Perturbation theory for the δ-correlated model of passive scalar advection near the Batchelor limit

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The third-order correlation function of the scalar field advected by a Gaussian random velocity, with a spatial scaling exponent $2-\epsilon$, and in the presence of a mean gradient, is calculated perturbatively in $\epsilon \ll 1$. This expansion corresponds to the regime close to Batchelor's advection by linear diffeomorphisms. The scaling exponent is found to be equal to 1 in dimensions 2 and 3, up to corrections smaller than $O(\epsilon)$, implying an anomalous scaling of the third-order correlation function and the persistence of small scale anisotropy. [S1063-651X(97)50302-3]

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The investigation of the statistics of the passive scalar field advected by random flow is interesting for the insight it offers into the origin of intermittency and anomalous scaling of turbulent fluctuations. The problem studied in this paper is stated simply by

$$\partial_t \Theta + (\vec{v} \cdot \vec{\nabla}) \Theta = \kappa \nabla^2 \Theta \tag{1}$$

with the scalar field Θ forced by the externally imposed gradient g. It is convenient to subtract out the gradient and study the fluctuating field, $\theta(r) = \Theta(r) - gr$. It turns out that even a Gaussian random, but scale invariant, velocity field results in nontrivial anomalous scaling of the passive scalar structure function, $\langle [\Theta(r) - \Theta(0)]^n \rangle$ for $n > 2$. This has been suggested by Kraichnan [1], on the basis of a closure scheme, for a δ-correlated velocity model where

$$\langle v_a(r,t) v_b(r',t') \rangle = \delta(t-t') C_{ab}(r-r'), \tag{2a}$$

with

$$D_{ab}(r) = C_{ab}(0) - C_{ab}(r) = D_0 \left((d-1+\zeta_v)\delta_{ab} - \zeta_v \frac{r^a r^b}{|r|^2} \right) |r|^{\zeta_v} \tag{2b}$$

(where ζ_v is the scaling exponent and d the space dimension), which he has introduced some 30 years ago [2]. The existence of the anomalous scaling has been demonstrated explicitly by Gawędski and Kupiainen [3], and Chertkov *et al.* [4] for certain limits of this δ-correlated model and by Shraiman and Siggia [5] for a generalized phenomenological model where temporal correlation of the advecting field is set by eddy turnover. These calculations are based on the so-called Hopf equations—the stationarity conditions of the equal-time multipoint correlators. For the white velocity case these can be derived exactly [6–8] extending the original analysis of the two-point function by Kraichnan [2]. They have the form

$$\sum_{i\neq j}^{N} [D_{ab}(r_i - r_j) + \kappa \delta_{ab}] \partial^a_{r_i} \partial^b_{r_j} \langle \theta(r_1) \cdots \theta(r_N) \rangle = \sum_{i\neq j}^{N} g_a g_b C_{ab}(r_i - r_j) \langle \theta \cdots \rangle^{N-2}_{ij} - 2 \sum_{i\neq j}^{N} g_a D_{ab}(r_i - r_j) \partial^b_j \langle \theta \cdots \rangle^{N-1}_i \tag{2c}$$

(with implicit summation over repeating indices). We restrict ourselves to the inertial range of scales, where r is large enough that the molecular diffusivity can be neglected: $r \gg \eta \equiv (\kappa/D_0)^{1/\zeta_v}$.

The analysis of Ref. [3] is based on the expansion of Eq. (2) in $\zeta_v \ll 1$ about the diffusion limit $\zeta_v = 0$, while we consider the complementary limit of $\zeta_v = 2 - \epsilon$, $\epsilon \ll 1$. Reality for the white velocity model, $\zeta_v = 4/3$, lies in between. The expansion in small ϵ is more involved than what was required in Refs. [3,4] for two reasons. There are an infinite number of degenerate modes for $\epsilon = 0$, which are all mixed by the perturbation, which itself is singular [5,9]. That is, the perturbation is formally small because of ϵ, but in certain restricted regions of configuration space it is the biggest term in the equation. It must be treated by the method of matched asymptotic expansions. The exponent we find for the third-order correlator [10,11] $\lambda_3 \approx 1$ implies that the anisotropy introduced by the mean gradient, g on the large scales, decays more slowly as one descends in scale than that predicted by Kolmogorov 1941 (K41) theory [12,13] (which for $\zeta_v = 2 - \epsilon$ predicts an exponent $1 + \epsilon$). Since the experimental exponent is also approximately one [14], it will be of interest to compare also the full coordinate dependence of the three-point correlation function when the latter becomes available from experiment or simulations. One way of expressing our results for this correlation function is as an expansion in the degenerate modes of the $\epsilon=0$ problem. Our matching determines all the coefficients explicitly.

Determination of the anomalous exponents reduces to finding the zero modes of the linear operator entering the Hopf equation [3–5], i.e., the left-hand side of Eq. (2), which in the present model (and $\kappa\to 0$ limit) is the generalized Richardson diffusion operator: $L(d,\zeta_v)\equiv\Sigma_{i\neq j}^{3}D_{ab}(r_i-r_j)\partial_{r_i}^{a}\partial_{r_j}^{b}$. The $\zeta_v=2$ case is the Batchelor limit [15], which is constrained by an overall SL(2)$\times$SO(d) symmetry so that the spectrum of $L_0\equiv L(d,2)$, also referred to here as the Batchelor-Kraichnan operator, can be completely constructed with the help of Lie algebraic methods [5,16]. Here it will serve as a starting point for the perturbation for $\zeta_v=2-\epsilon$:

$$L(d,2-\epsilon)=L_0(d)-\epsilon L_1(d) \tag{3}$$

to the leading order in $\epsilon\ll 1$. (Note that we are ultimately interested in the physical case of $\zeta_v=4/3$.) The perturbation expansion around $\zeta_v=2$ is a singular problem, which, however, can be addressed by the method introduced in Ref. [9], as we explain now.

Let us start with L_0. It is convenient to introduce the variables $\vec{\rho}_1=(\vec{r}_1-\vec{r}_2)/\sqrt{2}$ and $\vec{\rho}_2=(\vec{r}_1+\vec{r}_2-2\vec{r}_3)/\sqrt{6}$. (On occasion we shall refer to $i=1,2$ index labeling the ρ vectors as the ''pseudospace'' index to distinguish it from the d-dimensional real space.) Next we ''factorize'': $\rho_i^a=\Sigma_{i'}R_{ii'}(\chi)\xi_{i'}\eta_{i'}^{a}$, where R represents pseudospace rotations by χ, and $\hat{\eta}_{1,2}$ are two orthogonal unit vectors spanning the space of $\vec{\rho}_1,\vec{\rho}_2$. This factorization is just the singular value decomposition of the ρ_i^a matrix. In $d=3$, we also define $\hat{\eta}_3\equiv\hat{\eta}_1\times\hat{\eta}_2=\vec{\rho}_1\times\vec{\rho}_2/|\vec{\rho}_1\times\vec{\rho}_2|$ each component of which is invariant under the action of SL(2) [9]. Another important invariant is the area of the $\vec{r}_1,\vec{r}_2,\vec{r}_3$ triangle: $\zeta\equiv|\vec{\rho}_1\times\vec{\rho}_2|=\xi_1\xi_2$.

The zero modes of $L_0(d)$ for $d=2,3$ have been constructed in Refs. [5,9]; e.g., in $d=3$, the complete set of eigenfunctions has the form

$$\psi_{\nu,q,l,m,m'}^{\lambda}=e^{iq\chi}\zeta^{\lambda/2}P_{\nu}^{q,m'}(\xi)D_{m,m'}^{l}(\hat{\eta}), \tag{4}$$

where $\xi\equiv(\xi_1^2+\xi_2^2)/2\xi_1\xi_2$, $D_{m,m'}^{l}$ $(\hat{\eta})$ is the matrix element of the representation of the SO(3) group [17] of order l, and $P_{\nu}^{q,m'}(\xi)$ is the Legendre-Jacobi function [18]. (Note that quantum number m' corresponds to rotations of the $\hat{\eta}_i$ triad about $\hat{\eta}_3$ in pseudospace.) To ensure analyticity in the $\zeta=|\vec{\rho}_1\times\vec{\rho}_2|\to 0$ limit (which corresponds to all three points of the correlator being on one line), $\lambda/2-\max(\nu,-\nu-1)$ must be a positive integer. This is because as $\zeta\to 0$, $\xi\sim\zeta^{-1}\to\infty$, and $P_{\nu}^{q,m}(\xi)\sim\xi^{\max(\nu,-\nu-1)}$.

The third-order structure function, or the skewness, which is the physical object of our interest, has odd spatial parity and hence is only nonzero in as much as the mean scalar gradient g introduces a particular direction. Hence the relevant eigenfunctions are the p waves, $l=1$. The zero mode of $L_0(d)$ corresponds to the smallest exponent λ is obtained [5] for $\lambda/2=\nu$ yielding, in the $l=1$ sector, $\lambda=1$ independent of d.

We shall need the explicit form of the L_0 operator:

$$\frac{1}{2d}L_0\psi_{l=1}^{\lambda}(w,\chi,\hat{\eta})=\partial_\xi[(\xi^2-1)\partial_\xi\psi]+\frac{\partial_\chi^2-I_3^2-2i\xi I_3\partial_\chi}{4(\xi^2-1)}\psi-\nu(\nu+1)\psi, \tag{5}$$

where $\nu(\nu+1)\equiv[(d-2)/2d](\lambda^2/2d+\lambda)-[(d+1)/2d]l(l+d-2)$ and $I_3\equiv(1/i)(\hat{\eta}_1\cdot\vec{\partial}_{\eta_2}-\hat{\eta}_2\cdot\vec{\partial}_{\eta_1})$. In agreement with Eq. (5), the ∂_χ and I_3^2 are diagonalized by $\exp(iq\chi)$ and $\hat{\eta}_1\pm i\hat{\eta}_2$, the latter corresponding to the $l=1$, $m'=\pm 1$ sector. Requiring the left-hand side of Eq. (5) to vanish would make it into a Legendre equation [18] with ν and hence λ entering as an eigenvalue.

Next we define the perturbation operator in Eq. (3):

$$L_1(d)=\mathcal{L}+\frac{d-1}{2d}\left(l(l+d-2)+\lambda^2-d\lambda-\frac{1}{d-1}L_0\right), \tag{6a}$$

with

$$\mathcal{L}\equiv\sum_{S_3}-\ln(|\rho_1|)[(d+1)\rho_1^2(\partial_1^a\partial_1^a-\tfrac{1}{3}\partial_2^a\partial_2^a)-2\rho_1^a\rho_1^b(\partial_1^a\partial_1^b-\tfrac{1}{3}\partial_2^a\partial_2^b)]. \tag{6b}$$

In Eq. (6b), the summation extends over all the cyclic permutations of $(\vec{r}_1,\vec{r}_2,\vec{r}_3)$, resulting in the following symmetry for ρ: $\vec{\rho}_1\to-\vec{\rho}_1/2\pm(\sqrt{3}/2)\vec{\rho}_2$ and $\vec{\rho}_2\to\mp(\sqrt{3}/2)\vec{\rho}_1-\vec{\rho}_2/2$. The expansion breaks down for $-\ln|\rho_1|\ll\epsilon^{-1}$. When $\xi\to\infty$, i.e., when the three points $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$ are almost aligned, the operator $\epsilon\mathcal{L}$ becomes much larger than the Batchelor-Kraichnan operator L_0. This can be seen by expanding the full operator in the limit $\xi\to\infty$. Defining for $d=3$ and the $l=1$ sector $\vec{\psi}_{l=1}^{\lambda}\equiv\zeta^{\lambda/2}\vec{\varphi}(\xi,\hat{\eta},\chi)$ (where vector notation reflects the triplet nature of $l=1$ state) one finds

$$\mathcal{L}\vec{\varphi}=-\ln[1-(1-\xi^{-2})\cos(2\chi)][\xi^2\mathcal{L}_2\vec{\varphi}+\xi\mathcal{L}_1\varphi+\mathcal{L}_0\vec{\varphi}+O(1/\xi)]+\left(\chi\to\chi+\frac{2\pi}{3}\right)+\left(\chi\to\chi-\frac{2\pi}{3}\right), \tag{7a}$$

The operators $\mathcal{L}_i$ are

$$\mathcal{L}_2\vec{\varphi}=\tfrac{4}{3}(\cos^2\chi-1)(4\cos^2\chi-1)[-(2\xi\partial_\xi-\lambda)^2+4\hat{\eta}_1\cdot\vec{\partial}_{\eta_1}]\vec{\varphi}\varphi, \tag{7b}$$

$$\mathcal{L}_1\vec{\varphi}=\tfrac{32}{3}\cos\chi\,\sin\chi(\cos^2\chi-1)[(2\xi\partial_\xi-1)\times(\hat{\eta}_2\cdot\vec{\partial}_{\eta_1}-\hat{\eta}_1\cdot\vec{\partial}_{\eta_2})-2\hat{\eta}_2\cdot\vec{\partial}_{\eta_1}]\vec{\varphi}, \tag{7c}$$

and

$$\mathcal{L}_0\vec{\varphi} = -\tfrac{2}{3}(\cos^2\chi - 1)(4\cos^2\chi - 1)(\partial_{\chi^2} + 5)\vec{\varphi} + \tfrac{64}{3}\sin^3\chi\,\cos\chi\,\partial_\chi\vec{\varphi} + \tfrac{4}{3}(1 + 4\cos^2\chi - 8\cos^4\chi)\,\hat{\eta}_1\cdot\vec{\partial}_{\eta_1}\vec{\varphi}. \quad (7d)$$

In Eq. (7d), λ has been set equal to 1—its unperturbed value—since the corrections would be higher order in ϵ. The singular nature of the perturbation follows from the fact that the $\mathcal{L}_2$ term enters the prefactor ξ^2 so that when $\xi \gg (1/\epsilon)^{1/2}$, $\epsilon\mathcal{L} \gg L_0$. This situation calls for the ''boundary layer'' type matched asymptotic analysis, which we outline below.

Let us assume $\lambda = 1 + \epsilon\delta$, define the rescaled ''inner'' variable $z = \epsilon^{1/2}\xi$, and introduce the function $\vec{\varphi}(z,\chi,\eta) = z^{\lambda/2}[\phi_1(z,x)\,\hat{\eta}_1 + i\,\phi_2(z,\chi)\,\hat{\eta}_2]$. The prefactor is chosen to offset the scaling factor $\zeta^{\lambda/2}$ [see Eq. (4)], which vanishes for collinear points. Physics requires that ϕ_i is bounded when $z \to \infty$. With this change of variable and functions, the problem can be written, provided $\chi \gtrsim \xi^{-1}$, as

$$[((z^2\partial_z^2 + 3z\partial_z) + \tfrac{4}{9}z^2U(\chi) \times [(z\partial_z)^2 - \hat{\eta}_1\cdot\vec{\partial}_{\eta_1}]) + \epsilon^{1/2}\hat{L}_1 + \epsilon\hat{L}_2 + \cdots] \times (\phi_1\hat{\eta}_1 + i\,\phi_2\,\hat{\eta}_2) = 0, \quad (8a)$$

with

$$U(\chi) = \{(\cos^2\chi - 1)(4\cos^2\chi - 1)\ln[1 - \cos(2\chi)] + (\chi \to \chi + 2\pi/3) + (\chi \to \chi - 2\pi/3)\}. \quad (8b)$$

The operators $\hat{L}_1$ and $\hat{L}_2$ can be deduced from a systematic expansion of the operator in powers of ϵ starting from Eqs. (7a)–(7d).

Physically interesting solutions must be bounded (but nonzero) in the limit when the points in the correlator approach collinearity, which implies that when $z \to \infty$, the solution is a function of χ only. By direct substitution, one finds that $\phi_1 = 0$ and $\phi_2 = a(\chi)$, where $a(\chi)$ is an unknown function, decomposed for convenience as a Fourier series in χ: $a(\chi)\Sigma_q\hat{a}_q e^{iq\chi}$. When $z \to 0$, the problem reduces to the unperturbed Batchelor-Kraichnan operator, up to small corrections. In the matching region, defined by $z \to 0$ but $\xi \to \infty$, or equivalently, $1 \ll \xi \ll \epsilon^{-1/2}$, the ξ dependence of each Fourier mode in χ, q, must match with the asymptotic behavior of the eigenmodes of L_0 [Eqs. (4) and (5)], which is best found via their integral representation given in Refs. [5,9,17]. One finds that the functions $\phi_{i,q}$ must behave as $\phi_{1,q}\epsilon^{1/2}|q|/2z + \epsilon(1 - q^2)/4z^2 + \cdots$ and $\phi_{2,q} = (1 - \epsilon q^2/8z^2 + \cdots)\mathrm{sgn}(q)$. The crossover equation, Eq. (8), can be solved analytically, to the leading order in ϵ, determining the small z asymptotics of $\phi_{1,2}$ in terms of the, so far, free function $a(\chi)$ controlling the $z \to \infty$ behavior. The imposition of the matching conditions determines $a(\chi)$ via an eigenvalue equation for δ:

$$U(\chi)(\partial_\chi^2 + 1)a(\chi) + 6\,\delta a = 0. \quad (9)$$

The analysis in $d = 2$ can be carried out in an identical way. As in the three-dimensional case, the behavior for $\xi \to \infty$ is of the form $\phi = a_{2d}(\chi)\,\xi^{\lambda/2} + O(\xi^{\lambda/2-1})$, and the function a_{2d} is determined by a matching condition. Surprisingly, the equation determining $a_{2d}(\chi)$, and the correction to the scaling exponent $\delta = (\lambda - 1)/\epsilon$ is identical to Eq. (9).

Before solving Eq. (9), one needs to determine the appropriate boundary conditions. Because the three-point correlation function must be odd under $\vec{\rho}_i \to -\vec{\rho}_i$, it is implied that $a(\chi + \pi) = -a(\chi)$. This, together with the periodicity $a(\chi + 2\pi/3) = a(\chi)$, resulting from the invariance under cyclic permutation of $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$, implies that $a(\chi + \pi/3) = -a(\chi)$. The limit $\chi \to 0$ corresponds to the case where $\vec{r}_1$ and $\vec{r}_2$ come close together: $|\vec{r}_2 - \vec{r}_1| \ll |\vec{r}_3 - \vec{r}_1|, |\vec{r}_2 - \vec{r}_1|$. In this limit, the correlation function must be invariant when $\vec{r}_1$ and $\vec{r}_2$ are permuted, implying that $a(\chi)$ must be even. Since a is even near $\chi = 0$ and a is antiperiodic with period $\pi/3$, $a(\pi/6) = 0$.

At small, but finite ϵ, Eq. (9) reduces for $\chi \to 0$, to

$$-\chi^2\ln(\chi)a''(\chi) + \delta a(\chi) = 0. \quad (10a)$$

Introducing the change of variables $y \equiv -\ln(\chi)$ and $f(y) \equiv a(\chi)$, Eq. (10a) reduces to the following (Kummer) equation:

$$y(f'' + f') + \delta f = 0. \quad (10b)$$

The behavior of the solution when $\chi \to 0$ is

$$f(y) \sim y^{-\delta} \quad \text{when} \quad y \to \infty. \quad (10c)$$

This function diverges (goes to zero) when $\delta < 0$ ($\delta > 0$).

Since $\chi \to 0$ (for $\xi^{-1} = 0$) corresponds to $\rho_1 \to 0$, the perturbation expansion leading to Eq. (9) is valid only for $y = -\ln\chi \ll 1/\epsilon$. Hence, to determine the correct boundary condition as $y \to \infty$ the solution of Eq. (9) must be matched with the ''inner'' solution describing the correlator with two points near coincidence. The latter is governed by the equation derived directly from Eq. (3) by expanding in $\rho_1/\rho_2 \ll 1$ instead of ϵ and is written conveniently in the polar coordinates $|\rho_1/\rho_2|^2 = \xi^{-2}\chi^2/4$ and $\theta = \arctan(2/\xi\chi)$ (restricting here to $d = 2$ for simplicity). The natural radial variable in this ''inner'' equation turns out to be $Y = |\rho_1|^\epsilon$. The region of matching with Eq. (10a) corresponds to $1 - Y \ll 1$ and $\theta = 0$. Quite generally the solution near $Y = 1$ behaves as $A + B(1 - Y)^\alpha$ with $\alpha > 0$ required to keep the solution from diverging. In the matching region $Y \approx 1 + \epsilon\ln(\chi/2)$ so that only the constant term A must be kept when computing to the leading order in ϵ. Comparing with Eq. (10c) one concludes that matching the ''inner'' solution is only possible for $\delta = 0$.

For $\delta = 0$ the solution of Eq. (9) is $a(\chi) = \sin(\pi/6 - \chi)/\sin(\pi/6)$ for $0 < \chi < \pi/3$, which is continued over the full range of χ using reflection symmetry and periodicity defined above. One observes that $a(\chi)$ has an apparent $|\chi|$ singularity near $\chi = 0$ (and other points related by symmetry) which is regularized only for $\chi < e^{-1/\epsilon}$ via the crossover to the ''inner'' solution for nearly coincident points $\rho_1/\rho_2 < e^{-1/\epsilon}$ as discussed above. Note that although to $O(\epsilon)$ there is no correction to the $\lambda = 1$ eigenvalue, the computed eigenfunction is nontrivial: it is a superposition of many $\psi_{1,q}$ modes since $a_q \sim 1/q$ for large q. Also note that the calculations in two and three dimensions are identical and give the same result: $\delta = 0$. Of course one does not expect $\delta = 0$ to persist beyond the leading order considered here. The next order correction, according to Eq. (7), is expected to make $\delta \sim O(\epsilon^{1/2})$.

Thus, the main result of this paper is as follows: the scaling exponent of the $n=3$ structure function behaves as $\lambda=1$, up to corrections of order $\epsilon^{3/2}$. The exponent of the three-point correlation function is therefore smaller than the ''naive'' scaling exponent, equal to $1+\epsilon$, therefore demonstrating that the behavior of the skewness near the Batchelor limit of the Kraichnan's δ-correlated model is anomalous. Dispersion in the presence of a mean gradient has been shown experimentally [19,20] and numerically [21,22] to give rise to strong intermittency effects, resulting in a skewness that remains of order 1, independent of the Reynolds number. As is the case in real flows, it is interesting to notice that even for a white noise velocity field, the anisotropy induced at large scales decays more slowly than predicted by standard phenomenological arguments [12,13,23]. We conclude by mentioning numerical results demonstrating that a large scale anisotropy, such as a large scale shear, imposed on a turbulent velocity field, may also result in a large anisotropy at small scale [24], suggesting also the existence of an anomalous exponent for the $n=3$ structure function of the velocity field.

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