When coherent stochastic resonance appears

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We consider a system driven by a combination of Gaussian white noise and a periodic signal. It is demonstrated when the mean-exit time for this system necessarily exhibits a minimum as a function of the frequency of the periodic bias and gives rise to the phenomenon known as coherent stochastic resonance. This reasoning begins by calculating an exact expression for the mean-exit time of a system driven by noise and a telegraph signal and showing that no resonancelike behavior arises. This unexpected result lets us identify the sufficient conditions for resonance to appear. The origin of coherent stochastic resonance lies in the different behavior of low frequency and high frequency periodic signals. [S1063-651X(97)14505-6]

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I. INTRODUCTION

Coherent stochastic resonance is among those phenomena collected under the name stochastic resonance (SR) [1]. The ingredients for a system to exhibit SR are a noise, which is usually taken to be white Gaussian for simplicity, and a small driving periodic force added to the dynamics of the system. When these components mix properly, SR reveals by a non-monotonic behavior of some properties of the system if considered as functions of the noise intensity or of the periodic driving frequency. Systems showing stochastic resonance have been found in biology, magnetic systems, and other physical systems [1] and many studies have been devoted to it in recent years [2].

However, efforts have not concentrated similarly on coherent stochastic resonance (CSR). The paradigm of a system exhibiting CSR is a diffusion process driven by an oscillating force when the motion is constrained between two traps. The main properties of such a system were studied in [3]. The mean-exit time (MET) out of a region limited by two traps was analyzed through simulation of a random walk on a lattice for a sinusoid driving force. It was shown that the MET exhibits a minimum at some frequency of the driving force. The origin of this ''resonance'' was related to the coherent motion induced by the periodic bias. Thus, the phenomenon was generically called coherent stochastic resonance.

In a recent paper [4], the MET of a diffusion process driven by an oscillating force was considered when the sinusoid bias was substituted by a telegraph signal. This signal switches alternatively between the values $+v_0$ and $-v_0$ after a constant period of time τ. An expression for the MET was obtained but, due to numerical errors in the computation of its value [5], it was concluded incorrectly that the system presented CSR. In this paper, we calculate the MET for this system and show that it does not exhibit any resonant behavior. However, this result lets us identify the sufficient conditions for coherent stochastic resonance, which is one of the main goals of this paper.

Another kind of stochastic resonance phenomena is resonant activation [6] and it deals with the problem of potential barrier crossing due to thermal noise in the presence of fluctuations of the barrier itself. In this problem, barrier fluctuations are modeled by a Markovian colored noise, that is, by a noise with finite correlation time. It turns out that the activation rate as a function of the correlation time shows a maximum at some correlation time. The origin of this maximum lies in the different behavior of the colored noise at short and long correlation times as has been explained in [7]. Coherent stochastic resonance and resonant activation appear in quite different systems. Note, for instance, that in the former it exists with a periodic signal while in the second a colored noise replaces the periodic bias. However, as explained in this paper, CSR arises from the same source as resonant activation.

The paper is organized as follows. In Sec. II, we obtain an expression for the mean-exit time of a particle simultaneously driven by additive white noise and a periodic square wave. The next section is devoted to a discussion of these results. It is shown that the MET does not exhibit any resonant behavior. In addition, we compare the MET for this system with that of diffusion in the presence of dichotomous noise. Finally, we explain the origin of the CSR. Conclusions are drawn in the last section and some mathematical details of derivations in Sec. II are given in the Appendix.

II. ANALYSIS

Our explanation of coherent stochastic resonance arises from the analysis of a system driven by white Gaussian noise and a telegraph signal. The evolution of the system considered is governed by the one-dimensional equation

$$\dot{X}=\xi(t)+v(t), \tag{1}$$

where $\xi(t)$ is a zero-mean Gaussian white noise of intensity D, i.e.,

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{2}$$

and $v(t)$ is the telegraph signal:

$$v(t)\begin{cases} +v_0, & t\in[2n\tau,(2n+1)\tau] \\ -v_0, & t\in[(2n+1)\tau,(2n+2)\tau], \end{cases} \tag{3}$$

v_0 is a constant and $n=0,1,\ldots$. The period of the telegraph signal is 2τ and, therefore, its frequency $\omega=(2\tau)^{-1}$. We are interested in the mean survival time of the system when two traps are set at $x=0$ and $x=L$, that is to say, the MET out of region $(0,L)$.

Dimensionless units will be used in what follows. This is equivalent to setting the noise intensity D and the telegraph amplitude v_0 equal to 1. The probability density function for the position of a diffusive system driven by a bias $v=+1$, in the presence of two traps at $x=0$ and $x=L$, reads as follows [4]:

$$p_+(x,t|x_0)=\frac{2}{L}\exp[\tfrac{1}{2}(x-x_0)]\times\sum_{n=1}^{\infty}\exp[-(\beta_n^2+\tfrac{1}{4})t]\sin(\beta_n x)\sin(\beta_n x_0), \tag{4}$$

where $\beta_n=n\pi/L$. It is convenient to write this density as

$$p_+(x,t|x_0)=\frac{2}{L}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\psi_n^+(x)M_{nk}(t)\psi_k^-(x_0), \tag{5}$$

where the functions ψ_n^+ and ψ_n^- are defined as

$$\psi_n^-(x)=e^{-x/2}\sin(\beta_n x),\qquad \psi_n^+(x)=e^{x/2}\sin(\beta_n x), \tag{6}$$

and the matrix $\boldsymbol{M}(t)$ is

$$M_{nk}(t)=\exp[-(\beta_n^2+\tfrac{1}{4})t]\delta_{nk}, \tag{7}$$

where δ_{nk} is the Kronecker δ function ($\delta_{nk}=1$ if $n=k$, otherwise $\delta_{nk}=0$). The advantage of this notation will be explained below. In addition, functions without indexes will denote vectors, for instance,

$$\boldsymbol{\psi}^-(x_0)=\{\psi_1^-(x_0),\psi_2^-(x_0),\ldots,\psi_n^-(x_0),\ldots\}. \tag{8}$$

In order to shorten the notation, we will use the dot product to represent infinite sums over pairs of repeated indexes. The expression

$$\sum_{k=1}^{\infty}M_{nk}(t)\psi_k^-(x_0), \tag{9}$$

for example, will be represented by

$$\boldsymbol{M}(t)\cdot\boldsymbol{\psi}^-(x_0). \tag{10}$$

Therefore, the expression for $p_+(x,t|x_0)$ reduces to

$$p_+(x,t|x_0)=\frac{2}{L}\boldsymbol{\psi}^+(x)\cdot\boldsymbol{M}(t)\cdot\boldsymbol{\psi}^-(x_0). \tag{11}$$

When the bias is -1, the probability density in this notation is

$$p_-(x,t|x_0)=\frac{2}{L}\boldsymbol{\psi}^-(x)\cdot\boldsymbol{M}(t)\cdot\boldsymbol{\psi}^+(x_0). \tag{12}$$

In order to calculate the survival probability $S(x_0,t)$, it is necessary to know the probability for the position of the particle at the points where the telegraph signal changes. We define $P_k(x), k=0,1,\ldots,$ as the probability that the particle is found at a position between x and $x+dx$ at the kth change of the bias, that is to say, at time $t=k\tau$. Since the telegraph signal is piecewise constant it follows that $P_k(x)$ satisfies the recurrence relations

$$P_{2n+1}(x)=\int_0^L p_+(x,\tau|y)P_{2n}(y)dy,\qquad P_{2n+2}(x)=\int_0^L p_-(x,\tau|y)P_{2n+1}(y)dy, \tag{13}$$

where $n=0,1,\ldots$. The relation $P_0=\delta(x-x_0)$ arises because we assume that the system is initially at x_0. The mean-exit time $T(x_0)$ is related to $S(x_0,t)$ by

$$T(x_0)=\int_0^{\infty}S(x_0,t)dt. \tag{14}$$

It is convenient to rewrite this integral as

$$T(x_0)=\sum_{k=0}^{\infty}\int_{k\tau}^{(k+1)\tau}S(x_0,t)dt \tag{15}$$

because the survival probability for times $t\in[k\tau,(k+1)\tau]$ is related to $P_k(x)$ for k even by

$$S(x_0,t)=S(x_0,t'+k\tau)=\int_0^L dx\int_0^L p_+(x,t'|y)P_k(y)dy, \tag{16}$$

where $t'=t-k\tau$, and replacing p_+ by p_- when k is odd. After this substitution in Eq. (15), the following expression for the mean-exit time is obtained [4]:

$$T(x_0)=\sum_{n=0}^{\infty}\int_0^{\tau}dt'\int_0^L dx\int_0^L[p_+(x,t'|y)P_{2n}(y)+p_-(x,t'|y)P_{2n+1}(y)]dy. \tag{17}$$

In this paper we develop this expression further and obtain a method to calculate the mean-exit time up to a prescribed order of approximation.

The probabilities $P_k(x)$ are deduced in the Appendix and read as follows:

$$P_{2n+1}(x)=\frac{2}{L}\boldsymbol{\psi}^+(x)\cdot(\boldsymbol{M}\cdot\boldsymbol{I}\cdot\boldsymbol{M}\cdot\boldsymbol{J})^n\cdot\boldsymbol{M}\cdot\boldsymbol{\psi}^-(x_0),\qquad P_{2n+2}(x)=\frac{2}{L}\boldsymbol{\psi}^-(x)\cdot\boldsymbol{M}\cdot\boldsymbol{J}\cdot(\boldsymbol{M}\cdot\boldsymbol{I}\cdot\boldsymbol{M}\cdot\boldsymbol{J})^n\cdot\boldsymbol{M}\cdot\boldsymbol{\psi}^-(x_0), \tag{18}$$

with $n=0,1,\ldots$ and $\boldsymbol{M}\equiv\boldsymbol{M}(\tau)$. The matrices $\boldsymbol{I}$ and $\boldsymbol{J}$ are defined as follows:

$$I_{nk} \equiv \frac{2}{L}\int_0^L \psi_n^-(x)\psi_k^-(x)dx = \tfrac{1}{2}[1-(-1)^{i+j}e^{-L}]\left[\frac{4\beta_i\beta_j}{(1+\beta_i^2+\beta_j^2)^2-4\beta_i^2\beta_j^2}\right],$$

$$J_{nk} \equiv \frac{2}{L}\int_0^L \psi_n^+(x)\psi_k^+(x)dx = \tfrac{1}{2}[(-1)^{i+j}e^{L}-1]\left[\frac{4\beta_i\beta_j}{(1+\beta_i^2+\beta_j^2)^2-4\beta_i^2\beta_j^2}\right]. \tag{19}$$

When the functions $P_k(x)$ are introduced into Eq. (17), the mean-exit time is given by

$$T(x_0)=\sum_{n=1}^{\infty} e^{-x_0/2}\sin(\beta_n x_0)T_n(\omega), \tag{20}$$

with

$$T_n(\omega)=c_n^+ + [\boldsymbol{c}^+\cdot \boldsymbol{I}\cdot \boldsymbol{M}\cdot \boldsymbol{J}\cdot \boldsymbol{A}\cdot \boldsymbol{M}]_n + [\boldsymbol{c}^-\cdot \boldsymbol{J}\cdot \boldsymbol{A}\cdot \boldsymbol{M}]_n . \tag{21}$$

In this expression, the matrix $\boldsymbol{A}$ is defined implicitly as

$$\boldsymbol{A}^{-1}=\mathbf{1}-\boldsymbol{M}\cdot \boldsymbol{I}\cdot \boldsymbol{M}\cdot \boldsymbol{J}, \tag{22}$$

with **1** being the identity matrix, and the components of vectors $c^{\pm}$ are given by

$$c_n^+ = \frac{2}{L}\int_0^\tau dt\int_0^L [\boldsymbol{\psi}^+(x)\cdot \boldsymbol{M}(t)]_n dx,$$

$$c_n^- = \frac{2}{L}\int_0^\tau dt\int_0^L [\boldsymbol{\psi}^-(x)\cdot \boldsymbol{M}(t)]_n dx. \tag{23}$$

The components T_n depend on ω through the matrix $\boldsymbol{M}$ and the vectors $\boldsymbol{c}^+$ and $\boldsymbol{c}^-$ that are functions of τ and $\tau=(2\omega)^{-1}$.

Once dimensions are introduced, Eqs. (20) and (21) provide the exact solution to the mean-exit time out of region $(0,L)$ for a system driven by white Gaussian noise and a telegraph signal, Eq. (3). It is worth mentioning that it is possible to use perturbation methods [3] and linear response theory to deal with the same problem when the amplitude v_0 of the driving signal is small. The latter technique has proved very successful in the field of stochastic resonance [8] and may give a different insight into the question addressed in this paper. Nevertheless, the discussion included in Sec. III is based on our analytical work.

III. DISCUSSION

Expression (21) for $T_n(\omega)$ constitutes an important result of this paper. It has the advantage over the expression proposed in [4] that it can be calculated numerically up to a prescribed order. When an approximation of order k is considered, all matrices $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{A},\boldsymbol{M},\mathbf{1})$ and vectors $(\boldsymbol{c}^-,\boldsymbol{c}^+,\boldsymbol{\psi}^-,\boldsymbol{\psi}^+)$ are truncated to order k. Then, $T(x_0)$ is computed for different increasing k until it converges to a definite value. In Fig. 1, the mean-exit time is calculated as a function of the frequency for different orders. It can be seen that for $\omega<3$ in dimensionless units it is sufficient to truncate at order 10. As the values of ω increase, higher order approximations are needed to get good convergence.

As shown in Fig. 1, the mean-exit time results in a monotonic increasing function of frequency and does not show any resonant behavior when Eq. (20) is used to compute it. A physical argument shows why the MET decreases as the frequency approaches zero. When the frequency is slightly greater than zero, the bias reverses to -1 at some instant (precisely at $t=\tau$). By this time, only a small proportion of realizations of the process, those that contribute to the greater values of the MET, are found inside the interval $(0,L)$. Due to the inversion of the bias, these realizations will remain on average a longer time in the interval than they would if the bias had not reversed. Therefore, MET will increase due to this contribution. Moreover, as shown in the Appendix, the following expansion holds from Eq. (20) when $\omega\to 0$:

$$T(x_0)=T_{\text{bias}}(x_0)+\frac{64L^2\pi}{L^2+4\pi^2}\exp[-\tfrac{1}{2}x_0] \times\exp\left[-\frac{1}{8L^2\omega}(L^2+4\pi^2)\right]\sin(\pi x_0/L) \times\left[\frac{L(1+e^{3L/2})}{(9L^2+4\pi^2)(e^L-1)}-\frac{(1+e^{L/2})}{L^2+4\pi^2}\right], \tag{24}$$

where T_{bias} is the value of the MET at $\omega=0$ and corresponds to the mean-exit time of a region $(0,L)$ for a system driven by white noise of intensity $D=1$ plus a constant bias $+1$,

$$T_{\text{bias}}(x_0)=-x_0+L\frac{1-e^{-x_0}}{1-e^{-L}}. \tag{25}$$

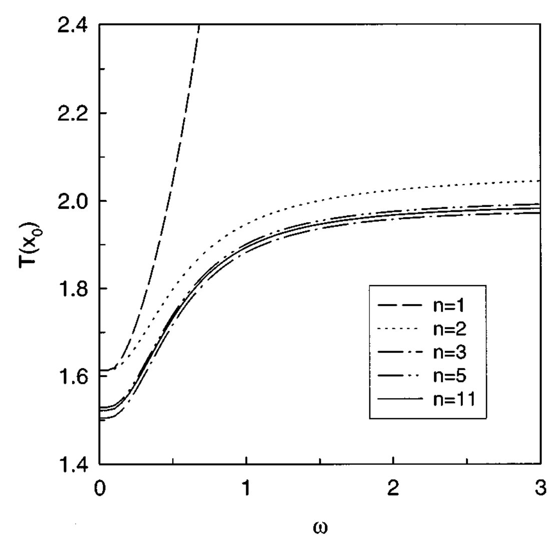

FIG. 1. Approximation of order n to the mean-exit time $T(x_0)$ as a function of ω when $L=4$ and $x_0=2$ in dimensionless units $(D=1,v_0=1)$ for (a) $n=1$ (short dashed line); (b) $n=2$ (long dashed line); (c) $n=3$ (dot-dashed line); (d) $n=5$ (double-dot-dashed line); (e) $n=11$ (solid line).

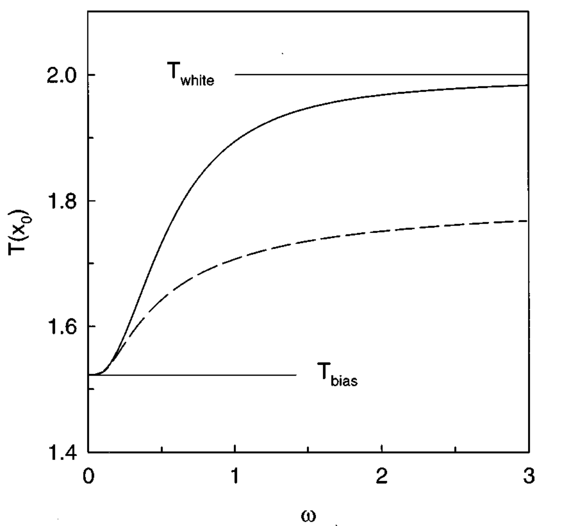

FIG. 2. Mean-exit time $T(x_0)$ as a function of the frequency ω of the telegraph signal for $L=4$ and $x_0=2$. Thin solid lines show the limiting values of the MET when $\omega \to \infty, T_{\text{white}}=2.0$, and $\omega \to 0, T_{\text{bias}}=1.523\ldots$. The approximate result Eq. (24) is displayed as a dashed line. All magnitudes are dimensionless $(D=1, v_0=1)$.

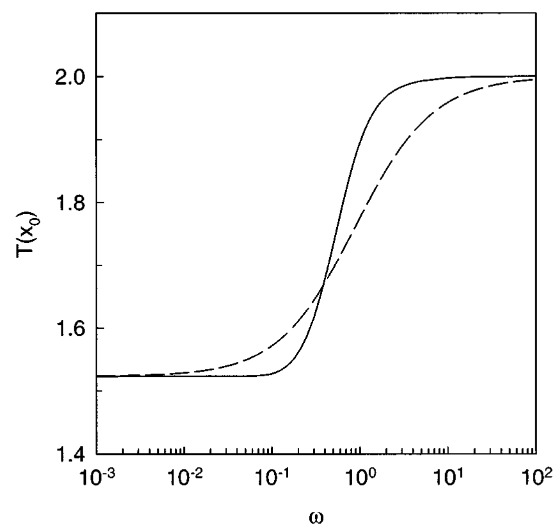

FIG. 3. Mean-exit time $T(x_0)$ as a function of the frequency ω for a system driven by a telegraph signal and white noise (solid line), and a system driven by dichotomous noise $\lambda=2\omega$ and white noise (dashed line). Magnitudes are represented in dimensionless units that correspond to a unit amplitude of the telegraph signal, $v_0=1$, and white noise intensity D equal to 1. Dimensionless units are used in both plots and parameter values are $L=4$ and $x_0=2$.

We note that the first order correction to the MET at zero frequency is of order $\exp[-\gamma/\omega]$. In Fig. 2, the low frequency approximation of $T(x_0)$, Eq. (24), is plotted.

In the opposite limit, $\omega \to \infty$, the evolution of the system is increasingly less influenced by the telegraph signal because of its higher rate of oscillation. Consequently, the mean-exit time when $\omega \to \infty$ converges to the MET of a pure diffusive system,

$$T_{\text{white}}(x_0)=\frac{x_0(L-x_0)}{2}, \tag{26}$$

that is, to the MET of system governed by Eq. (1) when $v(t)=0$.

The telegraph signal $v(t)$, Eq. (3), becomes a random noise when its half period is randomized every time the signal changes value. This randomized signal corresponds to a Markovian dichotomous noise if the telegraph signal half period τ is distributed according to an exponential density function

$$\phi(t)=\text{Prob}\{t<\tau \leq t+dt\}=\lambda e^{-\lambda t}. \tag{27}$$

The parameter λ is the inverse of the average time that the noise keeps the same value. Although the period of the dichotomous noise is random, it is possible to associate to it an average frequency, $\omega=\lambda/2$, which gives half the average number of noise reversals per unit time.

It has been shown recently [9] that the MET of a system driven by the superposition of white noise and dichotomous noise does not exhibit any resonantlike behavior. In Fig. 3, the MET of a system that evolves according to Eq. (1) is compared with the MET of a system driven by just the same white noise and a dichotomous noise instead of a telegraph signal. The amplitude of the random dichotomic signal is identical to that of the telegraph signal v_0 and the parameter λ is the inverse of the telegraph signal half period τ. The figure shows that the MET for the system driven by dichotomous noise (dashed line) grows faster from the $\omega=0$ value and converges slower to the high frequency limit, Eq. (26), than the MET of system (1). The reason for this behavior lies in the fact that a random value of the signal half period allows a wider range of extreme events than a deterministic period. Let us consider, for instance, the case of low frequencies. For the telegraph signal, it is not relevant that ω is not zero whenever the MET remains much smaller than $1/\omega$ because by the time the telegraph signal changes its value most realizations have already escaped. However, with a random period there is a probability that a change in the signal happens very soon and therefore the MET is more severely modified than in the previous case. An analogous reasoning explains the observed behavior at high frequencies.

We now turn to the main question of this paper: the origin of coherent stochastic resonance. As mentioned in the Introduction, it was shown in [3] that the MET of a diffusive system driven by a pure sinusoidal bias exhibits a minimum at some frequency of the driving force. However, we have determined in this paper that the resonance disappears when the sinusoidal signal is substituted by a telegraph signal. Both signals are periodic and therefore equally ''coherent.'' Hence the reason for coherent stochastic resonance cannot be the coherence of the periodic bias [4]. At high frequencies, the influence of any of the two signals on the MET diminishes until it disappears completely (we assumed the periodic force to be subthreshold, that is, the system cannot escape in the absence of noise). What makes the two signals different is the low frequency behavior. The sinusoidal function (with phase equal to zero) increases linearly with time at very low frequencies because

$$\sin\omega t \sim \omega t, \tag{28}$$

when $\omega \to 0$. Conversely, the telegraph signal (that only takes two values) keeps its value constant for a period τ. It turns out that this difference is crucial. The MET of the system driven by the sinusoidal bias necessarily decreases when the frequency increases from zero because at a low frequency (different from zero) it exhibits a growing bias, Eq. (28), that makes the MET smaller than its value at $\omega = 0$. In addition, the two limiting values of the MET at $\omega \to \infty$ and $\omega = 0$ coincide with T_{white}, Eq. (26). Consequently, we have demonstrated that if MET is a continuous function of the frequency of the sinusoidal driving force, it exhibits one minimum (at least) at some frequency. Using similar reasoning, we demonstrated that MET of a system driven by a telegraph signal must increase at low frequencies. This argument does not exclude the possibility of a minimum but implies that, if it existed, a maximum would appear first.

Therefore, the source of the so-called coherent stochastic resonance lies in the different behavior of a periodic signal at low and high frequencies. When the periodic signal behaves as an increasing function of time at low frequencies, the MET decreases. Then, coherent stochastic resonance appears, that is, the MET exhibits a minimum at some frequency, if MET at $\omega = 0$ is less than its value at the limit $\omega \to \infty$. This argument parallels the explanation of resonant activation given in [7]. There, however, the behavior of the system was known at high frequencies (short correlations times) and the condition for resonant behavior is that the escape time at zero frequency is greater than its value at infinite frequency (correlation time equal to zero). Therefore, we see that both phenomena, CSR and resonant activation, emerge because of the different nature of the driving force (a periodic bias in the case of CSR or a correlated Markovian noise in resonant activation) when its characteristic time scale (the period for the periodic bias and the correlation time for the noise) approaches zero and infinity.

IV. CONCLUSIONS

We have revisited the problem of the mean-exit time out of a region for a system driven by additive white Gaussian noise and a telegraph signal. The computation of the MET using the expression (21) comes out with a monotonically increasing function of the frequency and does not exhibit any resonant behavior.

This fact led us to conclude that the origin of coherent stochastic resonance cannot be the ‘‘coherence’’ of the periodic driving force. We show that CSR appears because of the behavior of the driving bias at low frequencies. This explanation is similar to that given recently to the phenomenon of resonant activation.

We also deduced that the MET of a system driven by a general periodic signal and a white Gaussian noise will exhibit coherent stochastic resonance whenever two conditions are satisfied. The conditions are, first, that the value of the MET at zero frequency is less than its limiting value at high frequency, and second, that the MET is a decreasing function of the frequency at low frequencies. The second condition is immediately satisfied when the periodic function behaves as an increasing function of time at low frequencies. Therefore, we have shown that coherent stochastic resonance can occur for a variety of periodic signals.

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APPENDIX A: DERIVATION OF EQ. (18)

We begin from Eq. (14) for $P_{2n+1}(x)$,

$$P_{2n+1}(x) = \int_0^L p_+(x,\tau|y) P_{2n}(y)\, dy. \tag{A1}$$

After using Eq. (14) successively, the following expression is obtained:

$$P_{2n+1}(x) = \int_0^L dy_1 p_+(x,\tau|y_1) \int_0^L dy_2 p_-(y_1,\tau|y_2) \int_0^L dy_3 \cdots \int_0^L dy_{2n} p_-(y_{2n-1},\tau|y_{2n}) P_1(y_{2n}). \tag{A2}$$

The initial condition $P_0(y) = \delta(y - x_0)$ leads to the identity $P_1(y_{2n}) = p_+(y_{2n},\tau|x_0)$. When this result and Eqs. (11) and (12) for $p_\pm(x,\tau|y)$ are introduced into Eq. (A2), we get

$$P_{2n+1}(x) = \frac{2}{L}\boldsymbol{\psi}^+(x)\cdot\boldsymbol{M}\cdot\int_0^L dy_1 \frac{2}{L}\boldsymbol{\psi}^-(y_1)\times\boldsymbol{\psi}^-(y_1)\cdot\boldsymbol{M}\cdot\int_0^L dy_2 \frac{2}{L}\boldsymbol{\psi}^+(y_2)\times\boldsymbol{\psi}^+(y_2)\cdot\boldsymbol{M}\cdot\int_0^L dy_3\cdots\boldsymbol{M}\cdot\int_0^L dy_{2n}\frac{2}{L}\boldsymbol{\psi}^+(y_{2n})$$
$$\times\boldsymbol{\psi}^+(y_{2n})\cdot\boldsymbol{M}\cdot\boldsymbol{\psi}^-(x_0). \tag{A3}$$

Note that in this expression the terms

$$\int_0^L dy_1 \frac{2}{L} \boldsymbol{\psi}^-(y_1) \times \boldsymbol{\psi}^-(y_1) \tag{A4}$$

define the matrix $\boldsymbol{I}$ whose components were set in Eq. (19) and read as follows:

$$I_{nk} \equiv \frac{2}{L}\int_0^L \psi_n^-(x)\psi_k^-(x)dx. \tag{A5}$$

The matrix $\boldsymbol{J}$ is equivalent to $\boldsymbol{I}$ when $\boldsymbol{\psi}^-(x)$ is replaced by $\boldsymbol{\psi}^+(x)$. This notation leads directly to Eq. (18) for $P_{2n+1}(x)$ and by a similar reasoning to $P_{2n+2}(x)$.

APPENDIX B: DERIVATION OF EQ. (21)

The derivation starts from Eq. (17) for the MET,

$$T(x_0) = \sum_{n=0}^{\infty} \int_0^\tau dt' \int_0^L dx \int_0^L [p_+(x,t'|y)P_{2n}(y) + p_-(x,t'|y)P_{2n+1}(y)]dy. \tag{B1}$$

Using Eqs. (11) and (12) for $p_\pm(x,\tau|y)$, this expression reads

$$T(x_0) = \sum_{n=0}^{\infty} \boldsymbol{c}^+ \cdot \int_0^L \boldsymbol{\psi}^-(y)P_{2n}(y)dy + \sum_{n=0}^{\infty} \boldsymbol{c}^- \cdot \int_0^L \boldsymbol{\psi}^+(y)P_{2n+1}(y)dy, \tag{B2}$$

where vectors $\boldsymbol{c}^+$ and $\boldsymbol{c}^-$ are given by

$$\boldsymbol{c}^+ = \frac{2}{L}\int_0^\tau dt \int_0^L \boldsymbol{\psi}^+(x)\cdot \boldsymbol{M}(t)dx,$$

$$\boldsymbol{c}^- = \frac{2}{L}\int_0^\tau dt \int_0^L \boldsymbol{\psi}^-(x)\cdot \boldsymbol{M}(t)dx. \tag{B3}$$

We then introduce Eq. (18) for $P_{2n+1}(x)$ and $P_{2n+2}(x)$ into Eq. (B2) and obtain

$$T(x_0) = \boldsymbol{c}^+ \cdot \boldsymbol{\psi}^-(x_0) + \sum_{n=0}^{\infty} \boldsymbol{c}^+ \cdot \boldsymbol{I}\cdot\boldsymbol{M}\cdot\boldsymbol{J}\cdot(\boldsymbol{M}\cdot\boldsymbol{I}\cdot\boldsymbol{M}\cdot\boldsymbol{J})^n\cdot\boldsymbol{M}\cdot\boldsymbol{\psi}^-(x_0) + \sum_{n=0}^{\infty} \boldsymbol{c}^-\cdot\boldsymbol{J}\cdot(\boldsymbol{M}\cdot\boldsymbol{I}\cdot\boldsymbol{M}\cdot\boldsymbol{J})^n\cdot\boldsymbol{M}\cdot\boldsymbol{\psi}^-(x_0). \tag{B4}$$

The definition of the matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \sum_{n=0}^{\infty} (\boldsymbol{M}\cdot\boldsymbol{I}\cdot\boldsymbol{M}\cdot\boldsymbol{J})^n \equiv (\boldsymbol{1} - \boldsymbol{M}\cdot\boldsymbol{I}\cdot\boldsymbol{M}\cdot\boldsymbol{J})^{-1}, \tag{B5}$$

with $\boldsymbol{1}$ being the identity matrix, simplifies the last expression, which can be written as

$$T(x_0) = \boldsymbol{c}^+\cdot\boldsymbol{\psi}^-(x_0) + \boldsymbol{c}^+\cdot\boldsymbol{I}\cdot\boldsymbol{M}\cdot\boldsymbol{J}\cdot\boldsymbol{A}\cdot\boldsymbol{M}\cdot\boldsymbol{\psi}^-(x_0) + \boldsymbol{c}^-\cdot\boldsymbol{J}\cdot\boldsymbol{A}\cdot\boldsymbol{M}\cdot\boldsymbol{\psi}^-(x_0). \tag{B6}$$

Equation (21) follows directly from this.

APPENDIX C: DERIVATION OF EQ. (24)

The MET, Eq. (20), depends on ω through the components $T_n(\omega)$,

$$T_n(\omega) = c_n^+ + [\boldsymbol{c}^+\cdot\boldsymbol{I}\cdot\boldsymbol{M}\cdot\boldsymbol{J}\cdot\boldsymbol{A}\cdot\boldsymbol{M}]_n + [\boldsymbol{c}^-\cdot\boldsymbol{J}\cdot\boldsymbol{A}\cdot\boldsymbol{M}]_n. \tag{C1}$$

The behavior of these components at low frequencies gives expansion (24). When $\omega\to 0$, c_n^+ behaves as

$$c_n^+ \sim \frac{2\beta_n[1-e^{L/2}(-1)^n]}{L(\beta_n^2+\frac{1}{4})^2} - \delta_{n1}\frac{2\pi(1+e^{L/2})}{L^2(\beta_1^2+\frac{1}{4})^2} \times \exp\left[-\frac{1}{8L^2\omega}(L^2+4\pi^2)\right] + o\left(\exp\left[-\frac{1}{8L^2\omega}(L^2+4\pi^2)\right]\right), \tag{C2}$$

and c_n^- has the same expression after replacing L by $-L$ in the exponential functions. This result follows directly from Eq. (20). The Kronecker δ function δ_{n1} indicates that the lowest-order correction term affects only to the component $T_1(\omega)$. To this order of approximation, the second term in the right-hand side of Eq. (C1) does not contribute and the third term reads as follows:

$$[\boldsymbol{c}^-\cdot\boldsymbol{J}]_1 \exp\left[-\frac{1}{8L^2\omega}(L^2+4\pi^2)\right]. \tag{C3}$$

Therefore, the leading term of the MET expansion around $\omega=0$ is of order $\exp[-\gamma/\omega]$, with $\gamma=(L^2+4\pi^2)/8L^2$. The first term in Eq. (C2) for c_n^+ gives the coefficients of the expansion of $T_{\rm bias}(x_0)$ in terms of functions $\psi_n^-(x_0)$, that is,

$$T_{\rm bias}(x_0) = -x_0 + L\frac{1-e^{-x_0}}{1-e^{-L}} = \frac{2}{L}e^{-x_0/2}\sum_{n=1}^{\infty}\frac{\beta_n[1-e^{L/2}(-1)^n]}{(\beta_n^2+\frac{1}{4})^2}\sin(\beta_n x_0). \tag{C4}$$

The correction arises from the second term in c_n^+ and from expression (C3). In order to calculate the later, we use Eq. (19) for matrix $\boldsymbol{J}$ and write

$$[\boldsymbol{c}^- \cdot \boldsymbol{J}]_1 = \sum_{k=1}^{\infty} c_k^- J_{k1} = \sum_{k=1}^{\infty} c_k^- \frac{2}{L} \int_0^L \psi_k^+(x) \psi_1^+(x) dx. \tag{C5}$$

Then, after reversing the order of summation and integration, the following result is obtained:

$$[\boldsymbol{c}^- \cdot \boldsymbol{J}]_1 = \frac{2}{L} \int_0^L T^-(x) e^{x/2} \sin(\beta_1 x) dx, \tag{C6}$$

where

$$T^-(x) = x - L \frac{1-e^x}{1-e^L} = \frac{2}{L} e^{x/2} \sum_{k=1}^{\infty} \frac{\beta_k [1 - e^{-L/2}(-1)^k]}{(\beta_k^2 + \frac{1}{4})^2} \sin(\beta_k x). \tag{C7}$$

Equation (24) follows from here after some algebra.

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