Iterative filtering procedure for the Vlasov equation

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An iterative filtering scheme is used for deriving the evolution of large scales in a plasma. The information lost by filtering out the small-scale fluctuations is accounted for by the introduction of an effective propagator and vertex operators in the Vlasov equation. These renormalizing terms correspond to large-scale diffusive effects. A general expression for the fluxes of energy and particles is obtained. The transport coefficients are explicitly derived for the quasilinear limit and the guiding center approximation. [S1063-651X(97)06201-6]

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I. INTRODUCTION

The Vlasov equation (VE) is the standard tool for studying a plasma when the collective effects dominate the collisional interactions between particles. Within the collisionless limit, this equation gives the evolution of the probability distribution $F_\alpha(\mathbf{r},t,\mathbf{v})$ of a particle of species α at position $\mathbf{r}$ and time t with a velocity $\mathbf{v}$:

$$\left[\partial_t+\mathbf{v}\cdot\nabla+\frac{e_\alpha}{m_\alpha}\left(\frac{\mathbf{v}\wedge\mathbf{B}}{c}+\mathbf{E}\right)\cdot\partial_{\mathbf{v}}\right]F_\alpha(\mathbf{r},t,\mathbf{v})=0, \qquad (1)$$

where $\mathbf{B}$ and $\mathbf{E}$ are, respectively, the magnetic and the electric fields. The mass and the charge of the particles of species α are denoted, respectively, m_α and e_α. Equation (1) is nonlinear because the electromagnetic field is related to the probability distribution $F_\alpha(\mathbf{r},t,\mathbf{v})$ through the Maxwell equations. The search for analytical solutions of the nonlinear VE is then a very difficult problem [1,2]. Apart from a perturbative treatment, it is usually not possible to obtain analytical results for Eq. (1). Actually, the study of this equation is already complicated when the electromagnetic field is assumed to be known independently of the distribution function (non-self-consistent treatment). Also, the numerical simulations of this equation require rather massive computational resources. For these reasons, the derivation of well-defined approximation schemes starting from the VE and leading to a simplified description of the plasma would be most welcome.

The purpose of this work is to reach such a simplified description by building up an analytical scheme to separate the large scales from the small ones in the VE. Indeed, in many situations the details of the small-scale fluctuations are somewhat unimportant from a practical point of view and only the knowledge of the large-scale physics is really required [3]. This idea has already prompted the development of the large eddy simulations in fluid turbulence [4]. In this work, the result of the overall procedure will be a spatially filtered VE that allows a simpler treatment. In Fourier space, the small-scale elimination corresponds to a reduction of modes needed to characterize the one-particle distribution function. In real space, this function is ''smoother'' and can be accurately described by a smaller set of data.

This will be particularly useful for studying problems related to microturbulence in plasma [5]. Indeed, if the stochastic electromagnetic field drives the plasma in a turbulent state, anomalous transport phenomena are observed even in a collisionless plasma. The large-scale behavior of the plasma is then dominated by these transport phenomena.

The procedure used to filter the VE is based on an iterative scheme that shares some of the features of the renormalization-group techniques developed for dynamical equations [6–9]. These techniques differ from usual renormalization schemes [10–12] in that the simplification comes from the small-scale elimination instead of a statistical truncation of the highest moments of the distribution function. In order to avoid too involved formalisms, we limit the present study to the non-self-consistent case. The filtering technique is thus formally similar to the procedure used for investigating the large-scale dynamics of a passive scalar [13]. It will be shown that the results are also comparable to those obtained for the non-self-consistent gyrokinetic equation [14]. However, the VE is more complex due to the dependence on the velocity derivatives.

In Sec. II the VE and the statistics of the external electromagnetic field are introduced. The systematic procedure used for filtering the small scales is presented in Sec. III. The resulting filtered and renormalized Valsov equation (FRVE) is derived in Sec. IV together with general expressions for the anomalous fluxes. Two particular limits of the FRVE are then discussed: the quasilinear limit (Sec. V) and the guiding center approximation (Sec. VI).

II. SYSTEM DESCRIPTION

As announced in the Introduction, we restrict our scope to situations where the self-consistent electromagnetic field can be either neglected or approximated by an external stochastic field. In that case, the distribution functions for different species (typically $\alpha=i$ for ions and $\alpha=e$ for electrons) are not coupled. For that reason, the explicit species dependence is omitted in the following discussions. Also, the plasma is considered to be embedded in an external electromagnetic field consisting of a constant magnetic component directed along the z axis, $\mathbf{B}=B\mathbf{b}$ and a stochastic electric field $\mathbf{E}$. The introduction of a stochastic variable in the VE suggests the following decomposition of the distribution function F into an average part and a fluctuation:

$$F(\mathbf{r},t,\mathbf{v})=\mathcal{F}(\mathbf{r},t,\mathbf{v})+f(\mathbf{r},t,\mathbf{v}), \tag{2}$$

where $\mathcal{F}$ represents the average of the distribution function $\mathcal{F}\equiv\langle F(\mathbf{r},t,\mathbf{v})\rangle$. The equation for the fluctuation is obtained from Eq. (1):

$$[\partial_t+\mathbf{v}\cdot\nabla+\Omega(\mathbf{v}\wedge\mathbf{b})\cdot\partial_{\mathbf{v}}]f(\mathbf{r},t,\mathbf{v})$$

$$=-\frac{e}{m}\partial_{\mathbf{v}}\mathcal{F}\cdot\mathbf{E}(\mathbf{r},t)-\frac{e}{m}\partial_{\mathbf{v}}\cdot\delta[f(\mathbf{r},t,\mathbf{v})\mathbf{E}(\mathbf{r},t)], \tag{3}$$

where δ represents the fluctuating part of a quantity ($\delta[x]\equiv x-\langle x\rangle$) and Ω is the gyrofrequency. The iterative filtering procedure developed in this paper is more suitably expressed in Fourier space. In order to Fourier transform Eq. (3), the averaged distribution function is assumed to have a slow spatial dependence in contrast with the rapid spatial variations of the fluctuations $f(\mathbf{r},t,\mathbf{v})$. A multiscale approximation [15] is then justified and consequently $\mathcal{F}$ is not Fourier transformed. It is convenient to introduce the following symbolic notations for Eq. (3) in Fourier space:

$$g_0^{-1}f+\mathbf{V}^0\cdot\mathbf{E}=\lambda N[\mathbf{E},f]. \tag{4}$$

The unperturbed propagator g_0 and vertex $\mathbf{V}^0$ are given, respectively, by

$$g_0^{-1}(\hat{\mathbf{k}},\mathbf{v})=[-i(\omega-\mathbf{k}\cdot\mathbf{v})+\Omega(\mathbf{v}\wedge\mathbf{b})\cdot\partial_{\mathbf{v}}] \tag{5}$$

and

$$\mathbf{V}^0=\frac{e}{m}\partial_{\mathbf{v}}\mathcal{F}, \tag{6}$$

where the symbol $\hat{\mathbf{k}}$ stands for both the wave vector $\mathbf{k}$ and the frequency ω. The term $N[\mathbf{E},f]$ is nonlinear with respect to the fluctuations. It is explicitly written as

$$N[\mathbf{E},f]=-\frac{e}{m}\partial_{\mathbf{v}}\cdot\int\frac{d\hat{\mathbf{q}}}{(2\pi)^4}\delta[\mathbf{E}(\hat{\mathbf{k}}-\hat{\mathbf{q}})f(\hat{\mathbf{q}},\mathbf{v})]. \tag{7}$$

The coupling constant λ is introduced in Eq. (4) for further convenience. As already mentioned, the statistical properties of the external electric field must be specified. Here the modes $\mathbf{E}(\hat{\mathbf{k}})$ are assumed to be independent stochastic variables with a Gaussian distribution and zero average. They are thus completely determined by their correlations

$$\langle\mathbf{E}(\hat{\mathbf{k}})\mathbf{E}(\hat{\mathbf{k}}')\rangle=\mathcal{E}(\hat{\mathbf{k}})\delta(\hat{\mathbf{k}}+\hat{\mathbf{k}}'). \tag{8}$$

The Dirac distribution $\delta(\hat{\mathbf{k}}+\hat{\mathbf{k}}')$ appears as a result of the additional assumption that the external electric field is a stationary and homogeneous stochastic process. Consequently, different modes are correlated only when the sum of the corresponding wave vectors is zero. Finally, the electric field is supposed to derive from a potential so that the tensor $\mathcal{E}(\hat{\mathbf{k}})$ takes the form

$$\mathcal{E}_{ij}(\hat{\mathbf{k}})=A^2k_i\zeta(\hat{\mathbf{k}})k_j, \tag{9}$$

where A represents the amplitude of the electric field and $\zeta(\hat{\mathbf{k}})$ characterizes the shape of its spectrum.

III. ITERATIVE FILTERING

A. Perturbative approach

Due to the nonlinear convolution operator N [see Eq. (7)], all Fourier modes are coupled in the VE. Therefore, it is impossible to obtain a closed set of exact equations for a subset of Fourier modes. As a consequence, the exact equations for the large-scale modes cannot be decoupled from the small scales. It is then necessary to use some approximations. The simplest one consists of treating the nonlinear operator N as a small quantity compared to the linear term. Actually, the introduction of the parameter λ in Eq. (4) anticipates this approximation, which will be referred to as the λ expansion.

Two important questions must be discussed at this point. On the one hand, the strong coupling between some modes does not justify the λ expansion in many situations where the nonlinear interaction between the stochastic electric field and the particles is the dominant dynamical effect. On the other hand, the coupling between modes should diminish for the very-high-wave-vector modes. In many physical systems, the small-scale phenomena appear to follow linear evolution laws. In this paper, the system is assumed to be characterized by these two properties. The nonlinearity is dominant for the large scales, while it is small in the small-scale range. Hence the λ expansion is justified mainly for the high wave vector.

B. First iteration

We first remark that the spectrum of excited modes never extends until $k\to\infty$ in a real physical system. We thus introduce a cutoff wave number Λ_0 that could be related to a molecular scale length or to the Debye length depending on the physical situations. Its actual value is unimportant for our purpose. However, the cutoff may be used as a starting point in an iterative elimination scheme based on the λ expansion. We have assumed that this expansion should be valid for the high wave vectors mainly. The lack of knowledge about the convergence properties of the λ expansion then suggests that one should proceed as carefully as possible. In a first step, this expansion will be used only in the infinitesimal domain defined by the wave vectors shell $\Lambda_1\equiv\Lambda_0-\delta\Lambda<|\mathbf{k}|\leqslant\Lambda_0$. We thus consider the following decomposition of the distribution function:

$$f(\hat{\mathbf{k}},\mathbf{v})=\begin{cases} f^{>}(\hat{\mathbf{k}},\mathbf{v}) & \text{if } \Lambda_1<|\mathbf{k}|\leqslant\Lambda_0 \\ f^{<}(\hat{\mathbf{k}},\mathbf{v}) & \text{if } |\mathbf{k}|\leqslant\Lambda_1. \end{cases} \tag{10}$$

The same decomposition is used for the electric field and Eq. (4) splits into two coupled equations

$$g_0^{-1}f^{<}+\mathbf{V}^0\cdot\mathbf{E}^{<}=\lambda N^{<}[\mathbf{E}^{<}+\mathbf{E}^{>},f^{<}+f^{>}], \tag{11}$$

$$g_0^{-1}f^{>}+\mathbf{V}^0\cdot\mathbf{E}^{>}=\lambda N^{>}[\mathbf{E}^{<}+\mathbf{E}^{>},f^{<}+f^{>}]. \tag{12}$$

The equation for $f^>$ is then solved perturbatively using the λ expansion. The approximate solution up to first order in λ is

$$f^> = -g_0 \mathbf{V}^0 \cdot \mathbf{E}^> + \lambda g_0 N^>[\mathbf{E}^< + \mathbf{E}^>, f^< - g_0 \mathbf{V}^0 \cdot \mathbf{E}^>] + O(\lambda^2). \tag{13}$$

This solution is now inserted into Eq. (11), giving (up to order λ^2) a closed equation for $f^<$, i.e., an equation that is independent of $f^>$. However, this equation still depends on $\mathbf{E}^>$. In order to eliminate totally the small-scale contribution, the equation for $f^<$ is averaged over all realizations of $\mathbf{E}^>$. This ''partial averaging operation'' will be denoted by $\langle\ \rangle_>$. With the assumptions made on the stochastic variables $\mathbf{E}(\hat{\mathbf{k}})$ in Sec. II there is no difficulty in averaging over one subset of these variables that corresponds to $\mathbf{E}^>$. Indeed, their probability distribution $p(\{\mathbf{E}(\hat{\mathbf{k}})\})$ has been assumed to be a Gaussian. Moreover, relation (8) shows that variables $\mathbf{E}(\hat{\mathbf{k}})$ belonging to different shells of wave vectors are not correlated. Hence the distribution $p(\{\mathbf{E}(\hat{\mathbf{k}})\})$ factorizes into terms corresponding to the different shells and consequently

$$p(\{\mathbf{E}(\hat{\mathbf{k}})\}) = p_<(\{\mathbf{E}^<(\hat{\mathbf{k}})\}) p_>(\{\mathbf{E}^>(\hat{\mathbf{k}})\}), \tag{14}$$

where $p_<$ and $p_>$ are, respectively, the probability distribution function (PDF) associated with the variables $\mathbf{E}^<$ and $\mathbf{E}^>$. The PDF of $\mathbf{E}^<$ remains unaffected by the partial averaging over all the realizations of $\mathbf{E}^>$. However, the explicit evaluation of this averaging requires some approximations for the terms depending on $f^<$. Contrary to the $\mathbf{E}^<(\hat{\mathbf{k}})$, the stochastic variables $f^<(\hat{\mathbf{k}})$ are not independent of $\mathbf{E}^>$. The nonlinear term in the VE induces a coupling between $f^<$ and $\mathbf{E}^>$. Here we make the additional assumption that the correlation between $f^<$ and $\mathbf{E}^>$ may be neglected since these variables are coupled only through the nonlinearity assumed to be small in this shell of wave vectors. This approximation and the factorization of the probability (14) implies a few useful properties for the partial averaging of the equation for $f^<$.

(i) The total averaging and the partial averaging are equivalent as far as only the variables $\mathbf{E}^<$ are concerned:

$$\langle \mathcal{W}[\mathbf{E}^>] \rangle = \langle \mathcal{W}[\mathbf{E}^>] \rangle_>, \tag{15}$$

where $\mathcal{W}$ is any functional of $\mathbf{E}^>$ only. This property also implies $\langle \delta(\mathcal{W}[\mathbf{E}^>]) \rangle_> = 0$. Consequently, the partial average of the nonlinear term $N[\mathbf{E}^>, \mathbf{E}^>]$ [see Eq. (7)] vanishes.

(ii) The partial averaging does not affect a functional $\mathcal{Q}$ depending on $f^<$ and $\mathbf{E}^<$ only:

$$\langle \mathcal{Q}[f^<, \mathbf{E}^<] \rangle_> = \mathcal{Q}[f^<, \mathbf{E}^<]. \tag{16}$$

(iii) The partial averaging of a mixed term depending simultaneously on $f^<$, $\mathbf{E}^<$, and $\mathbf{E}^>$ must be evaluated following the procedure

$$\left\langle \prod_{i=1}^{n} \prod_{j=1}^{m} \prod_{l=1}^{r} f^<(\hat{\mathbf{k}}_i) \mathbf{E}^<(\hat{\mathbf{k}}_j) \mathbf{E}^>(\hat{\mathbf{k}}_l) \right\rangle_> = \prod_{i=1}^{n} \prod_{j=1}^{m} f^<(\hat{\mathbf{k}}_i) \mathbf{E}^<(\hat{\mathbf{k}}_j) \left\langle \prod_{l=1}^{r} \mathbf{E}^>(\hat{\mathbf{k}}_l) \right\rangle_>. \tag{17}$$

These terms vanishes when r, the number of factor $\mathbf{E}^>$, is odd because the electric field has a Gaussian distribution with zero average.

The use of these properties in the systematic partial averaging of the equation for $f^>$ cancels all but three terms generated by the small-scale elimination. The first one is proportional to $\langle \mathbf{E}^> \mathbf{E}^> \rangle_> f^<$ and is included in the propagator. The second one is proportional to $\langle \mathbf{E}^> \mathbf{E}^> \rangle_> \mathbf{E}^<$ and renormalizes the vertex. Finally, a cubic nonlinearity [16] proportional to $\mathbf{E}^< \mathbf{E}^< f^<$ has to be included in the equation

$$g_1^{-1} f^< + \mathbf{V}^1 \cdot \mathbf{E}^< = \lambda N^<[\mathbf{E}^<, f^<] + \lambda^2 N^<[\mathbf{E}^<, g_0 N^>[\mathbf{E}^<, f^<]] + O(\lambda^3). \tag{18}$$

In this equation, the renormalized propagator g_1 and the renormalized vertex $\mathbf{V}^1$ are given, respectively, by

$$g_1^{-1} f^< = g_0^{-1} f^< - \lambda^2 \langle N^<[\mathbf{E}^>, g_0 N^>[\mathbf{E}^>, f^<]] \rangle_> \tag{19}$$

and

$$\mathbf{V}^1 \cdot \mathbf{E}^< = \mathbf{V}^0 \cdot \mathbf{E}^< - \lambda^2 \langle N^<[\mathbf{E}^>, g_0 N^>[\mathbf{E}^<, g_0 \mathbf{V}^0 \cdot \mathbf{E}^>]] \rangle_>. \tag{20}$$

The explicit evaluation of these terms using the property of the partial averaging yields, respectively,

$$g_1^{-1} = g_0^{-1} - \overline{\lambda^2} \partial_i \int_S d\hat{\mathbf{q}} q_i \zeta(\hat{\mathbf{q}}) q_j g_0(\hat{\mathbf{k}} - \hat{\mathbf{q}}, \mathbf{v}) \partial_j \tag{21}$$

and

$$V_i^1 = V_i^0 - \overline{\lambda^2} \partial_l \int_S d\hat{\mathbf{q}} q_l \zeta(\hat{\mathbf{q}}) q_j \times g_0(\hat{\mathbf{k}} - \hat{\mathbf{q}}, \mathbf{v}) \partial_i g_0(-\hat{\mathbf{q}}, \mathbf{v}) V_j^0(-\hat{\mathbf{q}}, \mathbf{v}), \tag{22}$$

where summation over repeated indices is implied. Here ∂_i represents the ith component of the vector $\partial_{\mathbf{v}}$ and $\overline{\lambda} \equiv \lambda A e / m (2\pi)^4$ is the effective coupling constant. Hence the λ expansion must be regarded as an expansion in the amplitude of the spectrum of the fluctuating electric field. The integration domain is $S = \{\Lambda_1 < (|\mathbf{q}|, |\mathbf{k} - \mathbf{q}|) \leq \Lambda_0\}$. At this point it must be stressed that both g and $\mathbf{V}$ include a velocity differential operator acting, respectively, on f and $\mathbf{E}$.

C. Higher-order iterations

The same procedure is then iterated and yields, after m steps, the renormalized VE

$$g_m^{-1}f^< + \mathbf{V}^m\cdot\mathbf{E}^< = \lambda N^<[\mathbf{E}^<,f^<] + \lambda^2\sum_{i=0}^{m-1} N^<[\mathbf{E}^<,g_iN^>[\mathbf{E}^<,f^<]] + O(\lambda^3), \tag{23}$$

where the propagator and the vertex are given by the recurrences

$$(g_m^{-1}-g_{m-1}^{-1})f^< = -\lambda^2\sum_{l=0}^{m-1}\langle N^<[\mathbf{E}^>,g_lN^>[\mathbf{E}^>,f^<]]\rangle_> \tag{24}$$

and

$$(\mathbf{V}^m-\mathbf{V}^{m-1})\cdot\mathbf{E}^< = -\lambda^2\sum_{l=0}^{m-1}\langle N^<[\mathbf{E}^>,g_lN^>[\mathbf{E}^<,g_{m-1}\mathbf{V}^{m-1}\cdot\mathbf{E}^>]]\rangle_> . \tag{25}$$

On the left-hand sides of these relations, the symbols g_m^{-1} and $\mathbf{V}^m$ represent, respectively, the inverse propagator and the vertex obtained after elimination of all modes $f(\mathbf{k},\mathbf{v})$ and $\mathbf{E}(\mathbf{k})$ with $|\mathbf{k}|\geqslant\Lambda_m$. For example, the action of the propagator on a function $a(\hat{\mathbf{k}},\mathbf{v})$ must be understood as

$$g_m^{-1}a \equiv g^{-1}(\hat{\mathbf{k}},\mathbf{v};\Lambda_m)a(\hat{\mathbf{k}},\mathbf{v}) \quad \text{for } |\mathbf{k}|\leqslant\Lambda_m . \tag{26}$$

On the right-hand sides of relations (24) and (25), g_l^{-1} and $\mathbf{V}^l$ have the property to act always on modes with $\Lambda_{l+1}\leqslant|\mathbf{q}|\leqslant\Lambda_l$. In the continuous limit of the iterative filtering ($\delta\Lambda=\Lambda_{i-1}-\Lambda_i\to 0$), these operators thus act on modes with wave vectors characterized by $|\mathbf{k}|=\Lambda_l$ and it is convenient to introduce compact notations

$$\tilde{g}^{-1}a \equiv g^{-1}(\hat{\mathbf{k}},\mathbf{v};|\mathbf{k}|)a(\hat{\mathbf{k}},\mathbf{v}). \tag{27}$$

The recursion (24) can then be explicitly written as

$$g_m^{-1} = g_{m-1}^{-1} - \overline{\lambda^2}\sum_{l=0}^{m-1}\partial_i\left[\int_{S_{lm}} d\hat{\mathbf{q}}\; q_i\zeta(\hat{\mathbf{q}})q_j\tilde{g}(\hat{\mathbf{k}}-\hat{\mathbf{q}},\mathbf{v})\right]\partial_j , \tag{28}$$

where $S_{lm}=\{\Lambda_{l+1}<|\mathbf{k}-\mathbf{q}|\leqslant\Lambda_l;\Lambda_m<|\mathbf{q}|\leqslant\Lambda_{m-1}\}$. This relation is simplified by noting that the summation index l appears in the volume integration only. The summation is thus equivalent to the integration over a larger volume corresponding to the union of the S_{lm}. This yields

$$g_m^{-1} = g_{m-1}^{-1} - \overline{\lambda^2}\partial_i\int_{S_m} d\hat{\mathbf{q}}\; q_i\zeta(\hat{\mathbf{q}})q_j\tilde{g}(\hat{\mathbf{k}}-\hat{\mathbf{q}},\mathbf{v})\partial_j , \tag{29}$$

where $S_m=\{\Lambda_m<|\mathbf{k}-\mathbf{q}|\leqslant\Lambda_0;\quad \Lambda_m<|\mathbf{q}|\leqslant\Lambda_{m-1}\}$.

The procedure developed in this section shows how the VE is modified by renormalization of both the propagator and the vertex after the small-scale elimination. We have mentioned at the beginning of this section that the use of the λ expansion for filtering the VE is better justified when we proceed iteratively. However, the details of this iteration should not influence the large-scale description of the plasma and it is desirable to obtain a FRVE in which the successive steps of the filtering are written in a compact form. Such an equation and some general results are presented in the next section.

IV. GENERAL RELATIONS BETWEEN FILTERED QUANTITIES

The main motivation for filtering the VE is to obtain a set of equations for spatially filtered (smoothed) quantities such as the distribution function or the electric field. In this section, we derive general relations involving these filtered quantities.

A. Filtered and renormalized Vlasov equation

The FRVE (23) depends on all steps of the filtering procedure through the summation index. However, the recurrence formula (29), which determines the propagator, may be formally solved and the result may be expressed in terms of the final cutoff $\Lambda\equiv\Lambda_m$ corresponding to the last step of the iterative procedure

$$g_\Lambda^{-1} = g_0^{-1} - \overline{\lambda^2}\partial_i\int_{S_\Lambda} d\hat{\mathbf{q}}\; q_i\zeta(\hat{\mathbf{q}})q_j\tilde{g}(\hat{\mathbf{k}}-\hat{\mathbf{q}},\mathbf{v})\partial_j . \tag{30}$$

This relation is now independent of the details of the iteration and the domain S_Λ is defined by the inequalities $\Lambda<|\mathbf{q}|\leqslant|\mathbf{k}-\mathbf{q}|\leqslant\Lambda_0$. A similar expression may be derived for the renormalized vertex

$$V_i^\Lambda = V_i^0 - \overline{\lambda^2}\partial_n\int_{S_\Lambda} d\hat{\mathbf{q}}\; q_n\zeta(\hat{\mathbf{q}})q_j \times\tilde{g}(\hat{\mathbf{k}}-\hat{\mathbf{q}},\mathbf{v})\partial_i\tilde{g}(-\hat{\mathbf{q}},\mathbf{v})\tilde{V}_j(-\hat{\mathbf{q}},\mathbf{v}), \tag{31}$$

where $\tilde{\mathbf{V}}$ is defined in the same way as $\tilde{g}$. When all wave vectors larger than Λ are filtered out the FRVE reads [see Eq. (23)]

$$g_\Lambda^{-1}(\hat{\mathbf{k}},\mathbf{v})f(\hat{\mathbf{k}},\mathbf{v}) + V_l^\Lambda(\hat{\mathbf{k}},\mathbf{v})E_l(\hat{\mathbf{k}}) = \frac{-\lambda^2 e}{m}\partial_l\int_{|\mathbf{q}|,|\mathbf{k}-\mathbf{q}|\geqslant\Lambda}\frac{d\hat{\mathbf{q}}}{(2\pi)^4}\delta[E_l(\hat{\mathbf{k}}-\hat{\mathbf{q}})f(\hat{\mathbf{q}},\mathbf{v})] + T. \tag{32}$$

Here T stands for the cubic nonlinearities. Its explicit form is

$$T = \frac{\lambda^2e^2}{m^2}\partial_l\int_{|\mathbf{k}-\mathbf{q}|\geqslant\Lambda\geqslant|\mathbf{q}|}\frac{d\hat{\mathbf{q}}}{(2\pi)^4}\delta\left[E_l(\hat{\mathbf{q}})\tilde{g}(\hat{\mathbf{k}}-\hat{\mathbf{q}},\mathbf{v})\partial_j \times\int_{|\mathbf{k}-\mathbf{q}-\mathbf{q}'|,|\mathbf{q}'|\leqslant\Lambda}\frac{d\hat{\mathbf{q}}'}{(2\pi)^4}\delta[E_j(\hat{\mathbf{k}}-\hat{\mathbf{q}}-\hat{\mathbf{q}}')f(\hat{\mathbf{q}}',\mathbf{v})]\right]. \tag{33}$$

Equation (32) may be regarded as the main result of this section. Although the structure has not been strongly modified by the iterative filtering, the differences with the VE are important. The propagator g_Λ is a solution of Eq. (30) and differs from the original propagator g_0 by a renormalizing term. Similarly, the vertex $\mathbf{V}^{\Lambda}$ is the solution of Eq. (31) and it is also renormalized. Up to order λ^2 new cubic nonlinearities T appear. The last, but not least, difference between the VE and the FRVE is that all physical quantities $\mathbf{E}$ and f in the renormalized equation are filtered and the integration domains are thus bounded by Λ.

B. Fluxes of energy and particles

Let us now consider the expression for the anomalous fluxes of energy and particles. They are defined by [5,17,18]

$$\Gamma_n = -\frac{c}{B}\int\int d\hat{\mathbf{q}}\, d\hat{\mathbf{q}}'\, q_y \mathrm{Im}\langle \delta n(\hat{\mathbf{q}})\delta\phi^*(\hat{\mathbf{q}}')\rangle, \quad (34)$$

$$\Gamma_p = -\frac{5c}{2BT}\int\int d\hat{\mathbf{q}}\, d\hat{\mathbf{q}}'\, q_y \mathrm{Im}\langle \delta P(\hat{\mathbf{q}})\delta\phi^*(\hat{\mathbf{q}}')\rangle, \quad (35)$$

where Im is the imaginary part of a complex variable and * is its complex conjugate. These fluxes are generated by the fluctuations of macroscopic quantities such as the density and the pressure, which are given, respectively, by

$$\delta n(\hat{\mathbf{q}}) = \int d\mathbf{v}\, f(\hat{\mathbf{q}},\mathbf{v}) \quad (36)$$

and

$$\delta P(\hat{\mathbf{q}}) = \int d\mathbf{v}\frac{m}{3}v^2 f(\hat{\mathbf{q}},\mathbf{v}). \quad (37)$$

The usual procedure in computing fluxes is to use the distribution function obtained as a solution of the linearized VE since the general solutions of the nonlinear VE are not available. The advantage of our method is that the linear FRVE gives a better description of the plasma evolution than the usual linearized VE. Indeed, the nonlinear effects have been partially taken into account in the renormalization of the linear terms. However, the nonlinear interactions between filtered modes also generate the cubic nonlinearity. This term cannot be evaluated explicitly. Replacing the product of $E_j E_l$ in the integration by its average gives a first approximation of this cubic term

$$T \approx \overline{\lambda^2}\partial_l \int_{K_\Lambda} d\hat{\mathbf{q}}\; q_j q_l \zeta(\hat{\mathbf{q}})\tilde{g}(\hat{\mathbf{k}}-\hat{\mathbf{q}},\mathbf{v})\partial_j f(\hat{\mathbf{k}},\mathbf{v}), \quad (38)$$

where the domain K_Λ is given by $|\mathbf{q}|\leqslant\Lambda\leqslant|\mathbf{k}-\mathbf{q}|$. Clearly, this approximation for the term (33) shows the same structure as the correction to the propagator. However, the integration domain is different. The domain S_Λ in Eq. (30) keeps on growing as modes are eliminated ($\Lambda\rightarrow 0$), while K_Λ tends to zero. This does not mean that the term T is always negligible. Indeed, when the peak of the energy spectrum of electric-field fluctuations $\zeta(\mathbf{k})$ is at wave number ($k=k_{\max}$) smaller than Λ, most of this spectrum contributes to T. Hence T should not be neglected. For that reason, the term T is explicitly taken into account in the filtering procedure developed in the previous sections. However, once enough modes have been eliminated ($k_{\max}\geqslant\Lambda$), the energy spectrum of electric-field fluctuations mostly contributes to the renormalization of the linear terms, while the cubic nonlinearity becomes small. In particular, for $\Lambda\rightarrow 0$, the following approximation should be justified:

$$f(\hat{\mathbf{k}},\mathbf{v}) = -g_\Lambda(\hat{\mathbf{k}},\mathbf{v})\mathbf{V}^{\Lambda}(\hat{\mathbf{k}},\mathbf{v})\cdot\mathbf{E}(\hat{\mathbf{k}}), \quad (39)$$

where the norm of wave vector $\mathbf{k}$ is smaller than the cutoff Λ. This expression is not truly linear since both g_Λ and $\mathbf{V}^{\Lambda}$ depend on the spectrum of $\mathbf{E}$. When this relation is inserted in the expressions for the fluxes, one obtains

$$\Gamma_\beta = A^2\int d\mathbf{v}\int_{q<\Lambda} d\hat{\mathbf{q}}\; q_y \zeta(\hat{\mathbf{q}})\mathcal{P}_\beta(\mathbf{v}) \times \mathrm{Im}[i g_\Lambda(\hat{\mathbf{q}},\mathbf{v})\mathbf{q}\cdot\mathbf{V}^{\Lambda}(\hat{\mathbf{q}},\mathbf{v})], \quad (40)$$

where the index β corresponds to n for the density and to p for the pressure. The functions $\mathcal{P}_n = c/B$ and $\mathcal{P}_p = 5mcv^2/6BT$ are directly derived from the expression (36) and (37), respectively. The vertex depends explicitly on the average distribution function $\mathcal{F}$, which itself depends on the density and temperatures profiles. For example, the local equilibrium may be described by the Gaussian distribution

$$\mathcal{F} = \left(\frac{1}{\pi V_T(\mathbf{r})^2}\right)^{3/2} n(\mathbf{r})\exp\left(-\frac{v^2}{V_T(\mathbf{r})^2}\right), \quad (41)$$

where the thermal velocity is given by

$$V_T(\mathbf{r}) = \sqrt{\frac{2T(\mathbf{r})}{m}}. \quad (42)$$

Equation (40) thus gives general expressions for the fluxes. In Sec. VI, it will be shown that these expressions can be transformed into explicit flux-forces relations. However, the derivation of explicit analytical results requires further approximations. Some simple cases are presented in the next section.

V. QUASILINEAR LIMIT

The FRVE (32) is coupled to Eqs. (30) and (31) for the propagator and the vertex, which are both an integro-differential equation in the velocity and an integral equation in the wave vector. Obviously, these equations are too complex to be solved exactly. Let us consider an approximation in which explicit expressions for both the propagator and the vertex can be derived. We first note that the equation for the propagator involves both the operators $g_\Lambda \equiv g(\hat{\mathbf{k}},\mathbf{v};\Lambda)$ and $\tilde{g}\equiv g(\hat{\mathbf{k}},\mathbf{v};\Lambda=|\mathbf{k}|)$. However, g is obtained directly from $\tilde{g}$ through Eq. (30). It is thus only necessary to obtain an explicit expression for $\tilde{g}$. A closed equation for this quantity can be obtained from Eq. (30) by considering the limit $\Lambda=|\mathbf{k}|$:

$$\tilde{g}^{-1} = g_0^{-1} - \overline{\lambda^2}\partial_i \int_{\tilde{\Upsilon}_k} d\hat{\mathbf{q}}\; q_i \zeta(\hat{\mathbf{q}}) q_j \tilde{g}(\hat{\mathbf{k}}-\hat{\mathbf{q}},\mathbf{v})\partial_j. \quad (43)$$

This equation is very similar to the equation for g_Λ except that no explicit dependence on Λ is needed since the cutoff is defined by the norm of $\mathbf{k}$. The domain $\tilde{\Upsilon}_k$ is defined by the inequalities $|\mathbf{k}|<|\mathbf{q}|\leqslant|\mathbf{k}-\mathbf{q}|\leqslant\Lambda_0$. This clarifies the meaning of the operator $\tilde{g}(\hat{\mathbf{k}},\mathbf{v})$, which is the renormalized propagator acting on $f(\hat{\mathbf{k}},\mathbf{v})$ when all Fourier modes with $|\mathbf{q}|>|\mathbf{k}|$ have been filtered out from the VE. Let us stress the difference with $g_\Lambda(\hat{\mathbf{k}},\mathbf{v};\Lambda)$, which is the propagator acting on modes $f(\hat{\mathbf{k}},\mathbf{v})$ with $|\mathbf{k}|<\Lambda$ when all Fourier modes with $|\mathbf{q}|>\Lambda$ have been filtered out from the VE.

The nonlinear equation (43) is written in a symbolic form as

$$\tilde{g}^{-1}=g_0^{-1}+\overline{\lambda^2}Z[\zeta,\tilde{g}], \tag{44}$$

where the explicit form of the nonlinear convolution Z is obtained from Eq. (43). The problem is now to find a solution of this equation. Here again it is impossible to obtain exact results and some approximations are unavoidable. The simplest choice consists in expanding $\tilde{g}$ in series of λ. The lowest order is simply $\tilde{g}=g_0$. The first nontrivial approximation corresponds to the quasilinear (QL) limit and is formally given by

$$\tilde{g}_{\mathrm{QL}}^{-1}=g_0^{-1}+\overline{\lambda^2}Z[\zeta,g_0]. \tag{45}$$

We thus need the expression of g_0. This problem is more easily treated using the cylindrical coordinates. Indeed, g_0^{-1} may then be written as

$$g_0^{-1}=-i[\omega-k_\parallel v_\parallel-k_\perp v_\perp\cos(\phi-\theta)]-\Omega\partial_\phi, \tag{46}$$

where the coordinate system is chosen so that the wave vector and the velocity are given, respectively, by $\mathbf{k}=(k_\perp\cos\theta,\ k_\perp\sin\theta,\ k_\parallel)$ and $\mathbf{v}=(v_\perp\cos\phi,\ v_\perp\sin\phi,\ v_\parallel)$. The explicit expression for g_0 is obtained by inverting the equation $g_0^{-1}f=s$ in order to transform it into $f=g_0 s$. The general solution of this first-order differential equation contains an arbitrary parameter that is usually determined by the initial conditions. Here this condition is naturally replaced by the periodicity condition $f(\phi+2\pi)=f(\phi)$. Then, the unique solution is given by

$$f(\phi)=-\frac{1}{\Omega}\int_0^\phi h(\phi,\phi')s(\phi')d\phi' + \frac{1}{\Omega[1-h(0,2\pi)]}\int_0^{2\pi}h(\phi,\phi')s(\phi')d\phi', \tag{47}$$

where the function h reads

$$h(\phi,\phi')=\exp-\frac{i}{\Omega}\int_{\phi'}^{\phi}d\phi'' \times[\omega-k_\parallel v_\parallel-k_\perp v_\perp\cos(\phi''-\theta)]. \tag{48}$$

For simplicity, the dependence in $\hat{\mathbf{k}}$, $v_\perp$, and $v_\parallel$ are not written explicitly in h, s, and f. The relation (47) defines the action of the operator g_0 on an arbitrary function. The operator $\tilde{g}_{\mathrm{QL}}^{-1}$ is then obtained by inserting the expression (47) into the nonlinear convolution Z [(43) and (44)]. However, this would lead to long formulas that are difficult to handle. A major simplification comes from the assumption of gyrotropy. The velocity distribution function and the fluctuation spectrum ζ considered here are supposed to be isotropic in the plane perpendicular to the magnetic field $f(\hat{\mathbf{k}},\mathbf{v})=f(\omega,k_\perp,k_\parallel,v_\perp,v_\parallel)$. Thus we consider the action of g_{QL}^{-1} on such gyrotropic functions. Also, the presentation is limited to the lowest order in the wave vector $\mathbf{k}$ (*Markovian approximation*). This leads to the following result, which is well known [19–25] in the context of the QL approximation:

$$\tilde{g}_{\mathrm{QL}}^{-1}=g_0^{-1}+\overline{\lambda^2}\begin{pmatrix}\partial_\perp+\dfrac{1}{v_\perp},\partial_\parallel\end{pmatrix}\begin{pmatrix}D_{\perp\perp} & D_{\perp\parallel}\\ D_{\parallel\perp} & D_{\parallel\parallel}\end{pmatrix}\begin{pmatrix}\partial_\perp\\ \partial_\parallel\end{pmatrix}, \tag{49}$$

where the matrix $\mathbf{D}$ is given explicitly by

$$\mathbf{D}=\sum_{n=-\infty}^{+\infty}\int_{|\mathbf{q}|<\Lambda_0}dq_\parallel dq_\perp q_\perp d\omega'\zeta(q_\parallel,q_\perp,\omega')\frac{2\pi iJ_n^2}{n\Omega-\omega'+q_\parallel v_\parallel+i\epsilon}\begin{pmatrix}\left(\dfrac{n\Omega}{v_\perp}\right)^2 & \dfrac{q_\parallel n\Omega}{v_\perp}\\ \dfrac{q_\parallel n\Omega}{v_\perp} & q_\parallel^2\end{pmatrix}, \tag{50}$$

where $\epsilon\to0^+$. The J_n represent the Bessel functions that are evaluated at the value $q_\perp v_\perp/\Omega$. The summation in Eq. (50) comes from the decomposition

$$e^{ia\sin b}=\sum_{n=-\infty}^{\infty}J_n(a)e^{inb}, \tag{51}$$

which has been used extensively in order to evaluate the angular integrals containing $h(\phi,\phi')$. Similarly, the renormalized vertex can be expressed in the QL limit as

$$\tilde{\mathbf{V}}_{\mathrm{QL}}=\mathbf{V}^0+\overline{\lambda^2}\begin{pmatrix}\partial_\perp+\dfrac{1}{v_\perp},\partial_\parallel\end{pmatrix}\cdot\mathbf{Q}\begin{pmatrix}\partial_\perp\\ \partial_\parallel\end{pmatrix}\frac{e}{m}\mathcal{F}, \tag{52}$$

where the vector $\mathbf{Q}$ is given by

$$\mathbf{Q}=\sum_{n,r,p,s=-\infty}^{+\infty}\int_{|\mathbf{q}|<\Lambda_0}dq_\parallel dq_\perp q_\perp d\omega'\zeta(q_\parallel,q_\perp,\omega') \times\frac{J_nJ_rJ_pJ_s\mathbf{d}_r\cdot\mathbf{A}_{nps}\mathbf{d}_n}{(\omega'-q_\parallel v_\parallel-n\Omega-i\epsilon)(\omega'-q_\parallel v_\parallel-r\Omega-i\epsilon)}. \tag{53}$$

The vectors **A** and **d** are defined by

$$\mathbf{A}_{nps}=\left\{\left[(p-n)\left(1-\frac{s\Omega}{q_\perp v_\perp}\right)+s\left(1-\frac{p\Omega}{q_\perp v_\perp}\right)\right]\frac{1}{v_\perp}+\frac{s\Omega}{q_\perp v_\perp}\partial_\perp\right\}\mathbf{e}_\perp+\partial_\parallel\mathbf{b} \tag{54}$$

and

$$\mathbf{d}_n=\frac{n\Omega}{v_\perp}\mathbf{e}_\perp+q_\parallel\mathbf{b}. \tag{55}$$

Here $\mathbf{e}_\perp$ denotes the unit vector following $\mathbf{v}_\perp$ in the cylindrical coordinates. The relations (49) and (52), together with the explicit form of the FRVE (32), may now be used in explicit calculations. However, a commonly used approximation for strongly magnetized plasmas allows further simplifications in the expressions for Eqs. (50) and (53). Indeed, when **B** is large, finite Larmor radius effects may be neglected and the argument of the Bessel functions $(q_\perp v_\perp/\Omega)$ is small. In that case, the following approximation may be used:

$$\mathbf{D}\approx-2\pi i\int_{|\mathbf{q}|<\Lambda_0}dq_\parallel dq_\perp q_\perp d\omega'\,\frac{q_\parallel^2\zeta(q_\perp,q_\parallel,\omega')}{\omega'-q_\parallel v_\parallel-i\epsilon}\mathbf{bb} \tag{56}$$

and

$$\mathbf{Q}\approx2\pi\int_{|\mathbf{q}|<\Lambda_0}dq_\parallel dq_\perp q_\perp d\omega'\,\frac{q_\parallel^2\zeta(q_\perp,q_\parallel,\omega')}{(\omega'-q_\parallel v_\parallel-i\epsilon)^2}\partial_\parallel\mathbf{b}. \tag{57}$$

These relations can now be reduced to a single integral that is computed explicitly for a given shape of the spectrum. An interesting example is the Lorentzian frequency spectrum for the fluctuating electric field

$$\zeta(q_\perp,q_\parallel,\omega')=\frac{\omega_c^2}{\omega'^2+\omega_c^2}\overline{\zeta(q_\perp,q_\parallel)}, \tag{58}$$

where ω_c is a characteristic frequency and $\overline{\zeta}$ is the wave-vector spectrum. The frequency integral in Eqs. (56) and (57) can then be performed analytically. This yields

$$\mathbf{D}=2\pi^2 i\omega_c\int_{|\mathbf{q}|<\Lambda_0}dq_\parallel dq_\perp q_\perp\overline{\zeta(q_\parallel,q_\perp)}q_\parallel^2\frac{-i\omega_c+q_\parallel v_\parallel}{(\omega_c^2+q_\parallel^2v_\parallel^2)}\mathbf{bb} \tag{59}$$

and

$$\mathbf{Q}=2\pi^2\omega_c\int_{|\mathbf{q}|<\Lambda_0}dq_\parallel dq_\perp q_\perp\overline{\zeta(q_\parallel,q_\perp)}\frac{q_\parallel^2v_\parallel^2-\omega_c^2}{(\omega_c^2+q_\parallel^2v_\parallel^2)^2}q_\parallel^2\partial_\parallel\mathbf{b}. \tag{60}$$

These expressions are strongly simplified in the limit $\omega_c\to\infty$. In that case the spectrum is independent of the frequency and the electric field is a white noise. The tensor **D** then reduces to

$$\mathbf{D}\approx\overline{s_\parallel}\pi\mathbf{bb}, \tag{61}$$

where

$$\overline{s_\parallel}=2\pi\int_{q<\Lambda_0}dq_\parallel dq_\perp q_\perp q_\parallel^2\overline{\zeta(q_\perp,q_\parallel)}. \tag{62}$$

Within the same limit **Q** vanishes. This example shows that simple results can be obtained from the general expressions for both the renormalized propagator (30) and vertex (31) when the explicit form of the spectrum of the electric field is known. Here the results are given in the quasilinear limit. However, the general expressions (30) and (31) can be useful in other situations by using another analytical treatment.

VI. GUIDING CENTER APPROXIMATION

We have presented in the preceding section an approximation that allows many simplifications for the FRVE. Let us now consider another approximation (guiding center [26–28]) that leads to explicit results for the flux-force relations and more specifically for the transport coefficients. The thermodynamical forces are assumed to originate from the spatial dependence of the particle density and the temperature

$$X_n=-\frac{\partial\ln n(x)}{\partial x}, \tag{63}$$

$$X_T=-\frac{\partial\ln T(x)}{\partial x}, \tag{64}$$

where the gradients are supposed to be directed along the axis x perpendicular to the magnetic field. Moreover, within the guiding center approximation, the macroscopic quantities such as the density and the temperature are assumed to depend on the guiding center position of the particle trajectory instead of the particle position itself. Hence the average distribution function is given by

$$\mathcal{F}=\left(\frac{1}{\pi V_T(\mathbf{R})^2}\right)^{3/2}n(\mathbf{R})\exp\left(-\frac{v^2}{V_T(\mathbf{R})^2}\right), \tag{65}$$

where $\mathbf{R}=\mathbf{r}-\mathbf{M}\cdot\mathbf{v}/\Omega$ and

$$\mathbf{M}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}. \tag{66}$$

In that case the velocity derivatives $\partial_\mathbf{v}$ are replaced by $\partial_\mathbf{v}-\mathbf{M}\cdot\partial_\mathbf{R}/\Omega$. When combined with the Gaussian velocity dependence of $\mathcal{F}$ in the QL approximation, the flux-force relations reduce to

$$\Gamma_\beta=\frac{A^2}{\Omega}\sum_{n=-\infty}^{+\infty}\int d\mathbf{v}dq_\parallel q_\perp dq_\perp d\omega' q_y\zeta(q_\perp,q_\parallel,\omega')\mathcal{P}_\beta(\mathbf{v})\times\mathrm{Im}\left\{\frac{J_n^2}{(\omega'-q_\parallel v_\parallel-n\Omega-i\epsilon)}\right\}\mathbf{q}\cdot\mathbf{M}\cdot\partial_\mathbf{R}\mathcal{F}(\mathbf{R},\mathbf{v}). \tag{67}$$

Here also the Lorentzian spectrum may be treated analytically. The following flux-force relations are then derived when finite Larmor radius effects are neglected:

$$\Gamma_n=DnX_n, \tag{68}$$

$$\Gamma_p = \frac{5}{2} D n (X_n + X_T), \tag{69}$$

where D is given by

$$D = \frac{2\pi^2 c^2 A^2}{B^2} \int dq_\parallel dq_\perp q_\perp^3 \overline{\zeta(q_\perp, q_\parallel)}. \tag{70}$$

It is interesting to remark that these results are the same as those obtained by Carati *et al.* following another approach [14]. They first considered the guiding center approximation of the original VE, i.e., the gyrokinetic equation (GKE). Then they applied an iterative filtering on the GKE and obtained the same transport coefficients. Hence the guiding center approximation and the iterative filtering commute. We also remark that the Onsager symmetry is broken for the anomalous flux-force relations in agreement with the results obtained in Ref. [14].

VII. CONCLUSION

An iterative scheme has been developed for small-scale modes filtering from the one-particle distribution function in a collisionless plasma. The evolution equation for the filtered distribution function appears to be of the Vlasov type with a renormalized propagator, a renormalized vertex, and a cubic nonlinearity. The general form of this equation has been derived in Sec. IV. In a sense, it corresponds to the central result of this work. It gives a possible starting point for the investigation of large-scale phenomena in a collisionless plasma submitted to an external stochastic electric field. However, this equation is too complicated to be used just as it stands. Indeed, both the renormalized vertex and propagator must be obtained by solving integro-differential equations. For that reason, we have presented with some details classical approximations that may be used to simplify the FRVE. Both the quasilinear and guiding center approximations are discussed. Explicit expressions for the propagator and the vertex are then derived.

The numerical solution of the FRVE represents a very difficult task. Indeed, it would require one to solve Eqs. (30) and (31) for the renormalized linear operators, which are nonlinear integro-differential equations. Moreover, the FRVE itself is nonlocal in time and its numerical solution would require the knowledge of all previous time states. However, as in the theoretical investigation of this equation, some additional approximations may be considered for the FRVE and reasonable simulations could become accessible.

In addition to its practical interest, we would like to stress the theoretical aspects of the method developed here. The iterative filtering procedure is a scheme that takes the greatest care in using the expansion in the nonlinearity. After each iteration, the linear terms are increased by a contribution coming from the nonlinearity. Hence the respective ''weights'' of linear and nonlinear terms are continuously modified towards more important renormalized linear interactions. Thus there is an intrinsic justification within the procedure itself for using the λ expansion for smaller and smaller wave vectors. Also, it is established that the anomalous transport coefficients are not affected by inverting the order of (i) the iterative small-scale elimination and (ii) the guiding center approximation. On the one hand, in Ref. [14] the iterative filtering is applied on the gyrokinetic equation, which itself results from the application of the guiding center approximation on the VE. On the other hand, the guiding center approximation is applied here after the iterative small-scale elimination from the VE. We have shown that these two operations are commutative as far as anomalous transport coefficients are concerned.

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