Dynamical scaling in dissipative Burgers turbulence

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An exact asymptotic analysis is performed for the two-point correlation function $C(\mathbf{r},t)$ in dissipative Burgers turbulence with bounded initial data, in arbitrary spatial dimension d. Contrary to the usual scaling hypothesis of a single dynamic length scale, it is found that C contains *two* dynamic scales: a diffusive scale $l_D \sim t^{1/2}$ for very large r and a superdiffusive scale $L(t) \sim t^{\alpha}$ for $r \ll l_D$, where $\alpha = (d+1)/(d+2)$. The consequences for conventional scaling theory are discussed. Finally, some simple scaling arguments are presented within the ''toy model'' of disordered systems theory, which may be exactly mapped onto the current problem. [S1063-651X(97)07606-X]

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I. INTRODUCTION

The Burgers equation (BE) has found many applications, in both theoretical and practical senses, over the years since its birth in 1940. It was originally proposed [1] to describe wave propagation in weakly dissipative media and, in fact, it is now appreciated [2] that within this large class of phenomena there are only two model descriptions in the limit of weakly nonlinear waves, namely, the BE and the Korteweg–de Vries equation [3]. In later years the BE was scrutinized by the turbulence community as a simplified model of Navier-Stokes turbulence and thus ''Burgulence'' was conceived. The applications of the BE were boosted again in 1986 when Kardar, Parisi, and Zhang proposed that the BE with a stochastic source described the nonequilibrium evolution of a class of interface models [4]. Under a nonlinear transformation, this noisy BE was seen to describe another rich class of systems, namely, directed polymers in random media that have applications in wetting [5], disordered magnets [6], and the pinning of flux lines in superconductors [7]. The BE has also received attention as an approximate model for the formation of large-scale structures in the universe [8,9].

Naturally, with such a wide range of physical applications, the BE has attracted a great deal of theoretical attention. In the years subsequent to the revolution in critical phenomena, when the ideas of scaling and universality have become so prevalent [10], most theoretical ideas concerning the BE are formulated within a ''scaling picture.'' Although this is a convenient language for many phenomena, it must be realized that without a formal renormalization-group (RG) description, scaling must be supported by strong physical insight and not merely ''hand-waving'' arguments. As an example, the physics of domain growth in quenched ferromagnets has been very well understood on the basis of scaling arguments [11], although no explicit RG calculations have been performed away from the critical temperature. The domain morphology of this problem provides an excellent basis for scaling since it is clear that the growing domain scale acts as a well-defined measure of dynamic correlations (with the caveat that scalar order parameter domain growth has more subtle scaling due to the existence of a microscopic scale: the domain wall thickness). The concept of dynamical scaling is also supported in this field by a number of exact calculations (e.g., the one-dimensional Glauber model [12], the large-n limit of the time-dependent Ginzburg-Landau equations [13]), and a large number of numerical simulations.

The existence of scaling is not so well established in the BE, although most workers would agree that it is a convenient hypothesis, given the complexity of the problem. The analytic approach used by Burgers [14] and later by Kida [15] certainly demonstrated the existence of an important length scale, which may be considered as the mean shock wave separation. The moot point is whether this is the single dominant length in the problem. If so, then one has dynamical scaling in its simplest form and many quantities may be subsequently obtained by scaling arguments. What is lacking in the previous work on the BE is an explicit solvable case in which scaling is seen to emerge cleanly. In order to achieve this it is necessary to calculate the form of some correlation function, which entails more difficulties than studying, for instance, the mean energy decay. Our intention here is to present such a calculation for a class of initial conditions in which the velocity potential is a bounded, discontinuous, random function. In this case exact calculations are possible and we may extract the form of the velocity-velocity correlation function $E(\mathbf{r},t)$ for arbitrary spatial dimension d. We find that there exists dynamical scaling, but that it is controlled by two length scales rather than one: a diffusive scale l_D for large distances and a superdiffusive scale $L(t)$ for small distances. The details underlying this remark will be given below, but the important conceptual point is that if two length scales are playing a scaling role, then their ratio (which is, of course, dimensionless) may play a hidden role in subsequent scaling arguments. Thus simple dimensional analysis is likely to fail. We shall see an explicit demonstration of this as we proceed.

The outline of the paper is as follows. In Sec. II we introduce the BE, discuss various choices of initial conditions, and briefly describe a few analytic steps that are required before the calculation proper. By adopting the initial condition mentioned above, along with some interesting analytic methods, we are able to calculate $E(\mathbf{r},t)$ and we give explicit forms for its asymptotic behavior for small and large distances. This is presented in the lengthy Sec. IV, Sec. III

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being a warm-up exercise to calculate the mean energy decay (which has previously appeared in print [16]). In Sec. V we make a (formally exact) connection between the BE and a popular ''toy model'' in disordered systems theory. We then present simple scaling arguments for the toy model that yield partial agreement with the more complicated scaling picture that emerges from Secs. III and IV. The paper concludes in Sec. VI.

II. DEFINITION OF THE MODEL

The BE is a partial differential equation written in terms of a velocity field $\mathbf{v}(\mathbf{x},t)$:

$$\partial_t \mathbf{v} = \nu \nabla^2 \mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v}. \tag{1}$$

The field is taken to be irrotational, which allows one to express the equation solely in terms of the velocity potential ϕ defined via $\mathbf{v} = -\nabla\phi$. Explicitly one has

$$\partial_t \phi = \nu \nabla^2 \phi + \tfrac{1}{2}(\nabla\phi)^2. \tag{2}$$

The equation is most commonly discussed in one spatial dimension, in the spirit of its application to nonlinear waves [8]. However, the d-dimensional generalization given above is the canonical choice. One is interested in the evolution of the velocity field from some given initial condition, in the limit of vanishing viscosity, i.e., $\nu \to 0$. This leads to strongly nonlinear behavior, otherwise known as the strong turbulence limit. We shall make this limit more explicit in terms of a dynamic Reynolds number as we proceed.

The initial conditions we shall study are random functions $\mathbf{v}_0(\mathbf{x})$ and as such are defined in terms of a distribution function $P[\mathbf{v}_0]$. Naturally, there is an enormous choice available for P. Burgers [14] studied perhaps the most natural, namely, a Gaussian distribution of velocities with δ-function correlations:

$$P_B[\mathbf{v}_0] \sim \exp\left(-\frac{1}{2D}\int d^d x\, \mathbf{v}_0^2 \right). \tag{3}$$

His analysis was confined to $d=1$, where a controlled analytic study was possible. The main result to emerge was that the velocity field forms into shock waves separated by smooth regions and that the shocks become more dilute as time proceeds, the mean shock wave separation increasing as $L_s \sim t^{2/3}$. Dimensional arguments indicate that above $d=2$ the asymptotic properties are dominated by diffusion (i.e., the shock waves disappear and the field diffusively vanishes), so that the dominant length scale is then a diffusion scale growing as $t^{1/2}$. In precisely two dimensions [17], diffusion is still the dominant process, but logarithmic corrections are expected for quantities such as the mean energy decay $\mathcal{E}(t) \equiv \langle \mathbf{v}^2 \rangle$ (where here and henceforth, angular brackets indicate an average over the ensemble of initial conditions). Generalizations of the Gaussian form of the initial conditions (for example, defining different power spectra in Fourier space) have been studied previously [8,15,18,19] and scaling arguments have provided a broad classification for the time dependence of the length scale $L_s(t)$.

One may also consider initial distributions in terms of the velocity potential. A particular class of these is to divide the system into cells of size l^d and to assign a value of ϕ_0 independently within each cell [16,17]. In this case we may write the distribution as

$$P[\mathbf{v}_0] = \prod_{\text{cells}} p(\phi_{0,\text{cell}}). \tag{4}$$

It is important to distinguish between cell distributions p that allow bounded or unbounded values of their argument. We shall see that distributions of the former class (such as a top-hat function or a cutoff exponential distribution) constitute a particular universality class, whereas those of the latter (such as a Gaussian or a power-law distribution) have different scaling properties. In the present work we shall be interested solely in random initial conditions of the former type, by demanding the cell distribution function to be defined only for a finite range of the velocity potential. Furthermore, one may show that all such distributions lead to the same asymptotic behavior when the width of the distribution is large (but still finite) and we therefore concentrate on the simplest case, namely, a top-hat function. (This is strictly true for distributions that fall to zero discontinuously.) Explicitly we choose

$$p(\phi_0) = \frac{\theta(\Phi - |\phi_0|)}{2\Phi}, \tag{5}$$

where $\theta(z)$ is the Heaviside unit function [20].

Analytic progress has been possible in the BE over the years, since for a given initial condition one may exactly integrate the equation. This is due to the Hopf-Cole [21,22] transformation that linearizes the BE. By defining $w(\mathbf{x},t) = \exp[\phi(\mathbf{x},t)/2\nu]$ and substituting into Eq. (2), one may see that w satisfies the linear diffusion equation, which is immediately solved in terms of the heat kernel $g(\mathbf{x},t) = (4\pi\nu t)^{-d/2}\exp[-x^2/4\nu t]$. Re-expressing the solution in terms of the velocity potential, one has the explicit solution of the BE in the form

$$\phi(\mathbf{x},t) = 2\nu \ln\left\{ \int d^d y\, g(\mathbf{x}-\mathbf{y},t)\exp[\phi_0(\mathbf{y})/2\nu] \right\}. \tag{6}$$

The main analytic effort is now to perform averages over the initial distribution P. We shall accomplish this by making the following integral representation of the logarithm function in the above expression:

$$\ln(z) = \int_0^\infty \frac{du}{u}(e^{-u} - e^{-uz}). \tag{7}$$

(This representation has proved useful [23] in calculations in disordered systems theory as an alternative to the replica method and has also been used previously in problems related to the BE [16,17,24]). We therefore have the solution of the BE in the form

$$\phi(\mathbf{x},t) = 2\nu \int_0^\infty \frac{du}{u}[e^{-u} - \psi(u,\mathbf{x},t)], \tag{8}$$

where

$$\psi = \exp\left\{ -u \int d^d y\, g(\mathbf{x}-\mathbf{y},t)\exp[\phi_0(\mathbf{y})/2\nu] \right\}. \tag{9}$$

Our main focus in this work is to calculate the velocity-velocity correlation function defined by

$$E(\mathbf{r},t)=\tfrac{1}{2}\langle \mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{0},t)\rangle, \tag{10}$$

which may be easily related (with the aid of translational invariance) to the two-point correlation function for the velocity potential via

$$E\mathbf{r},t)=\tfrac{1}{4}\nabla^2 C(\mathbf{r},t), \tag{11}$$

where

$$C(\mathbf{r},t)=\langle[\phi(\mathbf{r},t)-\phi(\mathbf{0},t)]^2\rangle. \tag{12}$$

The mean energy decay is given simply by $\mathcal{E}(t)\equiv E(\mathbf{0},t)$, and we shall present a condensed derivation of this quantity in the next section before tackling the much harder task of calculating $C(\mathbf{r},t)$. Results for $E(\mathbf{0},t)$ have been presented before [16], but it is useful to sketch the derivation here in order to set up the necessary formalism required in Sec. IV, along with revealing the important time scale in the problem.

III. CALCULATION OF THE ENERGY DECAY

In previous studies of the BE [8,14,15], it is more common to infer the energy decay from a scaling argument once one has calculated the important dynamic length scale $L_s(t)$. On dimensional grounds one would like to infer $\mathcal{E}\sim L_s^2/t^2$. This scaling relation certainly holds true in many situations, but it is by no means a universal result. We shall attack the problem from the opposite direction by first calculating the energy decay explicitly. We shall then read off important length scales from the correlation function in Sec. IV; comparing the two independent results will then allow us to see if dimensional analysis holds for our particular choice of the initial distribution.

Expressing $\mathcal{E}$ in terms of the velocity potential, one may see from averaging Eq. (2) over the initial distribution that

$$\mathcal{E}=\partial_t\langle\phi(\mathbf{x},t)\rangle. \tag{13}$$

So in order to determine the energy decay, we need only calculate the mean velocity potential, which in turn is related to the average of the function ψ from Eq. (8). In fact, a very similar function will be central in the evaluation of the correlation function, so it is useful to dedicate a few lines to deriving an explicit expression for $\langle\psi\rangle$.

In order to perform the average it is necessary to impose a lattice scale a, this is because, generally, the initial condition average has the form of a functional integral, which is only strictly defined on a lattice. We shall find that for all but the shortest times (set by t_0, the time for diffusion over the cell size l) this scale a disappears from all physical quantities and is replaced by the cell scale l, which defines the correlation scale of the initial conditions. Explicitly we define a diffusion length $l_D=(4\nu t)^{1/2}$ and work in the limit $a\ll l\ll l_D$. In other words, the spatial smearing of the heat kernel is much greater than the cell size. Performing the initial condition average over ψ using the distribution defined by Eqs. (4) and (5), we find

$$\ln\langle\psi\rangle=l^{-d}\int d^dy\ln\biggl(\frac{1}{K_0}\{E_1[ul^dg(\mathbf{y},t)e^{-K_0/2}]-E_1[ul^dg(\mathbf{y},t)e^{K_0/2}]\}\biggr), \tag{14}$$

where $E_1(z)$ is the exponential integral [25] and we have defined K_0 to be the effective Reynolds number at zero time, i.e., $K_0=(\text{typical velocity})\times(\text{typical length})/\nu=\Phi/\nu$. (We refer the reader to Appendix A where the initial condition average is performed explicitly.) We may simplify this expression in two steps. First, we make the rescaling $u\to u(\pi^{1/2}l_D/l)^d e^{-K_0/2}$ and change the integration variable to $s=y^2/4\nu t$. Second, we impose the strong turbulence limit by taking $K_0\gg 1$. This leads us to

$$\ln\langle\psi\rangle=-\frac{(\pi^{1/2}l_D/l)^d}{K_0\Gamma(d/2+1)}L_{d/2}(u)+O(1/K_0^2), \tag{15}$$

where $\Gamma(z)$ is the Γ function [25] and

$$L_p(u)\equiv\int_0^\infty ds\, s^p[1-\exp(-ue^{-s})]. \tag{16}$$

We refer the reader to Appendix B, where it is shown that the integral may be evaluated for both small and large values of u with the result

$$L_p(u)=\begin{cases} u\Gamma(p+1)[1-2^{-(p+2)}u+O(u^2)], & u\ll 1\\ \dfrac{[\ln(u)]^{p+1}}{(p+1)}+\gamma[\ln(u)]^p+O([\ln(u)]^{p-1}), & u\gg 1. \end{cases} \tag{17}$$

In order to find the mean velocity potential we must perform the u integral as given in Eq. (8). One may see that the u integral is dominated by $u\ll 1$ ($\gg 1$) when the ratio $(l_D/l)^d/K_0\gg 1$ ($\ll 1$). The former case occurs for very large times, and on performing the u integral one obtains a diffusion result, i.e., $\mathcal{E}\sim t^{-(d/2+1)}$. So there exists a crossover time $t_c\sim(l^2/\nu)K_0^{2/d}$ beyond which the nonlinearity is irrelevant and the velocity potential relaxes according to diffusion. By taking the initial Reynolds number to be arbitrarily large, we may push t_c to arbitrarily late times. The interesting nonlinear regime occurs for $t_0\ll t\ll t_c$ in which case one must perform the u integral using the large-u asymptotic form for $\langle\psi\rangle$. In this case one finds (using the variable change $\sigma=[\ln(u)]^{d/2+1}$ and imposing a lower cutoff of $O(1)$ to the u integral)

$$\mathcal{E}=C_d\frac{(K_0l^dl_D^2)^{2/(d+2)}}{t^2}\sim t^{-\sigma}, \tag{18}$$

where $\sigma=2(d+1)/(d+2)$ and C_d is a complicated d-dependent constant.

We may reinterpret this expression by defining a time-dependent Reynolds number $K(t)$. For a typical velocity we take the square root of the mean energy decay and for a typical (large) length scale we take l_D (which we will justify *a posteriori* in Sec. IV). Then we have

$$K(t)=\left[C_d'K_0\left(\frac{l}{l_D}\right)^d\right]^{1/(d+2)}, \tag{19}$$

where the constant $C_d'=\pi^{-d/2}\Gamma(d/2+2)$ has been chosen for future convenience. We can see that $K(t)$ decays from its initial (very) large value with the power law $t^{-d/2(d+2)}$ until it becomes of order unity when $t\sim t_c$. In this nonlinear regime [defined by $K(t)\gg 1$] we may write the energy decay in the form

$$\mathcal{E}\sim\frac{l_D^2}{t^2}K(t)^2, \tag{20}$$

which is cast into the form ''expected'' from dimensional analysis, except that the dimensionless (but time-dependent) Reynolds number is also present, which invalidates the prediction for the time dependence of $\mathcal{E}$ from dimensional considerations alone.

The introduction of the time-dependent Reynolds number is useful, but must be justified by independently proving that l_D is the typical (large) length scale in the nonlinear regime. Alternatively one could insist on the dimensional prediction, in which case one would infer the important length scale to be $L_s\sim l_D K(t)$. [In fact, we shall see that the dynamic length scale $\sim l_D/K(t)$.] To place these results and speculations into a proper context one is forced to evaluate the scaling properties of the correlation function, which is a much more difficult task than the calculation of the mean energy decay.

IV. CALCULATION OF THE CORRELATION FUNCTION

This section constitutes the heart of the paper in that we present the exact asymptotic forms for the correlation function $C(\mathbf{r},t)$ in the nonlinear regime $t_0\ll t\ll t_c$. Unfortunately, in order to arrive at the required result, one must wade through a very long and technical calculation. So as not to burden the reader with details, all technical remarks will be relegated to the Appendixes, with only the general flow of the analysis described in the main text. First, we shall derive a general expression for C in the nonlinear regime. In the subsequent subsections, we shall then analyze the asymptotic properties of C in the limits of $r\ll l_D$ and $r\gg l_D$. As hinted at before, the main result of this analysis is the emergence of a length scale that describes the small distance behavior of the correlation function.

It is convenient to define C in a symmetric way [cf. Eq. (12)]

$$\begin{aligned}C(\mathbf{r},t)&=\langle[\phi(-\mathbf{r}/2,t)-\phi(\mathbf{r}/2,t)]^2\rangle\\&=2\langle\phi(\mathbf{0},t)^2\rangle-2\langle\phi(-\mathbf{r}/2,t)\phi(\mathbf{r}/2,t)\rangle.\end{aligned} \tag{21}$$

By utilizing the integral representation of the logarithm function twice, we may rewrite the bilinear combinations of velocity potentials in terms of integrals. This yields

$$C(\mathbf{r},t)=8\nu^2\int_0^\infty\frac{du}{u}\int_0^\infty\frac{dv}{v}[\Psi(u,v,\mathbf{0},t)-\Psi(u,v,\mathbf{r},t)], \tag{22}$$

where

$$\Psi(u,v,\mathbf{r},t)=\left\langle\exp\left[-\int d^dy[ug(\mathbf{y}-\mathbf{r}/2,t)+vg(\mathbf{y}+\mathbf{r}/2,t)]e^{\phi_0(\mathbf{y})/2\nu}\right]\right\rangle \tag{23}$$

and we have utilized the property of translational invariance.

In an analogous fashion to the averaging performed in Sec. III, the average over the initial conditions may be performed in a straightforward manner (see Appendix A), yielding a complicated expression for Ψ in terms of the exponential integral function. However, great simplification may be made by taking the limit $K_0\gg 1$. In this case we reduce Ψ to the form

$$\ln[\Psi(u,v,\mathbf{r},t)]=-\epsilon I(u,v,\mathbf{R})+O(1/K_0^2), \tag{24}$$

where $\epsilon=\Gamma(d/2+2)/K(t)^{d+2}\ll 1$,

$$\begin{aligned}I(u,v,\mathbf{R})=\frac{2}{d\pi^{d/2}}\int d^dy&\left\{y^2-\frac{\mathbf{y}\cdot\mathbf{R}}{2}\left[\frac{ue^{-(\mathbf{y}-\mathbf{R}/2)^2}-ve^{-(\mathbf{y}+\mathbf{R}/2)^2}}{ue^{-(\mathbf{y}-\mathbf{R}/2)^2}+ve^{-(\mathbf{y}+\mathbf{R}/2)^2}}\right]\right\}\\&\times\{1-\exp[-ue^{-(\mathbf{y}-\mathbf{R}/2)^2}-ve^{-(\mathbf{y}+\mathbf{R}/2)^2}]\},\end{aligned} \tag{25}$$

and we have defined $\mathbf{R}\equiv\mathbf{r}/l_D$.

At this point of the discussion it is convenient to consider the small- and large-distance behaviors of C separately.

A. Small-distance scaling

To ascertain the small distance properties of C we need to perturbatively evaluate the above integrals in a power series in $R\ll 1$. Although one may attempt this directly on the form of the integrals as given by Eqs. (23) and (25), it is far more efficient to transform the function I beforehand into a natural power series in R^2. The procedure for this is described in Appendix C, with the result

$$I(u,v,\mathbf{R})=\sum_{p=0}^{\infty}(R^2)^pF_p(u,v), \tag{26}$$

where the functions F_p have the integral form

$$F_p(u,v)=\frac{(-uv\partial_u\partial_v)^p}{\Gamma(p+1)\Gamma(p+d/2+1)}L_{p+d/2}(u+v). \tag{27}$$

We are interested in the nonlinear regime $K(t)\gg 1$, and in this case the (u,v) integrals are dominated by $u\gg 1$ and $v\gg 1$. Therefore, we expand the integral $L_{p+d/2}$ appearing in Eq. (27) in powers of $\Delta\equiv\ln(u+v)\gg 1$ (see Appendix B). The function F_p may now be expressed as

$$F_p(u,v)=\frac{(-1)^p}{\Gamma(p+1)\Gamma(p+d/2+1)}\left[\frac{\chi_p(u,v;p+d/2+1)}{(p+d/2+1)}+\gamma\chi_p(u,v;p+d/2)+O(\Delta^{p+d/2-2})\right], \tag{28}$$

where we have defined

$$\chi_p(u,v;q)=(uv\,\partial_u\partial_v)^p\Delta^q=f_p(u,v;q)\Delta^{q-1}+g_p(u,v;q)\Delta^{q-2}+O(\Delta^{q-3}). \qquad (29)$$

More details of these steps, along with the explicit form of the coefficients $\{f_p\}$ and $\{g_p\}$, may be found in Appendix D.

Now that we have a workable series for I in the nonlinear regime, it is possible to expand $\Psi(u,v,\mathbf{r},t)$ in powers of R^2 such that the coefficients are various combinations of the functions F_p. The integrals over u and v may then be performed (see Appendix E for details) and one has the final result

$$\begin{aligned}\frac{C(\mathbf{r},t)}{8\nu^2}=&\frac{2}{(d+2)}\Gamma\left(\frac{d+4}{d+2}\right)K(t)^2R^2\\&-\left\{R^2+\frac{(d+3)}{3(d+2)^2}\Gamma\left(\frac{d+4}{d+2}\right)K(t)^2R^4\right.\\&-\frac{4(d+5)}{45(d+2)^2(d+4)}\Gamma\left(\frac{d+6}{d+2}\right)K(t)^4R^6\\&+\frac{4(d+5)(d+7)}{63(d+2)^2(d+4)^2(d+6)}\Gamma\left(\frac{d+8}{d+2}\right)K(t)^6R^8\\&\left.+O(R^{10})\right\}+\cdots. \qquad (30)\end{aligned}$$

Several remarks are now in order. First, the above result is given (after much effort) to quite high order in R^2. One is obliged to do this to determine unambiguously the scaling properties of the correlation function. Second, the result for C has been written in such a way as to stress the form of the scaling. It turns out that the dominant term at each order of R^2 vanishes exactly, except for the dominant term *at* order R^2: This explains why this term stands alone in the above expression. The subdominant terms from each order are nonzero and are grouped together within the curly brackets. The remaining terms play no role in determining the leading scaling behavior and are indicated by the ellipsis. The fact that the dominant terms vanish means that the leading R^2 term can play no part in the scaling form of the correlation function. However, the terms in curly brackets have a natural scaling form that allows us to read off a dynamic length scale. Explicitly we may recast the above expression into the scaling form (neglecting constants)

$$C(\mathbf{r},t)\sim\nu^2\left[\left(\frac{\mathbf{r}}{L(t)}\right)^2+\left(\frac{\mathbf{r}}{l_D}\right)^2S\left(\frac{\mathbf{r}}{L(t)}\right)\right], \qquad (31)$$

where $S(x)$ is the scaling function and the dynamic length scale is $L(t)=l_D/K(t)\sim t^\alpha$ with $\alpha=(d+1)/(d+2)$.

We see that in the nonlinear regime, the dynamic length scale is much smaller than the diffusive scale l_D, although it is growing faster. This gives us another view of the crossover from nonlinear to linear evolution; i.e., the dynamic Reynolds number becomes of order unity when the scale $L(t)$ becomes of the same order as l_D.

As a final remark in this section, we may obtain the velocity-velocity correlation function from Eq. (31) with the use of Eq. (11). One finds

$$E(\mathbf{r},t)\sim\mathcal{E}(t)+\left(\frac{\nu}{l_D}\right)^2\tilde{S}\left(\frac{\mathbf{r}}{L(t)}\right). \qquad (32)$$

Again, it is interesting to see that the mean energy decay $\mathcal{E}(t)$ is not part of the scaling function, which explains the difficulties encountered in Sec. III with simple dimensional analysis. It remains to show the scaling importance of the diffusive scale: This will be accomplished in the next subsection.

B. Large-distance scaling

The scaling form for the correlation function for very large $|\mathbf{r}|$ may be obtained with relatively little effort. Starting with $C(\mathbf{r},t)$ expressed in terms of the function $I(u,v,\mathbf{R})$ as given in Eqs. (22), (24), and (25), we may express I by the series (cf. Appendix B)

$$I(u,v,\mathbf{R})=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}n^{-(d/2+1)}\sum_{m=0}^{n}C_m^n u^m v^{n-m}\times\exp\left[-\frac{m}{n}(n-m)R^2\right]. \qquad (33)$$

As $C(\mathbf{r},t)$ is nonzero for $|\mathbf{r}|\to\infty$, it is convenient to measure correlations with respect to the asymptotic value $C(\infty,t)$. Thus we define

$$\begin{aligned}\delta C&\equiv C(\infty,t)-C(\mathbf{r},t)\\&=8\nu^2\int_0^\infty\frac{du}{u}\int_0^\infty\frac{dv}{v}[\Psi(u,v,\mathbf{r},t)-\Psi(u,v,\infty,t)]\\&=8\nu^2\int_0^\infty\frac{du}{u}\int_0^\infty\frac{dv}{v}\{\exp[-\epsilon I(u,v,\mathbf{R})]\\&\quad-\exp[-\epsilon I(u,v,\infty)]\}. \qquad (34)\end{aligned}$$

From Eq. (33) it is easy to see

$$\begin{aligned}I(u,v,\infty)&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}n^{-(d/2+1)}(u^n+v^n)\\&=\frac{[L_{d/2}(u)+L_{d/2}(v)]}{\Gamma(d/2+1)}, \qquad (35)\end{aligned}$$

where we have rewritten the sum in terms of the familiar integral $L_{d/2}$ [using the integral representation shown in Eq. (C6)]. In the nonlinear regime, we are interested in the large (u,v) behavior, which according to Eq. (17) gives us

$$I(u,v,\infty)\sim\frac{1}{\Gamma(d/2+2)}\{[\ln(u)]^{d/2+1}+[\ln(v)]^{d/2+1}\}. \qquad (36)$$

Returning to Eq. (33), the large-$|\mathbf{R}|$ form for I may be written as

$$I(u,v,\mathbf{R})=I(u,v,\infty)-2^{-(d/2+1)}uve^{-R^2/2}+\cdots. \qquad (37)$$

Substituting this result into Eq. (34) gives the leading term of δC as

$$\delta C \sim \frac{4\nu^2\epsilon}{2^{d/2}} J(K)^2 e^{-R^2/2}, \tag{38}$$

where

$$J(K) = \int_c^\infty du \exp\{-K(t)^{-(d+2)}[\ln(u)]^{d/2+1}\}, \tag{39}$$

(c being a number of order unity).

The integral may be performed by steepest descents in the nonlinear regime (details in Appendix B) with the result that δC has the final asymptotic form (neglecting overall constants)

$$\delta C \sim \nu^2 K(t)^{(4-d^2)/d} \exp[C''_d K(t)^{2(d+2)/d}] \exp[-r^2/2l_D^2], \tag{40}$$

where $C''_d = d(d/2+1)^{-(1+2/d)}$. We see from this expression that the diffusion scale l_D is the natural scaling length for the correlation function, for very large distances. One may ascertain the range of validity of the above expression by calculating the contribution from the next term in the series [from Eq. (33)], and one finds that the above form is valid for $|\mathbf{r}| \gg l K_0^{1/d}$ $[=(\nu t_c)^{1/2} \gg l_D]$.

Before ending this rather technical section, we shall recapitulate the main results obtained. By performing an exact analysis on the correlation function $C(\mathbf{r},t)$ in the nonlinear regime, we have been able to confirm that there indeed exists dynamical scaling, albeit of a rather subtle type. The small distance properties of C are governed by a scaling length $L(t) \sim l_D/K(t)$, but the dominant term in C is singular, i.e., it may not be included into the scaling form. This indicates why the form of the mean energy decay found in Sec. III was not obtainable by simple dimensional analysis. The scale $L(t)$ is much smaller than the diffusion scale, but grows faster: The nonlinear regime crosses over to simple diffusion when these two length scales become compatible. The large distance scaling was found to be more conventional in that the dominant part of C (with respect to its asymptotic value) is a simple function of r/l_D, albeit with a complicated prefactor, thus indicating that l_D acts as the dynamic scale for the correlation function at very large distances.

V. SCALING ARGUMENTS

This section has two purposes. First, we shall show an exact mapping between the solution of the BE and the free energy of a popular toy model [26] in the field of disordered systems. Second, we shall perform some simple scaling calculations [27] on the latter model to extract the form of the mean energy decay $\mathcal{E}(t)$ in the original BE problem. These scaling calculations are very similar in spirit to the original calculations of Burgers [14] and Kida [15].

The toy model in question is simply described. Consider a single particle in a potential composed of a harmonic background plus a random potential $V(\mathbf{x})$. If the particle is in contact with a thermal reservoir, we may write the partition function for the particle as

$$Z = \left(\frac{\beta\mu}{2\pi}\right)^{d/2} \int d^d x \exp\left\{-\beta\left[\frac{\mu}{2}x^2 + V(\mathbf{x})\right]\right\}, \tag{41}$$

where β is the inverse temperature and we have normalized Z with respect to the harmonic background. For a given realization of the disorder potential V, we may calculate the free energy of the particle from $F_V = -(1/\beta)\ln(Z)$.

At this stage we compare the expression for the free energy with the exact solution of the BE (evaluated at the origin) as given by Eq. (6). We see that there exists an exact equivalence between the two quantities if one makes the following connections: $\phi(\mathbf{0},t) \leftrightarrow -F_V, 2\nu \leftrightarrow 1/\beta, t \leftrightarrow 1/\mu$, and $\phi_0(\mathbf{x}) \leftrightarrow -V(\mathbf{x})$. This correspondence holds regardless of the particular distribution of initial conditions (or equivalently, disorder).

To proceed with the description of the toy model, two quantities one may be interested in calculating are the quenched free energy $F = \langle F_V \rangle$ and the mean-square displacement of the particle

$$\langle x^2 \rangle = \left\langle \frac{1}{Z} \int d^d x x^2 \exp\left\{-\beta\left[\frac{\mu}{2}x^2 + V(\mathbf{x})\right]\right\}\right\rangle$$
$$\approx \langle -(2/\beta)\partial_\mu \ln(Z)\rangle = 2\partial_\mu F. \tag{42}$$

By utilizing the correspondence with the BE, we may relate the mean-square displacement to the quantity $\mathcal{E}(t)$ in the BE. Explicitly we write

$$\langle x^2 \rangle = 2\partial_\mu F = -2\partial_{1/t}\langle \phi(\mathbf{0},t)\rangle = 2t^2 \partial_t \langle \phi(\mathbf{0},t)\rangle = 2t^2 \mathcal{E}(t), \tag{43}$$

where we have made use of Eq. (13). So we have been able to show that the dimensional prediction for the mean energy decay, namely, $\mathcal{E}(t) \sim L_s(t)^2/t^2$, has a formal interpretation in terms of the toy model as long as we interpret $L_s(t)$ as the root-mean-square displacement of the particle.

We shall now derive an approximate form for $L_s(t)$ within the toy model formulation. Consider first the top-hat distribution that has been the subject of the present work. We take $P[V] = \Pi_{\text{cells}} p(V_{\text{cell}})$, with $p(V) = \theta(D-|V|)/2D$. The strong turbulence limit of the BE corresponds to the zero-temperature limit of the toy model. In this case, the particle will be trapped in the lowest potential-energy minimum, within a given realization. In this case we may estimate the excursion of the particle by calculating the probability $q(r,U^*)$ for the lowest potential site to be located at a distance r from the origin and to have an energy U^*. This will be proportional to the probability that all sites within a radius r of the origin have a potential energy higher than U^*.

For a general potential distribution $p(V)$, we may write

$$q(r,U^*) \sim p(U^* - \mu r^2/2) \prod_{|\mathbf{x}|<r} \int_{U^* - \mu x^2/2}^\infty dV_{\mathbf{x}} p(V_{\mathbf{x}}). \tag{44}$$

In the BE analogy we are interested in long times, so within the toy model we need to take μ to be small, i.e., the harmonic background is taken to be very ''flat.''

Restricting our attention to the top-hat distribution, we see that a flat harmonic background implies that the minimal energy U^* will be close to the lower bound of the random potential $-D$. We therefore set $U^*=-D+\delta U$, where $|\delta U|\ll D$. The above expression then reduces to

$$q(r,U^*)\sim\frac{1}{2D}\prod_{|\mathbf{x}|<r}\left[1-\frac{\delta U}{2D}+\frac{\mu x^2}{4D}\right]. \tag{45}$$

It is now straightforward to exponentiate the term in square brackets so as to transform the product over $\mathbf{x}$ as a spatial integral in the exponential. Taking $\mu r^2\ll D$ (which is justified *a posteriori* by the form of $r_{\rm typ}$ given below) allows the integral to be performed simply, yielding the final result

$$q(r,\delta)\sim\frac{1}{2D}\exp\left[-c_1(d)\frac{\delta}{D}\left(\frac{r}{l}\right)^d+c_2(d)\frac{\mu r^{d+2}}{l^d D}\right], \tag{46}$$

where c_1 and c_2 are constants. For this distribution of potential minima at distance r from the origin, we can read off a scaling relation between the typical value of r and μ, namely,

$$r_{\rm typ}\sim\left(\frac{l^d D}{\mu}\right)^{1/(d+2)}. \tag{47}$$

Making the correspondence with the BE, we identify $r_{\rm typ}\leftrightarrow L_s(t)$ and $\mu/D\leftrightarrow(l_D^2K_0)^{-1}$. Combining the above result with Eq. (42), we see that we have derived the correct form of the mean energy decay as calculated previously in Sec. III [cf. Eq. (18)], although the precise value of the prefactor may not be obtained by this simple scaling argument.

The above result may also be cast into the form $L_s(t)\sim l_D K(t)$. This is guaranteed under the scaling hypothesis, but what is interesting is that such a length scale plays no role in the actual dynamical scaling as defined by the behavior of the two-point correlation function. In other words, although we may calculate L_s (or rather $r_{\rm typ}$) from scaling considerations of the toy model, this length scale is not a dynamic scaling length; for instance, it could not be used to collapse the correlation function in a scaling plot.

To end this section, we mention that the toy model may be analyzed for other types of distribution. If one takes the disorder distribution to be of the cellular type, with p a Gaussian, then one may rederive the result of Kida [15], namely, that $L_s(t)$ is a diffusive scale up to logarithmic corrections. Alternatively one may consider the toy model in $d=1$ with a disorder distribution corresponding to the original Burgers choice, namely, $P[V]\sim\exp[-\int dx(dV/dx)^2]$. This particular scenario has been studied in detail recently, using a replica approach [28]. Although the essential Burgers scaling $[L_s(t)\sim t^{2/3}]$ is easily recovered, the calculation of prefactors is more difficult. It is an important test of various approaches as to whether they can quantitatively predict the correct prefactor. As far as we are aware, this is still an open problem, although there are a number of approximate values in the literature [14,16,28].

VI. CONCLUSION

This paper has been concerned with proving dynamical scaling for Burgers equation with random initial conditions. Exact calculations have been possible for a distribution of the initial velocity potential, which has a large but finite region of support and is uncorrelated in space. In Sec. III we calculated the mean energy decay $\mathcal{E}(t)$ in the nonlinear regime [i.e., in the temporal regime in which the Reynolds number $K(t)\gg 1$]. Explicitly we found

$$\mathcal{E}\sim\frac{l_D^2}{t^2}K(t)^2. \tag{48}$$

Dimensional arguments applied at this stage would then predict that there exists a dynamical length scale $L_s(t)\sim l_D K(t)$, where $l_D\sim(\nu t)^{1/2}$ is the diffusion scale.

In Sec. IV we set out to establish this result by calculating the two-point correlation function $C(\mathbf{r},t)$. This task is nontrivial, but we were able to extract the small- and large-scale asymptotics of C. There were two unexpected results. First, the small- and large-scale forms of C, although assuming a scaling form, have different dynamic length scales. For the small-distance scaling, the dynamic length scale was found to be $L(t)\sim l_D/K(t)$ and the dominant part of C in this spatial regime is singular and cannot be included in the scaling function. In terms of the velocity-velocity correlation function $E(\mathbf{r},t)$, this singular piece is exactly $\mathcal{E}(t)$, with the scaling part of the correlation function describing the nonlocal properties of E, i.e.,

$$E(\mathbf{r},t)\sim\mathcal{E}(t)+\left(\frac{\nu}{l_D}\right)^2\tilde{S}\left(\frac{\mathbf{r}}{L(t)}\right). \tag{49}$$

The scaling function $\tilde{S}(x)$ has a power series expansion in x^2, the first four coefficients of which may be inferred from Eq. (30). The large-distance scaling was found to be described by the diffusive scale l_D, which confirmed *a posteriori* the choice of this length scale in constructing the dynamic Reynolds number. The leading term in this large-distance regime was found to be

$$\delta C\sim\nu^2K(t)^{(4-d^2)/d}\exp[C_d''K(t)^{2(d+2)/d}]\exp[-r^2/2l_D^2]. \tag{50}$$

The second unexpected result is that neither of the two dynamic length scales coincides with the scale L_s found from dimensional considerations of $\mathcal{E}(t)$. This is explained in part by the fact that the local energy decay is singular and not contained in the scaling form for E. This result is similar to the cases in critical phenomena where caution is required in reading off scaling dimensions from composite operators (such as $\mathbf{v}^2$) [10]. Generally, these composite operators have their own scaling dimension, which may be related to the scaling of a two-point correlation function only via a small distance expansion (otherwise known as an operator product expansion.)

In Sec. V we introduced a mapping between the BE and a toy model that is well known from the field of disordered systems, that being a thermal particle in a harmonic background along with a random potential. By considering simple scaling calculations on the toy model, we were able to reproduce the previous result for $\mathcal{E}$ as found in Sec. III (up to prefactors). The scale L_s appears naturally in the toy model (as the typical root-mean-square displacement of the particle) and it is interesting that this scale is not a true dynamical scaling length in that it may not be used to collapse the two-point correlation function in a scaling plot. It would be interesting to understand this result more fully either by performing similar calculations to those described here for other choices of initial distributions or by calculating higher-order correlation functions to see if more exotic forms of scaling (such as multiscaling or intermittency) are present in these simple models. The connection between the toy model and the BE may be of some mutual aid in both fields, at least in supplying an intuitive understanding of these complementary problems.

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APPENDIX A

In this appendix we outline the initial condition average over the function

$$\psi=\exp\left\{-u\int d^d y g(\mathbf{y},t)\exp[\phi_0(\mathbf{y})/2\nu]\right\}, \qquad (A1)$$

where we have taken advantage of translational invariance and set $\mathbf{x}=\mathbf{0}$. (In fact, the translational invariance holds only for times greater than t_0 since clearly the cellular initial conditions allow only invariance under discrete transformations over a period l. Once the heat kernel has diffused beyond the cell scale, the continuous translational invariance is recovered.) The initial distribution is as described in Sec. II, namely, we divide space into cells of volume l^d and within each cell we assign the velocity potential to be an independent random number drawn from a top-hat distribution of width 2Φ. Explicitly we have

$$\begin{aligned}\langle\psi\rangle &= \int \mathcal{D}\phi_0(\mathbf{y}_i)P[\phi_0]\psi[\phi_0]\\ &=\prod_{\mathbf{Y}}\frac{1}{2\Phi}\int_{-\Phi}^{\Phi}d\phi_0(\mathbf{Y})\\ &\quad\times\exp\left\{-u\exp[\phi_0(\mathbf{Y})/2\nu]\sum_{\mathbf{y}_i\in\mathbf{Y}}g(\mathbf{y},t)\right\}, \qquad (A2)\end{aligned}$$

where $\mathbf{Y}$ labels the cells and $\mathbf{y}_i$ labels the discretized points (on a scale of the lattice cutoff a) within a given cell. The integrals are easily performed in terms of the exponential integral [25]

$$\begin{aligned}&\frac{1}{2\Phi}\int_{-\Phi}^{\Phi}d\phi_0\exp[-A\exp(\phi_0/2\nu)]\\ &\quad=\frac{1}{K_0}[E_1(Ae^{-K_0/2})-E_1(Ae^{K_0/2})], \qquad (A3)\end{aligned}$$

where $K_0=\Phi/\nu$ is the initial Reynolds number as defined in the text. To reach the result shown in the text, namely, Eq. (14), two more steps are required. First, the above result is re-exponentiated, so that the product over cells in Eq. (A2) may be written in the exponent as a sum over cells. Second, we use the fact that for times greater than $t_0=l^2/\nu$, the heat kernel has smeared beyond the cell scale, so that the sum over cells (on a scale l) and points within cells (on a scale a) may be replaced once more by a continuum spatial integral. In this way one finally arrives at the result given in the main text.

APPENDIX B

This appendix is dedicated to the asymptotic evaluation of two integrals that appear in the main text. The first integral $L_p(u)$ is defined in Eq. (16) and is used repeatedly in the present work. We have

$$L_p(u)\equiv\int_0^\infty ds s^p[1-\exp(-ue^{-s})]. \qquad (B1)$$

We require both the small- and large-u forms for this integral. The former is trivially obtained by expanding the integrand as a power series in u. The asymptotic expansion of this integral for $u\gg 1$ is less simple. For the precision required in the calculations, we need both the dominant and subdominant terms. To extract these we proceed as follows. We notice that for large u, the second factor in the integrand behaves very much like a step function centered at $s=\ln(u)$. Thus the dominant term will arise from replacing this factor by $\theta(\ln(u)-s)$ and the subdominant term will arise from finding the leading error made by this approximation.

So explicitly we write

$$L_p(u)=\int_0^{\ln(u)}ds s^p+\int_0^\infty ds s^p T(s), \qquad (B2)$$

where $T(s)=[1-\exp(-ue^{-s})]-\theta(\ln(u)-s)$. The first term gives the dominant contribution to the integral, which equals $[\ln(u)]^{p+1}/(p+1)$. To extract the main contribution from the second term, we replace s^p by $[\ln(u)]^p$ and perform the integral over $T(s)$:

$$\begin{aligned}\int_0^\infty ds s^p T(s) &= [\ln(u)]^p\int_0^\infty ds T(s)+O([\ln(u)]^{p-1})\\ &=[\ln(u)]^p\left\{\int_{\ln(u)}^\infty ds[1-\exp(-ue^{-s})]\right.\\ &\quad\left.-\int_0^{\ln(u)}ds\exp(-ue^{-s})\right\}+O([\ln(u)]^{p-1}).\end{aligned} \qquad (B3)$$

The first integral in the curly brackets may be evaluated simply (using the variable change $q=e^{-s}$) to give $\Sigma_{n=1}^{\infty}(-1)^{n+1}/nn!$, while the same variable change reduces the second integral to

$$\int_0^{\ln(u)}ds\exp(-ue^{-s})=\int_1^u\frac{dq}{q}e^{-q}=E_1(1)+O(e^{-u}/u). \qquad (B4)$$

Using the series expansion for the exponential integral [25], we may combine the two results to give

$$\int_0^\infty ds s^p T(s) = \gamma[\ln(u)]^p + O([\ln(u)]^{p-1}), \qquad \text{(B5)}$$

where $\gamma = 0.577\,22\ldots$ is Euler's constant. This is the required result for the subdominant terms.

The second integral appears in the evaluation of the large distance scaling in Sec. IV. Referring to Eq. (39), we need to evaluate

$$J(K) = \int_c^\infty du \exp\{-K(t)^{-(d+2)}[\ln(u)]^{d/2+1}\}, \qquad \text{(B6)}$$

in the nonlinear regime $K(t) \gg 1$. The constant c is a number of order unity and arises since we have used the large-u form for the integrand, and so we must cut off the integral at the lower end. For notational convenience let us consider

$$M_b(N) = \int_1^\infty du \exp\{-(1/N)[\ln(u)]^b\} \qquad \text{(B7)}$$

for $N \gg 1$, with $b > 1$. Then we can retrieve the integral of interest from $J(K) = M_{d/2+1}(K^{d+2})$.

In order to cast the integral into a form suitable for steepest descents, we make the variable change $x = N^{-1/(b-1)}\ln(u)$. We then have

$$M_b(N) = N^{1/(b-1)} \int_0^\infty dx \exp[-N^{1/(b-1)}(x^b - x)]. \qquad \text{(B8)}$$

This integral is easily performed by steepest descents to give (neglecting overall b-dependent constants)

$$M_b(N) \sim N^{1/2(b-1)} \exp\left[\frac{(b-1)}{b}\left(\frac{N}{b}\right)^{1/(b-1)}\right], \qquad \text{(B9)}$$

from which one may retrieve the form given in Eq. (40).

APPENDIX C

In this appendix we give details of the manipulation of $I(u,v,\mathbf{R})$ into a series expansion for small and large R. As given in Eq. (25), we have

$$I(u,v,\mathbf{R}) = \frac{2}{d\pi^{d/2}} \int d^d y \left\{ y^2 - \frac{\mathbf{y}\cdot\mathbf{R}}{2}\left[\frac{ue^{-(\mathbf{y}-\mathbf{R}/2)^2} - ve^{-(\mathbf{y}+\mathbf{R}/2)^2}}{ue^{-(\mathbf{y}-\mathbf{R}/2)^2} + ve^{-(\mathbf{y}+\mathbf{R}/2)^2}}\right]\right\} \times \{1 - \exp[-ue^{-(\mathbf{y}-\mathbf{R}/2)^2} - ve^{-(\mathbf{y}+\mathbf{R}/2)^2}]\}. \qquad \text{(C1)}$$

As a first step, we expand the exponential term in the second factor of the integrand to obtain

$$I = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n!} \Lambda_n(u,v,\mathbf{R}), \qquad \text{(C2)}$$

where

$$\Lambda_n(u,v,\mathbf{R}) = \frac{2}{d\pi^{d/2}} \int d^d y \left\{ y^2 [ue^{-(\mathbf{y}-\mathbf{R}/2)^2} + ve^{-(\mathbf{y}+\mathbf{R}/2)^2}]^n - \frac{\mathbf{y}\cdot\mathbf{R}}{2}[ue^{-(\mathbf{y}-\mathbf{R}/2)^2} + ve^{-(\mathbf{y}+\mathbf{R}/2)^2}]^{n-1} \times [ue^{-(\mathbf{y}-\mathbf{R}/2)^2} - ve^{-(\mathbf{y}+\mathbf{R}/2)^2}]\right\}. \qquad \text{(C3)}$$

This cumbersome expression may be simplified greatly by expanding the terms in square brackets as binomial series in powers of u and v and then performing the Gaussian integrals over $\mathbf{y}$. One is then left with

$$\Lambda(u,v,\mathbf{R}) = n^{-(d/2+1)} \sum_{m=0}^n C_m^n u^m v^{n-m} \exp\left[-\frac{m}{n}(n-m)R^2\right]. \qquad \text{(C4)}$$

Combining Eqs. (C2) and (C4) reproduces the large-R form given in Eq. (33).

In order to cast Λ into a small-R form, we expand the exponential terms in Eq. (C4) and then binomially resum the series in u and v. This leaves us with

$$\Lambda(u,v,\mathbf{R}) = n^{-(d/2+1)} \sum_{p=0}^\infty \frac{(-R^2)^p}{n^p p!} (uv\partial_u\partial_v)^p (u+v)^n. \qquad \text{(C5)}$$

We now substitute this expression back into Eq. (C2) and make the integral representation

$$n^{-(p+d/2+1)} = [\Gamma(p+d/2+1)]^{-1} \int_0^\infty ds s^{p+d/2} e^{-ns}. \qquad \text{(C6)}$$

This allows the sum over n to be performed as that for a geometric series and we are left with

$$I(u,v,\mathbf{R}) = \sum_{p=0}^\infty \frac{(-R^2)^p}{\Gamma(p+1)\Gamma(p+d/2+1)} \times (uv\partial_u\partial_v)^p L_{p+d/2}(u+v), \qquad \text{(C7)}$$

as given by Eqs. (26) and (27) in the main text.

APPENDIX D

This appendix will give details of the rewriting of the function $F_p(u,v)$ in moving from Eq. (27) to Eq. (28) in the main text. As given by Eq. (27), we have defined

$$F_p(u,v) = \frac{(-uv\partial_u\partial_v)^p}{\Gamma(p+1)\Gamma(p+d/2+1)} L_{p+d/2}(u+v). \qquad \text{(D1)}$$

Since we are interested in the nonlinear regime, we may expand the integral L as shown in Appendix B, namely,

$$L_{p+d/2}(u+v)=\frac{\Delta^{p+d/2+1}}{(p+d/2+1)}+\gamma\Delta^{p+d/2}+O(\Delta^{p+d/2-1}), \tag{D2}$$

where $\Delta\equiv\ln(u+v)$. Then we define the quantities $\chi_p(u,v;q)=(uv\partial_u\partial_v)^p\Delta^q$, which allow us to rewrite the above expression as

$$F_p(u,v)=\frac{(-1)^p}{\Gamma(p+1)\Gamma(p+d/2+1)}\left[\frac{\chi_p(u,v;p+d/2+1)}{(p+d/2+1)}+\gamma\chi_p(u,v;p+d/2)+O(\Delta^{p+d/2-2})\right], \tag{D3}$$

which is of the form given in Eq. (28) in the main text.

The above steps are largely a matter of redefinition of various quantities. We must extract explicit forms for the functions $\chi_p(u,v;q)$. Clearly the zeroth function is just $\chi_0(u,v;q)=\Delta^q$. The subsequent functions may be written as

$$\chi_p(u,v;q)=(uv\partial_u\partial_v)^p\Delta^q=f_p(u,v;q)\Delta^{q-1}+g_p(u,v;q)\Delta^{q-2}+O(\Delta^{q-3}). \tag{D4}$$

In order to determine unambiguously the scaling properties of the correlation function, it is sufficient to explicitly evaluate C to $O(R^6)$. However, given the singular nature of the scaling in this problem, we shall proceed to calculate the $O(r^8)$ terms as well, as a useful check. This in turn necessitates calculating the coefficients f_p and g_p for $p=1,2,3,4$. With the application of brute force algebra, we obtain

$$f_1(u,v;q)=-q\frac{uv}{(u+v)^2},$$

$$f_2(u,v;q)=+q\frac{uv}{(u+v)^4}(u^2-4uv+v^2),$$

$$f_3(u,v;q)=-q\frac{uv}{(u+v)^6}(u^4-26u^3v+66u^2v^2-26uv^3+v^4),$$

$$f_4(u,v;q)=+q\frac{uv}{(u+v)^8}(u^6-120u^5v+1191u^4v^2-2416u^3v^3+1191u^2v^4-120uv^5+v^6), \tag{D5}$$

and

$$g_1(u,v;q)=-(q-1)f_1,$$

$$g_2(u,v;q)=-2(q-1)f_2-q(q-1)\frac{u^2v^2}{(u+v)^4},$$

$$g_3(u,v;q)=-3(q-1)f_3+q(q-1)\frac{u^2v^2}{(u+v)^6}(17u^2-52uv+17v^2),$$

$$g_4(u,v;q)=-4(q-1)f_4-q(q-1)\frac{u^2v^2}{(u+v)^8}\times(129u^4-1648u^3v+3538u^2v^2-1648uv^3+129v^4). \tag{D6}$$

APPENDIX E

In this final appendix we give a brief description of the final steps required in order to perform the u and v integrals so as to derive the final form of the correlation function given in Eq. (30). Referring to the main text, we see that combining Eqs. (22), (24), and (26), we may write the correlation function in the form

$$C(\mathbf{r},t)=8\nu^2\int_c^\infty\frac{du}{u}\int_c^\infty\frac{dv}{v}\exp[-\epsilon F_0(u,v)]\times\left\{1-\exp\left[-\epsilon\sum_{p=1}^{\infty}R^{2p}F_p(u,v)\right]\right\}, \tag{E1}$$

where c is a number of order unity, required simply to cut off the integrals at their lower limits, given we are using the asymptotic form for the integrand in the nonlinear regime. The remaining steps are easy to describe, although rather tedious to perform in practice. We expand the exponential in the last factor of the above integrand in powers of R and then integrate over u and v. We shall explicitly demonstrate this for the dominant part of the $O(R^2)$ term. Using the asymptotic forms of the functions F_0 and F_1 from Appendix D, we have

$$O(R^2)\sim\frac{-8\nu^2\epsilon}{(d/2+2)\Gamma(d/2+2)}\int_c^\infty\frac{du}{u}\int_c^\infty\frac{dv}{v}f_1(u,v;2+d/2)\times\Delta^{d/2+1}\exp\left[-\frac{\epsilon}{\Gamma(d/2+2)}\Delta^{d/2+1}\right]. \tag{E2}$$

Remembering that $\epsilon=\Gamma(d/2+2)/K(t)^{d+2}$ and using the form of f_1 from Appendix D, we may rewrite this term as

$$[O(R^2)\ \ \text{term}]\sim8\nu^2K(t)^{-(d+2)}\int_c^\infty du\int_c^\infty dv(u+v)^{-2}\times\Delta^{d/2+1}\exp[-K(t)^{-(d+2)}\Delta^{d/2+1}]. \tag{E3}$$

Now the double integral has the form

$$\int_c^\infty du\int_c^\infty dv(u+v)^{-2}A(u+v)=\int_c^\infty du\int_u^\infty dw w^{-2}A(w)\simeq\int_c^\infty\frac{du}{u}A(u), \tag{E4}$$

where the last step was achieved using integration by parts on u. The final integral may be easily performed by substituting $x=\ln(u)$, thus yielding the first term in Eq. (30).

The integrals required at higher orders in R^2 may all be performed by changing variables from (u,v) to $(u,w=u+v)$ and performing the appropriate number of integrations by parts on u. In general, the ''boundary'' terms do not vanish, but they are negligible in the limit of interest, namely, $K(t)\gg 1$. (This is because the general form of the function $A(w)$, which appears as a spectator in these integral manipulations, is proportional to $\exp\{-K(t)^{-(d+2)}[\ln(w)]^{d/2+1}\}$, which decays faster than any power.) A final point worth mentioning is that the subdominant terms (i.e., those terms involving the coefficients g_p) are required since the (u,v) integrals over the leading terms containing f_p all vanish for $p>1$. This leads to the unusual scaling form described in the text, in which the dominant term may not be cast as part of the scaling function, and explains our cautionary calculation of the $O(R^8)$ term, which indeed confirms this singular scaling, i.e., the integrals over f_4 vanish exactly.

[1] J. M. Burgers, Proc. Acad. Sci. Amsterdam **43**, 2 (1940).

[2] V. I. Karpman, *Non-Linear Waves in Dispersive Media* (Pergamon, Oxford, 1975).

[3] D. J. Korteweg and G. de Vries, Philos. Mag. **39**, 422 (1895).

[4] M. Kardar, G. Parisi, and Y-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[5] D. B. Abraham, Phys. Rev. Lett. **44**, 1165 (1980); G. Forgacs *et al.*, *ibid.* **57**, 2184 (1986); B. Derrida *et al.*, J. Stat. Phys. **66**, 1189 (1992).

[6] D. A. Huse and C. L. Henley, Phys. Rev. Lett. **54**, 2708 (1985).

[7] G. Blatter *et al.*, Rev. Mod. Phys. **66**, 1125 (1994).

[8] S. N. Gurbatov *et al.*, *Non-Linear Random Waves and Turbulence in Nondispersive Media* (Manchester University Press, Manchester, 1991).

[9] M. Vergassola *et al.*, Astron. Astrophys. **289**, 325 (1994).

[10] D. J. Amit, *Field Theory, The Renormalization Group and Critical Phenomena*, 2nd ed. (McGraw-Hill, New York, 1984).

[11] A. J. Bray, Adv. Phys. **43**, 357 (1994).

[12] R. J. Glauber, J. Math. Phys. **4**, 294 (1963).

[13] T. J. Newman and A. J. Bray, J. Phys. A **23**, L279 (1990).

[14] J. M. Burgers, *The Non-Linear Diffusion Equation* (Reidel, Boston, 1974).

[15] S. Kida, J. Fluid Mech. **93**, 337 (1979).

[16] S. E. Esipov and T. J. Newman, Phys. Rev. E **48**, 1046 (1993).

[17] S. E. Esipov, Phys. Rev. E **49**, 2070 (1994).

[18] S. N. Gurbatov and A. I. Saichev, Zh. Éksp. Teor. Fiz. **80**, 689 (1981) [Sov. Phys. JETP **53**, 347 (1981)].

[19] Ya. G. Sinai, Commun. Math. Phys. **148**, 601 (1991).

[20] H. Jeffreys and B. Jeffreys, *Methods of Mathematical Physics*, 3rd ed. (Cambridge University Press, Cambridge, 1980).

[21] E. Hopf, Commun. Pure Appl. Math. **3**, 201 (1950).

[22] J. D. Cole, Q. Appl. Math. **9**, 225 (1951).

[23] B. Derrida, Phys. Rev. B **24**, 2613 (1981).

[24] T. J. Newman, Phys. Rev. E **49**, R2525 (1994); **51**, 4212 (1995).

[25] *Handbook of Mathematical Functions*, 10th ed., edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).

[26] J. Villain *et al.*, J. Phys. C **16**, 2588 (1983); A. Engel, J. Phys. (Paris) Lett. **46**, L409 (1985); J. Villain, J. Phys. A **21**, L1099 (1988); M. Mézard and G. Parisi, J. Phys. (France) I **2**, 2231 (1992).

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[28] V. Dotzenko and M. Mézard (unpublished).