, which are approximately two-dimensional Newtonian systems [8]. It has also been used to determine the masses of roundish galaxies, in the modified dynamics, from their observed velocity dispersions, when typical accelerations in them are very low (see [7,9] and references therein). In these cases, where q_i are the constituent masses, and where $N \gg 1$, $\sum |q_i|^d$ can be neglected as it is smaller than $|Q|^2 = M^2$ by a factor $\sim N^{-1/(D-1)}$. In the limit $N \rightarrow \infty$

$$\langle \mathbf{V}^2 \rangle = \frac{D-1}{D} (GM)^{1/(D-1)}. \quad (5.8)$$

D. Symmetric configurations

The force on bodies in some symmetric configurations can be calculated from expression (4.4): consider a configuration comprising a charge q_0 at the center, and n equal charges, q , at positions \mathbf{r}_i that are equivalent with respect to

the center—equivalent in the sense that each of the points \mathbf{r}_i can be interchanged with any other by an element of the symmetry group of the system, H : a rotation, a reflection about the center, a reflection about some hyperplane through the center, or combinations thereof, that is also a symmetry of the system. Examples are the corners of a rectangular hyperbox, the vertices of any perfect solid of dimension D or less (such as a perfect polygon, any hypercube, etc.), the vertices of polygonal prisms of different types, etc. Clearly, the q charges are then all at the same distance from the center, call it r . Also, they are subject to equivalent forces \mathbf{F}_i ; to wit, each of the \mathbf{F}_i 's can be transformed to any other by an element of the symmetry group. In particular, the radial components of these forces $(\mathbf{F} \cdot \mathbf{r})_i$ are all equal, because scalars are invariant under the point group. (The force on the charge at the center vanishes.) This common value can be deduced from Eq. (4.4) (the symmetry automatically ensures that $\mathbf{F} = \mathbf{0}$, $\mathcal{M} = \vec{0}$, and $\mathbf{I} = \mathbf{0}$):

$$\mathbf{F} \cdot \mathbf{r} = - \frac{1}{n} [\mathcal{V}_D(q_0 + nq) - \mathcal{V}_D(q_0)] + \mathcal{V}_D(q). \quad (5.9)$$

When the forces are radial—as when the point are the vertices of a perfect solid—the full force is obtained since in this case $\mathbf{F} \cdot \mathbf{r} = -Fr$ (F is positive when \mathbf{F} acts towards the center).

In the limit $n \rightarrow \infty$, $nq \rightarrow Q$, Eq. (5.9) gives the force on the elements of a spherical shell of any dimension smaller than D (e.g., a ring) having radius r , total charge Q , and a charge q_0 at its center. The force-per-unit-charge on the shell is

$$F = \frac{1}{Qr} [\mathcal{V}_D(q_0 + Q) - \mathcal{V}_D(q_0)]. \quad (5.10)$$

Additional results are described in Appendix A.

E. Point charges in the presence of conducting boundaries

When (equipotential) bodies of infinite conductivity are present, the full conformal symmetry enjoyed by the N -charge problem is destroyed (the boundaries remain in place when applying the transformation to the charges). So, for example, homogeneity is always lost, and the total force now does not vanish in general; rotational symmetry about an arbitrary center is lost, and so the total moment does not vanish, etc. Some symmetry may, however, be left, in which case the corresponding identities are still valid. If the arrangement of conductors is invariant under translations in a certain direction, the component of the total force on the charges in this direction vanishes. If the conductors are spherically symmetric about some center, the total moment on the charges with respect to this center vanishes, and so on. An interesting example involves boundaries that are invariant to rescaling about a certain point (taken at the origin). This happens when for every point \mathbf{r} on the boundary $\lambda \mathbf{r}$ is also on the boundary for every $\lambda \geq 0$ —the boundary is an arbitrary-cross-section cone (including a hyperplane, a corner, etc.). In this case, expression (4.4) for the reduced virial holds with respect to the origin (but $\mathbf{I} = \mathbf{0}$ does not); Q now is the total charge including that on the conductors. When the conductors are grounded they automatically take up a charge

that makes $Q=0$. Thus, for example, the force on a single charge q , in the presence of an arbitrary, grounded, conic conductor is always subject to a force \mathbf{F} , satisfying

$$\mathbf{F} \cdot \mathbf{r} = \mathcal{V}_D(q). \quad (5.11)$$

This can be extended to conductors in the shapes of disks or spherical caps, as they can be transformed into half planes by inversions.

VI. OTHER CONFORMAL ACTIONS?

Rotation invariance dictates that an action containing only first derivatives of φ is a function of $(\nabla\varphi)^2$, and scale invariance further requires that it be of the form $S^p \propto \int [(\nabla\varphi)^2]^p d^D r$. By themselves, all such actions are invariant under the scale transformation $\varphi(\mathbf{r}) \rightarrow \hat{\varphi}(\mathbf{r}) = \lambda^\alpha \varphi(\lambda\mathbf{r})$, with $\alpha = (D/2p) - 1$, namely, with φ having conformal dimension α . If it is to be fully CI, S^p must also be invariant under inversion at the origin: $\varphi(\mathbf{r}) \rightarrow \hat{\varphi}(\mathbf{r}) = (a/r)^{2\alpha} \varphi(a^2\mathbf{r}/r^2)$. This can be shown not to be the case unless either $p=1$ (the linear case), or $p=D/2$, which gives our action S_f^D . To see this note that starting with $\varphi \propto r^{-(D-2p)/(2p-1)}$, which is the only spherical vacuum solution of the theory (beside $\varphi = \text{const}$), the corresponding $\hat{\varphi}$ is not a solution, unless $p=1, D/2$. Similarly, if we start with $\varphi \propto z$, which is a vacuum solution for all S^p , the corresponding $\hat{\varphi}$ is not, unless $p=1, D/2$.

Field theory lore has it that scale-invariant theories tend to be CI, but this is not a theorem (see, e.g., [10], and references therein). The above nonlinear theories constitute counterexamples.

For the interaction action to be invariant we need φ to transform according to $\varphi(\mathbf{r}) \rightarrow \varphi(\mathbf{R}(\mathbf{r}))$, since $\rho d^D r$ is invariant; i.e., φ is then of conformal dimension zero. As explained in Sec. II B, the Poisson action in $D > 2$ is not CI because for the free action to be invariant φ has to have dimension $D/2 - 1$. We are thus left with S^D as the only CI action containing only first derivatives in the field part.

Consider now actions with a field part containing higher derivatives written for curved space

$$S = - \int g^{1/2} L(g^{ij}, \mathcal{D}\varphi, \dots, \mathcal{D}^k \varphi) d^D r - \int g^{1/2} \rho \varphi d^D r. \quad (6.1)$$

The field Lagrangian L depends on covariant derivatives of φ : $\varphi_{;i_1 \dots i_m}$, $1 \leq m \leq k$ (collectively designated $\mathcal{D}^m \varphi$), and is a coordinate scalar. Scale invariance alone is tantamount to the homogeneity requirement

$$L(g^{ij}, \lambda \mathcal{D}\varphi, \dots, \lambda^k \mathcal{D}^k \varphi) = \lambda^D L(g^{ij}, \mathcal{D}\varphi, \dots, \mathcal{D}^k \varphi). \quad (6.2)$$

This is because coordinate invariance dictates that every derivative in L is contracted with another by one g^{ij} . As φ has zero dimension, under dilatations, $\mathbf{r} \rightarrow \lambda^{-1} \mathbf{r}$, an m th covariant derivative is multiplied by λ^m , which is the same as multiplying g^{ij} by λ^2 , and conformal invariance tells us that $L(\lambda^2 g^{ij}, \dots) = \lambda^D L(g^{ij}, \dots)$.

Consider, hereafter, the Euclidean version of the action (where g_{ij} is put to δ_{ij} , and \mathcal{D} to ∂). As stated above, scale invariance does not ensure full CI. The quadratic Lagrangians—leading to linear field equations—with φ of zero scaling dimension $L = \varphi \Delta^{D/2} \varphi$ (for even D) are CI. The nonlinear actions with φ of zero scaling dimensions are probably not. In fact, I have not been able to find any that is (without having a general proof that none is). For example, I have shown that all the Lagrangians, in even D dimensions, of the form

$$L = [(\nabla\varphi)^2]^m (\Delta\varphi)^k, \quad (6.3)$$

with $m = D/2 - k$, and with $k=1$ for $D \geq 4$, $k=2$ for $D > 4$, or $k \geq 3$ for $D \geq 2k$, which are scale invariant, are not CI.

The homogeneity condition (6.2) implies that the field equation necessarily has a vacuum, spherically symmetric solution of the form $\varphi = A \ln r$. The Euler-Lagrange equation can be written in the form $\partial_i J_i = \rho$, where J_i is a vector that is a function of the first $2k-1$ derivatives of φ , with the homogeneity property [easily derived from Eq. (6.2)]

$$J_i(\lambda \partial\varphi, \dots, \lambda^{2k-1} \partial^{2k-1} \varphi) = \lambda^{D-1} J_i(\partial\varphi, \dots, \partial^{2k-1} \varphi). \quad (6.4)$$

When φ depends only on the radial coordinate r , \mathbf{J} has only an r component, and, from Eq. (6.4), for $\varphi = A \ln r$, $J_r = r^{-(D-1)} j(A)$. Since in the spherical case $\partial_i J_i = r^{-(D-1)} \partial_r [r^{D-1} J_r]$, clearly $\varphi = A \ln r$ is a solution. The coefficient A is determined via the Gauss theorem: $j(A) \propto Q$, where Q is the total charge at the center. If L is homogeneous in φ of degree β , then J is homogeneous of degree $\beta-1$ in φ , and then the coefficient A is given by $A \propto \text{sgn}(Q) |Q|^{1/(\beta-1)}$.

The purely logarithmic potential outside a single spherical body is, however, valid for only very specific density runs. When the Lagrangian depends on derivatives up to the k th, the generic, spherically symmetric, vacuum solution is characterized by $2k$ constants, which are determined by the exact density run in the central body. In general, the potential diverges at large r as a power in r . The notion of a point charge is thus not useful as the external solution does not depend only on the total charge. We cannot even speak of “the field of a point charge” as this is not well defined. For example, the quadratic theory with φ of zero dimensions, with $L \propto \varphi \Delta^{D/2} \varphi + A \rho \varphi$, with the field equation $\Delta^{D/2} \varphi \propto \rho$, has D independent spherically symmetric vacuum solutions of the form $\varphi = \text{const}, \ln r, r^{\pm\alpha}$, with $\alpha = 2, 4, \dots, D-2$.

VII. MULTIPOTENTIAL THEORIES

The above theory is straightforwardly extended to describe K (coupled) scalar potentials, φ_a , which couple to K types of charges with densities ρ_a ($a=1, \dots, K$). The action is

$$S = - \int \sum_a \rho_a \varphi_a d^D r - \frac{1}{2\alpha_D} \int \mathcal{L}(x_1, \dots, x_K) d^D r, \quad (7.1)$$

with $x_a \equiv (\nabla\varphi_a)^2$, and $G=1$. Conformal invariance is now equivalent to homogeneity of \mathcal{L} :

$$\mathcal{L}(\lambda x_a) = \lambda^{D/2} \mathcal{L}(x_a). \quad (7.2)$$

(One could actually generalize further by taking $x_{ab} = \nabla \varphi_a \cdot \nabla \varphi_b$ as variables, but I keep to the simpler form.) The K field equations are

$$\nabla \cdot [\mu_a(x_1, \dots) \nabla \varphi_a] = \alpha_D \rho_a, \quad (7.3)$$

with $\mu_a = \partial \mathcal{L} / \partial x_a$, $1 \leq a \leq K$. The spherical vacuum solution for a system with total charges q_a is

$$\varphi_a = s(q_a) Q_a^{1/(D-1)} \ln r, \quad (7.4)$$

with $Q_a \geq 0$, giving $x_a = (Q_a)^{2/(D-1)} r^{-2}$; $s(q_a) Q_a$ may be viewed as the asymptotically observed charges. They are determined from the actual charges, $q_a = \int \rho_a$, as follows: Inserting the above form of φ_a in the Gauss theorem $\mu_a(x_1, \dots) d\varphi_a / dr = q_a r^{-(D-1)}$, and making use of the homogeneity property of the μ_a (derived from that of \mathcal{L}): $\mu(\lambda x_1, \lambda x_2, \dots) = \lambda^{(D/2-1)} \mu(x_1, x_2, \dots)$, we get the K equations:

$$\mu_a(Q_1^{2/(D-1)}, Q_2^{2/(D-1)}, \dots, Q_a^{1/(D-1)}) = |q_a|. \quad (7.5)$$

All our results in the previous sections can be carried through *mutatis mutandis* to this, more general, case. The virial integral, which controls the change in system energy under dilatations, can now be shown to be given by

$$\begin{aligned} \mathcal{V} &\equiv \int \sum_a \rho_a \mathbf{r} \cdot \nabla \varphi_a d^D r \\ &= \frac{D-1}{2} \mathcal{L}(Q_1^{2/(D-1)}, Q_2^{2/(D-1)}, \dots, Q_K^{2/(D-1)}). \end{aligned} \quad (7.6)$$

Using the homogeneity of \mathcal{L} , Eq. (7.2), by which $\mathcal{L}(x_1, \dots, x_K) = (2/D) \sum_a x_a \mu_a$, and with Eq. (7.5), we can also write

$$\mathcal{V} = \frac{D-1}{D} \sum_a |q_a| Q_a^{1/(D-1)}. \quad (7.7)$$

This expression for \mathcal{V} is to replace $\mathcal{V}_D(q)$ in all our results (remember that the q_a 's and Q_a 's are charges of different types of the same body). For example, the powers β_{ij} appearing in the two- and three-body energy functions and defined in Eq. (5.3) are now given by

$$\begin{aligned} \beta_{ij} &= \frac{D-2}{2} \{ \mathcal{L}(\hat{Q}_1^{2/(D-1)}, \dots) - \mathcal{L}((Q_1^i)^{2/(D-1)}, \dots) \\ &\quad - \mathcal{L}((Q_1^j)^{2/(D-1)}, \dots) \}, \end{aligned} \quad (7.8)$$

where Q_a^i , $a=1, \dots, K$, are the ‘‘asymptotic’’ charges for the point body i , and \hat{Q}_a are those for the two bodies i, j taken together, i.e., as calculated from Eq. (7.5) with the charges $q_a = q_a^i + q_a^j$.

As a special case, assume that the theory is invariant under rotations in the internal space of φ 's, i.e., under $\varphi_a \rightarrow \hat{\varphi}_a = O_{ab} \varphi_b$, where O is an orthogonal matrix (the charges are then rotated by the same matrix, leaving

$\sum_a \rho_a \varphi_a$ invariant). This means that \mathcal{L} must be a function of $\sum_a x_a$, and the required homogeneity then dictates that

$$\mathcal{L} = \frac{2}{D} \left(\sum_a x_a \right)^{D/2} \quad (7.9)$$

(the constant in front is chosen to match the single-field case).

Equation (7.5) can now be solved to give the asymptotic charges as

$$Q_a = |q_a|^{D-1} q^{-(D-2)}, \quad (7.10)$$

where $q \equiv (\sum_a q_a^2)^{1/2}$ is the root mean square over all the charge types of a body. The virial now takes the single-field form, only with the system's single charge replaced by its root mean square over all the charge types: $\mathcal{V} = \mathcal{V}_D(q)$.

VIII. VECTOR AND HIGHER-FORM THEORIES

Maxwell's electromagnetism is governed by the action

$$S = -\frac{1}{4} \int g^{1/2} F^{\mu\nu} F_{\mu\nu} d^D r + \int g^{1/2} J^\mu A_\mu d^D r. \quad (8.1)$$

It describes the electromagnetic field $F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$, derived from the vector potential A_μ , in the presence of conserved currents J^μ . The theory is gauge invariant, and is conformally invariant in four dimensions only ($F^{\mu\nu} = F_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu}$, and $J^\mu = g^{-1/2} j^\mu$, where the vector density j^μ contains only matter degrees of freedom, but not the metric). In a vein similar to that in our treatment of the scalar case we can construct nonlinear, CI generalizations for $D > 4$: take the field action to be

$$S_f = - \int g^{1/2} \mathcal{L}(F_{\mu\nu}, g_{\mu\nu}) d^D r, \quad (8.2)$$

where \mathcal{L} is homogeneous of degree $D/2$ in F : $\mathcal{L}(\eta F_{\mu\nu}, g_{\mu\nu}) = \eta^{D/2} \mathcal{L}(F_{\mu\nu}, g_{\mu\nu})$ (and, of course, is a coordinate scalar). It is CI because multiplying $g_{\mu\nu}$ by $\lambda(\mathbf{r})$ in \mathcal{L} is tantamount to multiplying $F_{\mu\nu}$ by $\lambda^{-1}(\mathbf{r})$ (every two lower case indices in $F_{\mu\nu}$ must be contracted using one $g^{\mu\nu}$). A factor $\lambda^{-D/2}$ is then pulled out of \mathcal{L} to cancel the factor from $g^{1/2}$. In $D > 4$ dimensions we may take

$$\mathcal{L} \propto (F_{\mu\nu} F^{\mu\nu})^{D/4}, \quad (8.3)$$

which gives a CI, vector theory when coupled to the currents as above, but there are others. For example, in eight dimensions

$$\mathcal{L} \propto F_{\mu\alpha} F^{\alpha\beta} F_{\beta\gamma} F^{\gamma\mu} \quad (8.4)$$

is such a theory.

More generally, we have linear, gauge-invariant, CI theories in even D dimensions involving an antisymmetric-tensor gauge potential of rank $n = D/2 - 1$ (an n -form potential), $A_{\alpha_1, \dots, \alpha_n}$. The field tensor $H_{\alpha_1, \dots, \alpha_{n+1}}$ is the totally antisymmetrized derivative of A , and the current density is also an n -form, in analogy with the Maxwellian case. The field Lagrangian is $\mathcal{L} \propto H_{\alpha_1, \dots, \alpha_{n+1}} H^{\alpha_1, \dots, \alpha_{n+1}}$, and the interac-

tion Lagrangian is $A_{\alpha_1, \dots, \alpha_n} J^{\alpha_1, \dots, \alpha_n}$. In dimensions higher than $2(n+1)$ we get a CI theory by taking $\mathcal{L}(H)$ that is homogeneous of degree $D/(n+1)$ in H , e.g., $\mathcal{L} \propto (H_{\alpha_1, \dots, \alpha_{n+1}})^{D/2(n+1)}$. There is no known linear, CI generalization in $D > 2(n+1)$ (see, e.g., [11]).

Specializing now to flat spaces, I find that, as in the scalar case, the effect of rescaling on the energy of some current distribution is given by

$$E_\lambda - E = \mathcal{V} \ln \lambda, \quad (8.5)$$

where E_λ is the energy of the rescaled current distribution. Again, \mathcal{V} can be written as a surface integral. For example, for the vector-potential case, where the rescaled current density is $\lambda^{-(D-1)} J^\mu(\mathbf{r}/\lambda)$, we have

$$\mathcal{V} = \int d\sigma_\nu \left(r^\nu \mathcal{L} - 2r^\beta A_{\alpha\beta} \frac{\partial \mathcal{L}}{\partial F_{\nu\alpha}} \right). \quad (8.6)$$

This can be shown to vanish for configurations with spatially bounded currents (bounded in space and time in higher- D Minkowski spaces). We do not have the appropriate analogues of point charges in the scalar case, with finite \mathcal{V} , for which to calculate and manipulate N -point energies. The situation is akin to having, in the scalar case, a system of point bodies of null charge but a finite higher multipole. As explained in Sec. IV, our treatment of point-charge systems does not apply to such systems.

We can still employ the CI to create new solutions of the nonlinear theory from other known solutions. For example, the vector potential $A_\mu = (1/2) B_{\alpha\mu} r^\alpha$, with $B_{\alpha\mu}$ constant and antisymmetric, gives a constant field $F_{\mu\nu} = B_{\mu\nu}$, and is thus a vacuum solution of all the above vector theories in D dimensions. Inversion at the origin, under which $A_\mu(\mathbf{r}) \rightarrow r^{-2} A_\mu(\mathbf{r}/r^2)$ (A_μ is of dimension one in these theories) gives $\hat{A}_\mu = (1/2) B_{\alpha\mu} r^\alpha r^{-4}$, which must then also be a solution.

IX. DISCUSSION

It was pointed out to me by David Kutasov (private communication) that our results for the classical field theory evoke, and might well be related to, results known to hold in conformal field theory. Perhaps there is such a conformal QFT whose classical limit is our theory. Comparing with the definition of a conformal QFT as given, e.g., in [12], to make such a connection we will have to identify the so-termed ‘‘quasiprimary’’ fields of the QFT with $e^{iq\varphi(\mathbf{r})}$, where $\varphi(\mathbf{r})$ is now a quantum field. The N -point-charge energy function is the classical limit of the correlation function of these quasiprimary fields

$$E(\mathbf{r}_1, \dots, \mathbf{r}_N) = \ln \langle e^{iq_1\varphi(\mathbf{r}_1)} \dots e^{iq_N\varphi(\mathbf{r}_N)} \rangle, \quad (9.1)$$

and the constraints I found such as (4.4) and (4.12) are the Ward identities for the correlator. What I then proved amounts to showing that the (anomalous) dimension of the operator $e^{iq\varphi(\mathbf{r})}$ is $\mathcal{V}_D(q)$.

In the vector- and higher-form potential case we cannot identify such an infinite set of ‘‘quasiprimary’’ fields.

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APPENDIX: SOME MORE EXAMPLES OF FORCE CALCULATIONS

The forces on the charges can also be calculated for symmetric configurations of the following type: start with the symmetric configuration described in Sec. V D, with an even number of sites, n . Put the charge at the center to 0. Place an equal number of positive and negative charges $\pm q$ at the sites in such a way that all charges of the same sign are equivalent: every two charges of the same sign can be interchanged by a point symmetry that does not mix charges of a different sign. The corners of a $3-D$ rectangular box, for example, can be decorated in three inequivalent ways that satisfy the above. The vertices of a perfect $4m$ polygon can be decorated in two ways: alternating charges, and in a two-pluses–two-minuses pattern. Here again the forces on all charges, positive as negative, are equivalent (can be transformed to each other by elements of the full, site point group H). Again, $\mathbf{F} \cdot \mathbf{r}$ takes the same value for all the charges. Using expression (4.4) we get

$$\mathbf{F} \cdot \mathbf{r} = \mathcal{V}_D(q). \quad (A1)$$

A charge configuration of the above description results when we apply the method of images to the problem of a charge q in a region bounded by two intersecting, grounded, $(D-1)$ -dimensional hyperplanes. When the angle between the hyperplanes is $\alpha = \pi/m$, the field in the region bounded by the planes is the same as that of a system of images. This has m pairs of charges $\pm q$ arranged alternately on the vertices of a polygon in a configuration as above (the polygon is not perfect but has edges of alternating lengths). Equation (A1) thus gives the radial force on the charge; it is a special case of Eq. (5.11).

Now an example of the use of the constraint $\mathbf{I} = \mathbf{0}$, where \mathbf{I} is defined by Eq. (4.14). Consider a planar, perfect polygon of n equal charges q (a uniform ring in the limit $n \rightarrow \infty$), and a charge $-nq$ on the symmetry axis of the polygon, a distance ℓ from the origin at the polygon’s center. We want the force F on the large charge (which acts along the axis), and the force \mathbf{f} on the small charges. From $\mathbf{I} = \mathbf{0}$ we have

$$F = - \frac{2\mathcal{V}_D(nq)\ell}{r^2 + \ell^2}, \quad (A2)$$

where r is the radius of the polygon. The axial component of the force on the small charges is $-F/n$, and the radial component f_r is gotten from $V = 0$:

$$f_r = -r^{-1} \left[\mathcal{V}_D(q) + \frac{1}{n} \mathcal{V}_D(nq) \frac{r^2 - \ell^2}{r^2 + \ell^2} \right]. \quad (A3)$$

(F and f_r are positive when towards the center.) Since this configuration is obtained by inversion from one in which the

large charge is at the center of the polygon, the above results also follow by applying the force-transformation formula Eq. (4.15) to the results of the symmetric case. Inversion about a point placed on a uniformly charged ring (limit of a polygon)

transforms it into a line with charge density $\rho(x) = \rho_0 [1 + (x/A)^2]^{-1}$, and the point charge at the center can be moved to a point at an arbitrary distance in the symmetry plane of the wire.

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- [1] S. L. Adler and T. Piran, *Rev. Mod. Phys.* **56**, 1 (1984).
[2] J. Bekenstein and M. Milgrom, *Astrophys. J.* **286**, 7 (1984).
[3] M. Milgrom, *Astrophys. J.* **302**, 617 (1986).
[4] R. Blumenfeld and D. J. Bergman, *Physica A* **157**, 428 (1989).
[5] M. Milgrom (unpublished).
[6] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1994).
[7] M. Milgrom, *Astrophys. J.* **429**, 540 (1994).
[8] D. J. Eisenstein, A. Loeb, and E. L. Turner, *Astrophys. J.* **475**, 421 (1997).
[9] M. Milgrom, *Astrophys. J.* **455**, 439 (1995).
[10] J. Polchinski, *Nucl. Phys. B* **303**, 226 (1988).
[11] S. Deser and A. Schwimmer, *Int. J. Mod. Phys.* **B8**, 3741 (1994).
[12] P. Ginsparg, in *Fields, Strings, and Critical Phenomena*, edited by E. Brézin and J. Zinn-Justin (Elsevier, New York, 1989).