Velocity-difference probability density functions for Burgers turbulence

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In this paper the Polyakov equation [Phys. Rev. E **52**, 6183 (1995)] for the velocity-difference probability density functions, with the random Gaussian external force, with the correlation function $\kappa(y)\sim 1-y^{\alpha}$, is analyzed. Solutions for the cases $\alpha=\{2,1/2,1\}$ are found, which agree very well with available numerical results. It is also argued that the stationary regime of Burgers turbulence can depend not only on the distribution of the external force, but also on the dissipative regularization. [S1063-651X(97)03306-0]

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The Burgers equation is attracting considerable attention as a model for one-dimensional turbulence without pressure, which captures, in a simple manner, some of its characteristic features. More precisely, when supplemented by a random external force correlated at large distances,

$$u_t+uu_x=\nu u_{xx}+f(x,t), \tag{1}$$

this equation can be used to describe the stationary turbulence with nonlinear energy transfer over scales from the pumping region (external force) to the dissipative one (shocks). It is always assumed that the regions of source and sink are very well separated, i.e., the formal limits of the large dimension of the system and small viscosity are considered. This is in accord with the general picture of developed turbulence, first proposed by Kolmogorov [1].

Recently, methods of quantum field theory became available for treating such a problem. They were developed and first applied to the Burgers turbulence by Polyakov [2], which allowed the qualitative explanation of numerical observations [3,4]. This indicates that the randomly driven Burgers equation is a possible candidate for an exactly solvable model. In this paper we present rather strong evidence in favor of such an assertion. We show that the methods of [2] allow one to obtain quantitatively accurate results. The methods we are using in this paper can be applied with some modifications to the turbulence with pressure, passive scalar advection, problems of self-organized criticality, etc. [2,12].

In its usual formulation the problem is specified by choosing the force to be Gaussian with zero mean and white in time variance

$$\langle f(x,t)f(x',t')\rangle=\kappa(x-x')\,\delta(t-t'). \tag{2}$$

Equation (1) thus becomes the Langevin equation, which leads uniquely to the Fokker-Planck equation for the probability distribution functional at time t

$$\frac{\partial}{\partial t}P([u],t)=\int dx\,\frac{\delta}{\delta u(x)}\Bigg[uu_x-\nu u_{xx}+\frac{1}{2}\int dy\;\kappa(x-y)\,\frac{\delta}{\delta u(y)}\Bigg]P([u],t). \tag{3}$$

The problem is completely defined if we assume $u(x)\equiv 0$ at some initial moment and consider the stationary limit of $P[u]$ as $t\to+\infty$. After that, we have to go to the limit $\nu\to 0$. We assume that both limits, taken in the specified order, exist. The solution for this equation can be represented in the form of a path integral, though it seems to be rather difficult to calculate it in a closed form. The saddle point approximations (instanton contributions) have been found in a number of works [7–9], which allowed the tails of the probability density function (PDF) for velocity differences and for velocity gradients to be calculated in some cases. Other direct approaches to the same calculations have been developed in [5,6].

The approach proposed by Polyakov [2] allows one to calculate not only the tails, but also the *whole* velocity-difference PDF. It is based on the self-consistent conjectures on the operator product expansion, Galilean and scaling invariance. Starting with the Burgers' equation with a random Gaussian stirring force [Eqs. (1) and (2)] it was obtained that the characteristic function for the velocity-difference PDF (Z function), determined as

$$Z(\mu,y)=\langle\exp(\mu[u(x+y/2)-u(x-y/2)])\rangle, \tag{4}$$

obeys the following differential master equation:

$$\left(\frac{\partial}{\partial\mu}-\frac{2b}{\mu}\right)\frac{\partial Z}{\partial y}-(\kappa(0)-\kappa(\mathbf{y}))\mu^2 Z=a(\mu)Z. \tag{5}$$

The correlation function of the external force $\kappa(y)$ can be chosen at our discretion, and b and $a(\mu)$ are undetermined coefficients, the so-called ‘‘anomalies.’’ The μ dependence of the a anomaly must be chosen to conform to the scaling invariance and can be different depending on the scaling properties of the force correlation function. If for a large-scale-correlated force this function can be expanded as $\kappa(y)\sim 1-y^{\alpha}$, then the a anomaly must depend on μ as follows: $a(\mu)=a\mu^{\sigma},\sigma=(2-\alpha)/(1+\alpha)$. Using the scaling ansatz $Z(\mu,y)=\Phi(\mu y^{\gamma}),\ \gamma=(\alpha+1)/3$, one can rewrite Eq. (5) in the form

$$\gamma x\Phi''+\gamma(1-2b)\Phi'-x^2\Phi=ax^{\sigma}\Phi, \tag{6}$$

where $x=\mu y^{\gamma}$.

The unknown parameters a and b should be determined from the main requirement that the PDF be a positive, finite, and normalized function. Other possible restrictions for the theory are discussed later. Polyakov considered the case α

$=2$ and found the solution for Eq. (6), corresponding to $a=0$. It gives the left tail for the velocity-difference PDF $\sim 1/u^{5/2}$, while the numerical results of [4] show $\sim 1/u^2$. [Here and in what follows we will refer to the velocity difference $\Delta u \equiv u(x+y/2)-u(x-y/2)$ simply as u.]

In fact, it was mentioned already in [2], that $a=0$ should not be the only possible choice. Strong evidence for this nonuniversality was also found in [4]. It was observed that the left tail of the PDF depends on the external force, while $a=0$ would exclude such a possibility. This indicates that there must be other solutions of the Polyakov equation, corresponding to $a\neq 0$.

In the present paper we find such solutions. We consider the correlator of the force in its general form $\kappa(y)\sim 1-y^{\alpha}$, and analyze several solvable cases ($\alpha=2,\alpha=1,\alpha=1/2$). The found solutions turn out to be in a remarkable agreement with numerical results of [4]. We then argue that in addition to the dependence on the external force, the stationary regime could also depend on the structure of the dissipation term (see also [12]).

To begin with, we write down the asymptotics of the solutions of Eq. (6) for small x

$$\Phi(x)\sim 1+\frac{a\gamma}{1-2b\gamma}x^{3/(\alpha+1)}+cx^{2b}+\cdots , \tag{7}$$

and for large positive x

$$\Phi(x)\propto \exp\frac{2}{3\sqrt{\gamma}}x^{3/2}, \tag{8}$$

where a, b, and c should be determined from the conditions mentioned above.

We note that the most restrictive condition, the condition of normalizability of the PDF, can be reformulated directly in terms of the Φ function. Indeed, the function Φ must be analytical in the right half of the complex plane Re $x\geqslant 0$, and must vanish for $x\to\rho\pm i\infty$, $\rho\geqslant 0$. This, along with the condition of normalization $\Phi(0)=1$, gives the quantization rule for a and b.

Let us denote the Laplace transform of $\Phi(x)$ as $\widetilde{w}(z)$, the velocity-difference distribution function being $w(u,y)=\widetilde{w}(u/y^{\gamma})/y^{\gamma}$. The integral representation for $\widetilde{w}(z)$ is

$$\widetilde{w}(z)=\int_{\rho-i\infty}^{\rho+i\infty}e^{-xz}\Phi(x)dx. \tag{9}$$

The asymptotics of $\widetilde{w}(z)$ for large positive z is determined by large x and is given by $\widetilde{w}(z)\propto\exp(-\gamma z^3/3)$. To find the asymptotics for large negative z, we deform the tails of the integration contour to coincide with the negative real axis. Since e^{-xz} decays rapidly as $x\to-\infty$, the asymptotics is determined by the leading singularity in the expansion (7).

In general, two cases are possible. If $3/(\alpha+1)<2b$, the asymptotics is $\widetilde{w}\sim z^{-1-3/(1+\alpha)}$. Such behavior is observed in numerical simulations [4], which indicates that this inequality usually holds, and the b anomaly does not affect the asymptotics. For $2b<3/(\alpha+1)$ the asymptotics should in general be determined by the b anomaly, $\widetilde{w}(z)\sim z^{-2b-1}$, if $2b$ is not an integer. The asymptotics (7) also shows that there exist two degenerate cases. These are the cases when $3/(\alpha+1)$ is an integer, and the corresponding term does not contribute to the integral. These cases ($\alpha=1/2,2$) are solvable and will be considered below. We will also consider another solvable case, with $\alpha=1$. By solvability we mean that the problem can be either solved exactly or reduced to finding the ground state in some potential, which can be done numerically.

We start with the case $\alpha=2$. Let us Laplace transform Eq. (6) to get an equation for the probability distribution $w(u,y)=\widetilde{w}(z)/y$

$$\widetilde{w}''+z^2\widetilde{w}'+(1+2b)z\widetilde{w}=-a\widetilde{w}, \tag{10}$$

where $z=u/y$, assuming the notation of [2]. All derivatives in this equation are with respect to z. Below we consider only the function $\widetilde{w}$ and drop the tilde sign. Asymptotics of the solution at $|z|\to\infty$ can be easily found from Eq. (10)

$$w\propto e^{-z^3/3},\quad w\sim\frac{1}{z^{2b+1}}. \tag{11}$$

We are looking for a physically reasonable solution, with the asymptotics

$$w\propto e^{-z^3/3},\quad z\to+\infty,$$

$$w\sim\frac{1}{z^{2b+1}},\quad z\to-\infty. \tag{12}$$

For the w function to be normalizable we should consider only $b>0$. Upon writing $w=\Psi e^{-z^3/6}$, we exclude the first derivative from Eq. (10) and get the Schrödinger equation for the Ψ function,

$$-\Psi''+\left(\frac{z^4}{4}-2bz\right)\Psi=a\Psi, \tag{13}$$

mentioned in [2]. The ground state of this equation is a positive and normalizable function. This is the only solution satisfying the general requirements for the PDF. Thus, for any $b>0$ we find the PDF as the ground state of the potential (13), a being the energy of the ground state. Note that the case $b=1/2$ corresponds to the left tail of the PDF $\sim 1/u^2$, and the PDF obtained as a solution of Eq. (13), fits well the numerical observations [4] (Fig. 1). A numerical estimate in this case gives for the a anomaly $a\simeq 0.354$.

An important remark should be made here. Integrating Eq. (10) from $-\infty$ to $+\infty$ for the case $b>1/2$, we get

$$(2b-1)\int zw(z)=-a\int w(z).$$

We would like to stress that this expression does not contradict the requirement $\int w(u,y)u\,du=0$. A significant contribution to the latter integral can come from nonuniversal tails of the distribution function, not described by Eq. (10). These tails are due to spontaneous breakdown of the Galilean symmetry [2]. This fact should be taken into account when one compares the theoretical results with experimental observations.

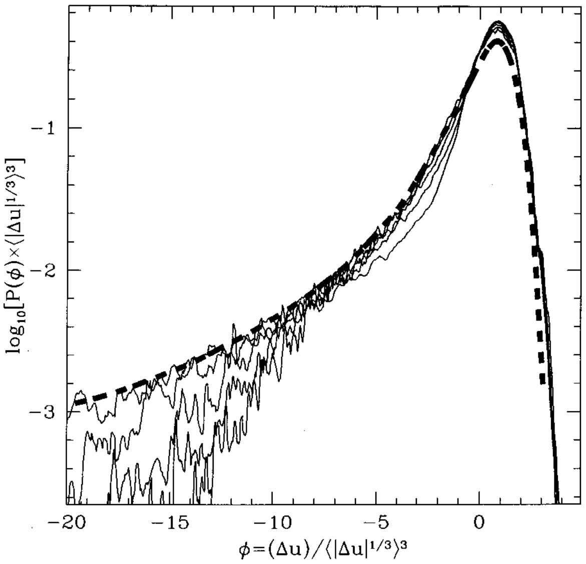

FIG. 1. Collapse of the PDFs in the universal region of Δu, for $\alpha=2$. The solution of Eq. (10) for $b=1/2$ is depicted by the dashed line. (Courtesy of V. Yakhot and A. Chekhlov [4].)

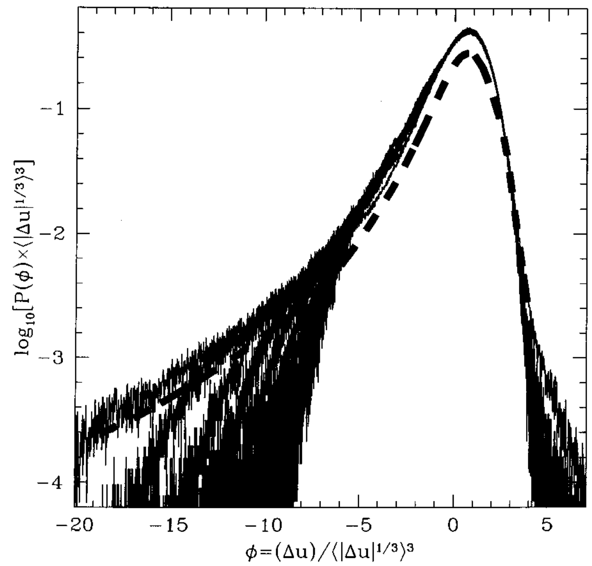

FIG. 2. Collapse of the PDFs in the universal region of Δu, for $\alpha=1/2$. The solution of Eq. (16) for $b=1$ is depicted by the dashed line. (Courtesy of V. Yakhot and A. Chekhlov [4].)

Nevertheless, a case exists for which $\langle z\rangle=0$, which corresponds to $a=0$. To consider it, we set $a=0$ in (10), and by the substitution $s=-z^3/3$ arrive at the degenerate hypergeometric equation

$$sw''+(\gamma-s)w'-\alpha w=0, \tag{14}$$

with parameters $\gamma=\frac{2}{3}$, $\alpha=\frac{1}{3}(2b+1)$ [do not confuse these parameters, used only in the analysis of Eq. (14), with the parameters α and γ, introduced in Eq. (6)]. The positive, finite and normalizable solution for this case has been found in [2]. This solution can be constructed in the following way: the only solution, exponentially decaying at $s\to-\infty$ and having powerlike asymptotics at $s\to+\infty$, has the form

$$w(s)=\int_{-\infty}^{(s+)} e^t(t-s)^{-\alpha}t^{\alpha-2/3}dt, \quad s<0,$$

$$w(s)=\int_{-\infty}^{(0+)} e^t(t-s)^{-\alpha}t^{\alpha-2/3}dt, \quad s>0, \tag{15}$$

where in each integral the contour of the integration starts from $-\infty$, goes around only one of the two singular points (denoted as the upper limits) in a positive direction and ends up at $-\infty$ again. One of these solutions can be analytically continued to the other one only if $\alpha=n-1/6$, where n is any integer number. It is interesting to note that this exact quantization rule can also be obtained as the Bohr-Sommerfeld condition for quantum mechanics considered above, with zero energy (I would like to thank V. Gurarie for pointing this out). Positivity of the solution requires $n=1$.

For the other degenerate case, the correlator of the external force has the form: $\kappa(y)=1-y^{1/2}$. This force leads to a differential equation for the $w(z)$ function, analogous to Eq. (10)

$$w''+\tfrac{1}{2}z^2w'+(\tfrac{1}{2}+b)zw=aw', \tag{16}$$

where $z=u/y^{1/2}$. Asymptotics of the left tail of the solution is given by Eq. (12). Excluding the first derivative from this equation, we obtain the Schrödinger equation for the function $\Psi=w\ \exp(z^3/12-az/2)$

$$-\Psi''+\left(\frac{z^4}{16}-\frac{a}{4}z^2-bz\right)\Psi=-\frac{a^2}{4}\Psi. \tag{17}$$

As in the previous case, one can find the solutions as the ground states of this equation. The numerically observed PDF [4] has the left tail $\sim 1/u^3$ in the considered case. The same PDF can be obtained from our equation if we set $b=1$, i.e., when the β anomaly is absent. One can then numerically obtain $a\simeq-0.473$. A comparison of the whole PDF with the numerical results [4] reveals a very good agreement (Fig. 2).

To analyze the last case, $\alpha=1$, let us work in the x representation. Note that by the substitution $\zeta=x^{3/2}$ one can cast Eq. (6) into the form

$$\tfrac{3}{2}\zeta\Phi''+(\tfrac{3}{2}-2b)\Phi'-(\zeta+a)\Phi=0. \tag{18}$$

This equation can be solved by the Laplace transform. The solution with the correct asymptotics is

$$\Phi(x)=Cx^{-3/2[\alpha_1+\alpha_2+1]}e^{\sqrt{2/3}x^{3/2}} \times\int_{-\infty}^{(0+)} e^{\tau}\tau^{\alpha_1}(\tau+2\sqrt{2/3}x^{3/2})^{\alpha_2}d\tau. \tag{19}$$

with

$$\alpha_1=-\tfrac{1}{2}[1+4b/3]+a/\sqrt{6}, \quad \alpha_2=-\tfrac{1}{2}[1+4b/3]-a/\sqrt{6}.$$

$\Phi(x)$ will be an analytical function for $\mathrm{Re}\,x\geqslant 0$, and a decaying function for $x\to\rho\pm i\infty$ only when $\alpha_1=n$ or $\alpha_2=m$, where n is any negative integer number and m is any non-negative integer number. The only possibility of getting

$\Phi(0)=1$ is $\alpha_1=n$, which gives the following quantization rule: $a/\sqrt{6}-2b/3=n+1/2$. Positivity of the solution forces us to select $n=-1$, and Eq. (19) reduces to $\Phi=\exp\sqrt{2/3}x^{3/2}$.

Finally, we discuss an important general restriction that can be imposed on the theory. This follows from the physical condition of positivity of dissipation and was proposed by Polyakov [10]. It can be obtained if one notes that the operator

$$\frac{\partial^2}{\partial x^2} e^{(\lambda u(x)+\lambda_1 u(x_1)+\cdots)} \tag{20}$$

is not singular if x does not coincide with any other x_i. Therefore,

$$\lim_{\nu\to 0}\nu \frac{\partial^2}{\partial x^2} e^{\lambda u(x)}=0, \tag{21}$$

which leads to

$$\alpha(\lambda)Z+\frac{\tilde{\beta}}{\lambda}\frac{\partial}{\partial x}Z=-\lim_{\nu\to 0}\nu\langle\lambda^2 u_x^2 e^{\lambda u(x)+\cdots}\rangle, \tag{22}$$

where we use the notation of [2]. The right-hand side of this expression is nonpositive. The function $\alpha(\lambda)$ is analytical in the right half of the complex plane, and may have a discontinuity at the imaginary axis. Summing up corresponding expressions for $\lambda_1=\mu/2$, $x_1=x+y/2$ and $\lambda_2=-\mu/2$, $x_2=x-y/2$, we get the following necessary condition, that must be valid for all non-negative x:

$$ax^{\sigma}\Phi-2(1-b)\Phi'\leqslant 0, \tag{23}$$

where $b=\tilde{\beta}+1$.

One can easily see that this condition is rather strong and allows one to considerably restrict the possible solutions of Eq. (5). For example, it prohibits the solutions with $b<3/4$ for the case $\alpha=2$ and, probably, forces the β anomaly to vanish for $\alpha\leqslant 1/2$.

Nevertheless, this inequality is absent (or, at least, the above arguments do not work) if we consider the dissipation in the form $(-1)^{p+1}\nu\partial^{2p}/\partial x^{2p}$, with $p>1$ (the so-called hyperdissipation). This is the case for which the numerical simulations [4] have been performed. The structure of the shock fronts is changed qualitatively for $p>1$ (see, e.g., numerical simulations in [11]), which could lead to different stationary regimes of the Burgers turbulence. In the framework of the Polyakov method the possibility of such a nonuniversality can be simply explained: any new small dissipative operator, added to the system, has to be expandable into UV-finite ones, which are conjectured to be Z and Z_x', with some new coefficients a and b.

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