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Stability of temporal chirped solitary waves in quadratically nonlinear media

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A two-parameter family of moving solitary waves in nonlinear quadratic media is obtained and discussed in different parameter spaces, where they are unstable in certain domains. The stability criterion is derived and confirmed by means of a beam propagation method. [S1063-651X(97)14905-4]

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I. INTRODUCTION

Recently solitary waves based on a quadratically nonlinear interaction attracted a lot of attention in nonlinear optics [1–14]. They were found to exist as self-guided beams in bulk materials [8] and planar waveguides [9]. Both spatial and temporal solitary waves are described by the same evolution equations. Until recently only a one-parameter family of such solutions has been identified. They can be calculated as real valued. For one value of the ratio of the diffraction or dispersion coefficients of the fundamental and second harmonics the evolution equations are invariant under a certain transformation, thus yielding a two-parameter family of solitary wave solutions that can be directly obtained from the real-valued one-parameter family (see, e.g., [12]). With the exception of one element of the family, which is known analytically [1,3,6], the solitary wave solutions have to be determined numerically [5]. It was found that they exhibit a narrow domain of instability [11]. The collision of two such waves shows that they merge for sufficiently small relative velocities [12]. This is due to the fact that the evolution equations are not integrable.

In the spatial case the real-valued family of solitary wave solutions corresponds to the fundamental and second harmonics propagating in the same direction. No spatial walk-off is induced, e.g., by birefringence. The spatial walk-off can be avoided in a planar waveguide, whereas in the temporal case the situation is much more complicated. In this case the one-parameter family of solitary wave solutions corresponds to a certain carrier frequency where the group velocities of the two pulses (fundamental and second harmonics) are equal. In most materials such a frequency does not exist. Here we concentrate on the more common situation where the two pulses tend to move away from each other due to a group velocity difference. The family of solitary wave solutions obtained is two-parametric and complex valued. It has been identified independently very recently, focusing on the spatial case [14]. Here we concentrate on the temporal case using a different scaling from [14] to employ a minimum number of parameters for clarity. But the central role in our analysis is the stability of the moving solitary wave solutions, which is a crucial issue for potential applications. We derive the manifold in parameter space where the solitary waves destabilize.

This paper is organized as follows. In Sec. II we introduce the basic sets of equations and point out the number of parameters of the family of solutions under investigation. We discuss the solitary wave solutions with the emphasis on the temporal case. Section III deals in detail with their stability. In Sec. IV the solitary wave solutions are discussed in terms of their parameters and Sec. V concludes the paper.

II. BASIC EQUATIONS AND SOLITARY WAVE SOLUTIONS

We consider pulse propagation, e.g., in a channel waveguide with a quadratically nonlinear medium. The evolution equations for the slowly varying amplitudes A_1' and A_2' of the fundamental and second harmonics in a reference frame moving with the group velocity of the fundamental are

$$i\frac{\partial A_1'}{\partial Z'}+\frac{D_1}{2}\frac{\partial^2 A_1'}{\partial T'^2}+\chi_1 A_1'^* A_2'=0\,,$$

$$i\frac{\partial A_2'}{\partial Z'}+i\gamma'\frac{\partial A_2'}{\partial T'}+\frac{D_2}{2}\frac{\partial^2 A_2'}{\partial T'^2}-\beta' A_2'+\chi_2 A_1'^2=0\,, \qquad (1)$$

with

$$\beta'=2\beta_1(\omega_0)-\beta_2(2\omega_0)\,,$$

$$\gamma'=\left.\frac{\partial\beta_2}{\partial\omega}\right|_{\omega=2\omega_0}-\left.\frac{\partial\beta_1}{\partial\omega}\right|_{\omega=\omega_0}\,, \qquad (2)$$

$$D_1=-\left.\frac{\partial^2\beta_1}{\partial\omega^2}\right|_{\omega=\omega_0}\,,\quad D_2=-\left.\frac{\partial^2\beta_2}{\partial\omega^2}\right|_{\omega=2\omega_0}\,.$$

Here $\beta_1(\omega)$ and $\beta_2(\omega)$ are the dispersion relations of the two guided modes corresponding to the fundamental and second harmonics and ω_0 is the carrier frequency. The profiles of the two guided modes enter into the effective nonlinear coefficients χ_1 and χ_2. The quantities derived from the dispersion relations are the phase mismatch β' between the fundamental and second harmonics, the negative difference of their inverse group velocities γ', and their dispersion coefficients D_n, $n=1,2$. Positive and negative D_n correspond to anomalous and normal dispersion, respectively. The term proportional to γ' in the second of Eqs. (1) is commonly referred to as walk-off. Applying the transformation

$$T=\frac{T'}{T_0}, \quad Z=\frac{Z'}{L_D}, \quad A_1=\sqrt{\chi_1\chi_2}L_D A_1', \quad A_2=\chi_1 L_D A_2',$$

$$\gamma=\frac{\gamma' L_D}{T_0}, \quad \beta=\beta' L_D, \quad \sigma=\frac{D_2}{D_1}, \quad L_D=\frac{T_0^2}{|D_1|}, \tag{3}$$

where T_0 denotes the pulsewidth and L_D the dispersion length, leads for anomalous dispersion (both waves) to

$$i\frac{\partial A_1}{\partial Z}+\frac{1}{2}\frac{\partial^2 A_1}{\partial T^2}+A_1^* A_2=0,$$

$$i\frac{\partial A_2}{\partial Z}+i\gamma\frac{\partial A_2}{\partial T}+\frac{\sigma}{2}\frac{\partial^2 A_2}{\partial T^2}-\beta A_2+A_1^2=0. \tag{4}$$

The above equations describe the spatial case with walk-off as well. They are similar for the case of normal dispersion. The solutions of Eqs. (4) we are interested in are solitary waves with bright shapes. Due to their group velocity difference fundamental and second harmonic components of a solution tend to separate from each other. On the other hand, for a solitary wave to exist it is required that both constituents move with a common velocity. Thus the question is whether the nonlinear interaction is able to prevent the separation and which common velocity such a symbiotic object attains. It is expected that the complex term introduced by the walk-off in general leads to a chirped solution. Thus another parameter of the solutions is the average frequency (or momentum).

Even if solitary waves are determined for a fixed carrier frequency corresponding to a certain walk-off the resulting solution need not be centered at this chosen frequency. Thus the whole frequency domain has to be considered. Solitary waves are expected to be found in certain domains of the velocity-average frequency plane. Though being an important parameter the average frequency of a solitary wave is difficult to handle because it depends on the solution in a nontrivial way. To obtain a coherent description of the solitary waves we introduce a wave number coming back to the more relevant average frequency later. Thus we are looking for a two-parameter family of solutions with parameters κ (wave number) and v (velocity). To this end we introduce the following transformation of Eqs. (4):

$$t=\sqrt{\kappa-\frac{v^2}{2}}(T-vZ), \quad z=\left(\kappa-\frac{v^2}{2}\right)Z,$$

$$a_1=\frac{1}{\kappa-v^2/2}e^{-i(\kappa-v^2)Z}e^{-ivT}A_1,$$

$$a_2=\frac{1}{\kappa-v^2/2}e^{-2i(\kappa-v^2)Z}e^{-2ivT}A_2, \tag{5}$$

$$\delta=\frac{(2\sigma-1)v+\gamma}{\sqrt{\kappa-v^2/2}}, \quad \alpha=\frac{\beta+2\kappa+2(\sigma-1)v^2+2\gamma v}{\kappa-v^2/2},$$

giving

$$i\frac{\partial a_1}{\partial z}+\frac{1}{2}\frac{\partial^2 a_1}{\partial t^2}-a_1+a_1^* a_2=0,$$

$$i\frac{\partial a_2}{\partial z}+i\delta\frac{\partial a_2}{\partial t}+\frac{\sigma}{2}\frac{\partial^2 a_2}{\partial t^2}-\alpha a_2+a_1^2=0. \tag{6}$$

Bright solitary waves are now calculated as stationary solutions of Eqs. (6), i.e., equating the z derivatives to zero and applying the appropriate boundary conditions. They depend on the rescaled phase mismatch α and the rescaled walk-off δ. The solitary wave solutions calculated in this way can be considered as moving ones of Eqs. (4) with propagation constant κ and velocity v, thus establishing a two-parameter family of solutions. The scaling defined by Eqs. (5) reduces the number of parameters from four (γ, β, κ, and v) to two (α and δ) without loss of generality. Note from Eqs. (5) the interplay between the velocity v, the walk-off γ, and the dispersion coefficient σ in the expression for the rescaled walk-off δ. This indicates that the group velocity mismatch and the velocity of a solitary wave solution act similarly. This fact can be conveniently exploited, since a shift of the velocity $v\rightarrow v+v_0$ [$v_0=-\gamma/(2\sigma-1)$] and of the propagation constant $\kappa\rightarrow\kappa+(v+v_0/2)v_0$ allows the restriction to $\gamma=0$, provided that $\sigma\neq 1/2$ and the phase mismatch is renormalized as $\beta\rightarrow\beta+\gamma^2/(2\sigma-1)$.

For $\delta=0$ the real-valued families of solitary wave solutions are recovered. The one-parameter family (parameter κ) corresponds to $\sigma\neq 1/2$, i.e., $v=0$ for $\gamma=0$ or $v=-\gamma/(2\sigma-1)$ otherwise. It is just a limiting case within a broader class of solutions. The two-parameter family (parameters κ and v) is obtained in the case $\sigma=1/2$, i.e., $\gamma=0$. As can be seen from the transformation of Eqs. (5) the moving solutions of this family can be generated directly from the resting ones. The solitary wave solutions obtained for nonvanishing δ are complex-valued with a nontrivial phase; i.e., they have a chirp (see Fig. 1). Obviously there is no limitation with respect to the rescaled walk-off δ for solitary waves to exist. The lower boundary of the rescaled phase mismatch α depends on δ [$\alpha>\delta^2/(2\sigma)$, see next section].

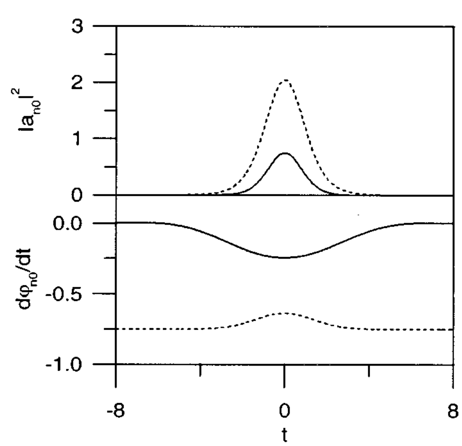

FIG. 1. Intensities and time derivatives of the phases of a solitary wave solution for $\delta=0.6$, $\alpha=0.5$, and $\sigma=0.8$. Solid and dashed lines correspond to the fundamental and second harmonics, respectively.

Equations (6) yield two conservation laws that play a central role in our analysis, the energy p and the momentum q of a solitary wave solution a_{n0}, $n=1,2$:

$$q=\int dt\ (|a_{10}|^2+|a_{20}|^2),$$

$$p=\frac{1}{2i}\int dt\left[a_{10}\frac{\partial a_{10}^*}{\partial t}-a_{10}^*\frac{\partial a_{10}}{\partial t}+\frac{1}{2}\left(a_{20}\frac{\partial a_{20}^*}{\partial t}-a_{20}^*\frac{\partial a_{20}}{\partial t}\right)\right]. \tag{7}$$

It should be mentioned that there is a third conserved quantity, a Hamiltonian, which is not given here.

Figure 2 shows a contour plot of the energy q in the (δ,α) plane. If the lower limit of existence is approached the width of the solitary waves diverges. All the energy is carried by the second harmonic and an infinite power is required to maintain the balance within the solitary wave (compare Fig. 6 for solitary waves close to the limit of existence).

More meaningful than the momentum itself is the scaled average frequency $\overline{\omega}=p/q$, which is conserved as well. For an unchirped pulse $\overline{\omega}$ is the difference from the carrier frequency. Thus the excitation of a solitary wave with $\overline{\omega}$ requires an initial pulse with that frequency. The averaged frequency $\overline{\omega}$ changes only marginally with α for fixed scaled walk-off δ (Fig. 3) and is fairly small in terms of these parameters. As in the case of the Schrödinger soliton, where the frequency is proportional to the velocity, $\overline{\omega}$ also increases with increasing δ, without giving rise to a strong additional chirp. For $\alpha=\delta^2/(2\sigma)$, i.e., at the limit of existence, the second of Eqs. (6) with the last term neglected yields $\overline{\omega}=\delta/(2\sigma)$. This is half the frequency of a second harmonic wave that travels with the group velocity corresponding to the scaled walk-off δ. Another limit is the Schrödinger case (see [5]) for $\alpha\to\infty$. In this limit $\overline{\omega}=0$ since in the stationary case considered here the Schrödinger soliton has no frequency.

III. STABILITY ANALYSIS

For the one-parameter family of solitary waves $(\delta=0)$ a narrow instability region was identified [11]. We show that this is the case for the moving solutions as well. To this end Eqs. (6) are linearized around a solitary wave solution $\mathbf{a}_0$. Here we introduced the real four-component vector $\mathbf{a}=(\mathrm{Re}a_1,\mathrm{Re}a_2,\mathrm{Im}a_1,\mathrm{Im}a_2)^T$, where T denotes the transposed and $\mathrm{Re}a_n$, $\mathrm{Im}a_n$, $n=1,2$, the real and imaginary parts of the fields, respectively. Substituting the ansatz $\mathbf{a}=\mathbf{a}_0+\delta\mathbf{a}e^{\lambda z}$ into Eqs. (6) and linearizing with respect to $\delta\mathbf{a}$ yields an eigenvalue problem for λ:

$$L\,\delta\mathbf{a}=\lambda\,\delta\mathbf{a}, \tag{8}$$

where the operator L is defined as

$$L=\begin{pmatrix} -\mathrm{Im}a_{20} & \mathrm{Im}a_{10} & -\frac{1}{2}\frac{\partial^2}{\partial t^2}+1+\mathrm{Re}a_{20} & -\mathrm{Re}a_{10}\\ -2\mathrm{Im}a_{10} & -\delta\frac{\partial}{\partial t} & -2\mathrm{Re}a_{10} & -\frac{\sigma}{2}\frac{\partial^2}{\partial t^2}+\alpha\\ \frac{1}{2}\frac{\partial^2}{\partial t^2}-1+\mathrm{Re}a_{20} & \mathrm{Re}a_{10} & \mathrm{Im}a_{20} & \mathrm{Im}a_{10}\\ 2\mathrm{Re}a_{10} & \frac{\sigma}{2}\frac{\partial^2}{\partial t^2}-\alpha & -2\mathrm{Im}a_{10} & -\delta\frac{\partial}{\partial t} \end{pmatrix}. \tag{9}$$

Asymptotically, i.e., for $t\to\pm\infty$, Eq. (8) can be solved by means of $\delta\mathbf{a}\sim e^{\pm i\Omega t}$ yielding the dispersion relations

$$\lambda=\pm i\left(\frac{\Omega^2}{2}+1\right),$$

$$\lambda=\pm i\left[\frac{\sigma}{2}\left(\Omega\pm\frac{\delta}{\sigma}\right)^2+\alpha-\frac{\delta^2}{2\sigma}\right]. \tag{10}$$

From the above equations the limit of the continuous spectrum is given by $\min\{1,\alpha-\delta^2/(2\sigma)\}$. For $\alpha=\delta^2/(2\sigma)$ the gap of the continuous spectrum vanishes. This also marks the limit of existence for bright solitary waves (see dashed line in Fig. 4), which is due to the requirement that these solutions have evanescent tails.

The linear problem has two localized or bound states with corresponding eigenvalue $\lambda=0$: $\delta\mathbf{a}_t=\partial\mathbf{a}_0/\partial t$ and $\delta\mathbf{a}_p=(\mathrm{Im}a_{10},2\mathrm{Im}a_{20},-\mathrm{Re}a_{10},-2\mathrm{Re}a_{20})^T$, which correspond to translational and phase invariance. Apart from these two bound eigenstates we identified numerically a third, nontrivial bound state $\delta\mathbf{a}_b$ (Fig. 5). The corresponding squared eigenvalue λ_b^2 is always real. At a critical value $\alpha=\alpha_c$ for fixed δ and σ it changes its sign and the solitary waves become unstable. Stability corresponds to a purely imaginary λ_b and instability to a positive real λ_b. The nonexistence of a second nontrivial bound state, which always has a positive real eigenvalue, is crucial for the following analysis. The curve in parameter space that separates stable and unstable domains (see solid line in Fig. 4) is derived by a similar asymptotic method as used in [11]. Our numerical analysis suggests that at the critical point the bound state $\delta\mathbf{a}_b$ is a linear combination of the two zero eigenmodes (i.e., $\delta\mathbf{a}_b=c\,\delta\mathbf{a}_t+d\,\delta\mathbf{a}_p$). This is due to the fact that the number of basic symmetries with a corresponding zero eigenmode does

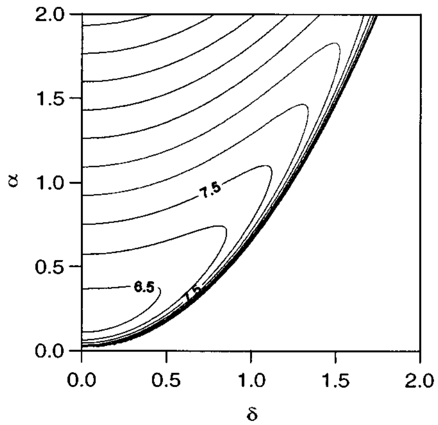

FIG. 2. Contour plot of the energy q in the (δ,α) plane for $\sigma=0.8$.

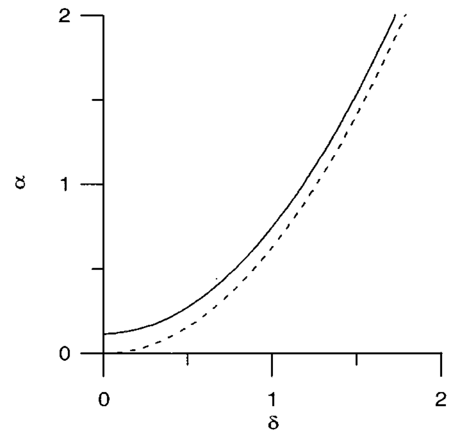

FIG. 4. Domains of stability (above solid line) and instability (between dashed and solid line) in the (δ,α) plane for $\sigma=0.8$.

not change and is not dependent on the soliton parameters. Thus an expansion around α_c for fixed δ and σ is introduced:

$$\alpha=\alpha_c+\alpha_2\epsilon^2\ ,\quad \lambda_b=\lambda_1\epsilon+\lambda_2\epsilon^2+\cdots\ ,$$

$$\delta\mathbf{a}_b=\delta\mathbf{a}_b^{(0)}+\delta\mathbf{a}_b^{(1)}\epsilon+\delta\mathbf{a}_b^{(2)}\epsilon^2+\cdots\ , \tag{11}$$

with $\alpha_2=\pm1$ and $\delta\mathbf{a}_b^{(0)}=c\,\delta\mathbf{a}_t+d\,\delta\mathbf{a}_p$. This and all the following α dependencies have to be taken at α_c. Substituting the above expansion into the linear problem of Eqs. (8) we get up to order ϵ^2:

$$O(1):\quad L^{(0)}\delta\mathbf{a}_b^{(0)}=0\ ,$$

$$O(\epsilon):\quad L^{(0)}\delta\mathbf{a}_b^{(1)}=\lambda_1\delta\mathbf{a}_b^{(0)}\ , \tag{12}$$

$$O(\epsilon^2):\quad L^{(0)}\delta\mathbf{a}_b^{(2)}-\alpha_2L^{(0)}\frac{\partial\,\delta\mathbf{a}_b^{(0)}}{\partial\alpha}=\lambda_2\delta\mathbf{a}_b^{(0)}+\lambda_1\delta\mathbf{a}_b^{(1)}\ .$$

Here $L^{(0)}$ is the operator L taken at α_c. The first of Eqs. (12) is trivially fulfilled since $\delta\mathbf{a}_b^{(0)}$ is a linear combination of the two zero eigenmodes. In the third of Eqs. (12) the relation $(\partial/\partial\alpha)(L\,\delta\mathbf{a}_b^{(0)})=0$ was used. The inhomogeneous equation at order ϵ in Eqs. (12) is solved by

$$\delta\mathbf{a}_b^{(1)}=\lambda_1c\left(t\,\delta\mathbf{a}_p-2\,\delta\frac{\partial\mathbf{a}_0}{\partial\alpha}-(2\sigma-1)\frac{\partial\mathbf{a}_0}{\partial\delta}\right)-\lambda_1d\left(\mathbf{a}_0+\frac{1}{2}t\,\delta\mathbf{a}_t+(2-\alpha)\frac{\partial\mathbf{a}_0}{\partial\alpha}-\frac{1}{2}\delta\frac{\partial\mathbf{a}_0}{\partial\delta}\right)\ . \tag{13}$$

Now from the third of Eqs. (12) a solvability condition is derived. This arises from the fact that the adjoint operator L^+ of L has a nontrivial null space:

$$L^+\delta\mathbf{a}_t'=0\ ,$$

$$\delta\mathbf{a}_t'=\left(-\frac{\partial\ \mathrm{Im}a_{10}}{\partial t},-\frac{1}{2}\frac{\partial\ \mathrm{Im}a_{20}}{\partial t},\frac{\partial\ \mathrm{Re}a_{10}}{\partial t},\frac{1}{2}\frac{\partial\ \mathrm{Re}a_{20}}{\partial t}\right)^T\ ,$$

$$L^+\delta\mathbf{a}_p'=0\ ,\quad \delta\mathbf{a}_p'=\mathbf{a}_0\ . \tag{14}$$

Calculating the scalar products between $\delta\mathbf{a}_t'$, $\delta\mathbf{a}_p'$, and the third of Eqs. (12) yields

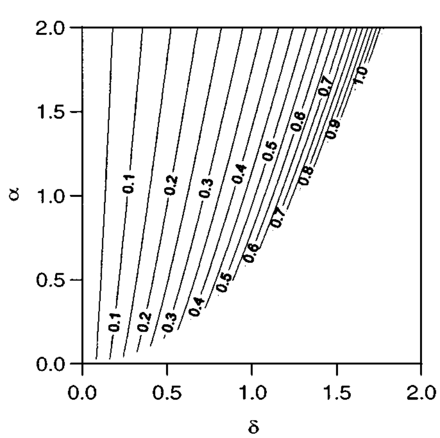

FIG. 3. Contour plot of the average frequency $\overline{\omega}$ in the (δ,α) plane for $\sigma=0.8$.

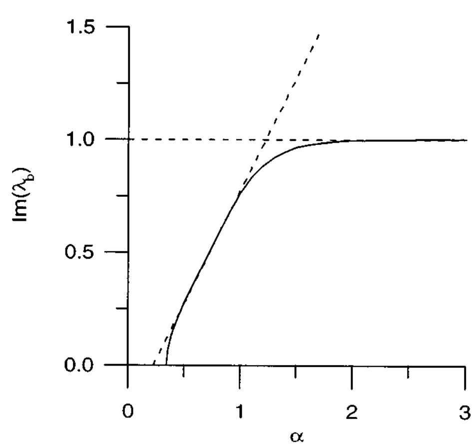

FIG. 5. Imaginary part of eigenvalues λ_b vs α for $\delta=0.6$ and $\sigma=0.8$. The dashed lines mark the limit of the continuous spectrum.

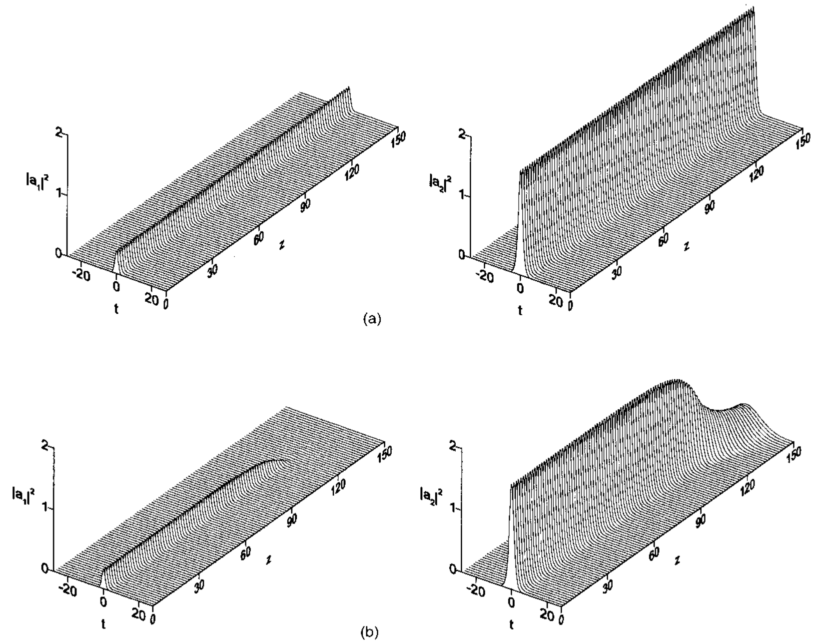

FIG. 6. Intensity of the fundamental and second harmonics showing stable (a) and unstable (b) propagation of a solitary wave at $\alpha=0.2$, $\delta=0.33$ (a), $\delta=0.39$ (b), and $\sigma=0.8$.

$$\langle \delta\mathbf{a}_t' , \delta\mathbf{a}_b^{(1)} \rangle = 0 \,, \quad \langle \delta\mathbf{a}_p' , \delta\mathbf{a}_b^{(1)} \rangle = 0 \,. \tag{15}$$

Here the scalar product is defined as $\langle \delta\mathbf{a}_1 , \delta\mathbf{a}_2 \rangle = \int_{-\infty}^{\infty} dt \;\, \delta\mathbf{a}_1^{*T} \cdot \delta\mathbf{a}_2$ with the dot denoting matrix multiplication. Equations (15) establish a system of linear equations for the constants c and d. The requirement that c and d should be nontrivial (or that the determinant of the coefficients vanishes) finally yields the solvability condition

$$\left(\frac{3}{2} q + (2-\alpha) \frac{\partial q}{\partial \alpha} - \frac{\delta}{2} \frac{\partial q}{\partial \delta} \right) \left(\frac{1}{2} q - \delta \frac{\partial p}{\partial \alpha} - \frac{1}{2} (2\sigma - 1) \frac{\partial p}{\partial \delta} \right) + \left(2p + (2-\alpha) \frac{\partial p}{\partial \alpha} - \frac{\delta}{2} \frac{\partial p}{\partial \delta} \right) \left(\delta \frac{\partial q}{\partial \alpha} + \frac{1}{2} (2\sigma - 1) \frac{\partial q}{\partial \delta} \right) = 0 \,, \tag{16}$$

where q is the energy and p the momentum of a solitary wave solution a_{n0}, $n=1,2$ as defined in Sec. II. Note that the above procedure applied to the second of Eqs. (12) leads to a trivial result since $\langle \delta\mathbf{a}_t' , \delta\mathbf{a}_b^{(0)} \rangle = \langle \delta\mathbf{a}_p' , \delta\mathbf{a}_b^{(0)} \rangle = 0$. The solvability condition of Eq. (16) describes the manifold in parameter space where the squared eigenvalue λ_b^2 of the nontrivial bound state changes its sign. Since there is only one nontrivial bound state, the stability changes at this manifold.

Finally, the stable and unstable propagation of the solitary wave solutions was confirmed by means of a beam propagation method (Fig. 6). In this example the unstable solitary wave decays with decreasing amplitude and increasing width while it moves away from the center.

IV. PHYSICAL INTERPRETATION OF SOLITARY WAVES (SOLITARY WAVE PARAMETERS)

In the two previous sections the solitary wave solutions and their stability have been characterized in terms of the (mathematically) convenient system parameters α and δ. Now we interpret our results by means of more accessible parameters (v and κ). The velocity v, the propagation constant κ, the energy Q, and the momentum P with respect to a solution A_{n0}, $n=1,2$, and the average frequency $\overline{\Omega}=P/Q$ are related to the scaled quantities α, δ, q, p, and $\overline{\omega}$ in the following way:

$$\kappa = \mu + \frac{1}{2} \left(\frac{\sqrt{\mu}\,\delta - \gamma}{2\sigma - 1} \right)^2 , \quad v = \frac{\sqrt{\mu}\,\delta - \gamma}{2\sigma - 1} \,,$$

$$Q = \mu^{3/2} q \,, \quad P = \mu^2 p \,, \quad \overline{\Omega} = \sqrt{\mu} \left(\overline{\omega} - \frac{\delta}{2\sigma - 1} \right) + \frac{\gamma}{2\sigma - 1} \,, \tag{17}$$

$$\mu = \kappa - \frac{v^2}{2} = - \frac{\beta(2\sigma-1) - \gamma^2}{(2-\alpha)(2\sigma-1) + \delta^2} \,.$$

Note that the scaling factor μ has to be positive [cf. Eqs. (5)] and that we may assume $\gamma=0$ as pointed out previously. As to the average frequencies the transformation defined by Eqs. (5) introduces a frequency shift. The relation between these frequencies in Eqs. (17) may be also written as $\overline{\Omega} = \overline{\omega} \sqrt{\mu} - v$. In terms of parameters α and δ there are no discontinuities, whereas for parameters κ and v there is a divergence at $\alpha = 2 + \delta^2/(2\sigma - 1)$ that separates domains of positive $[\alpha > 2 + \delta^2/(2\sigma-1)]$ and negative phase mismatch β $[\alpha < 2 + \delta^2/(2\sigma-1)]$ $(\gamma=0)$. Here the limitation to $\beta = \pm 1$ is sufficient.

We first express the stability criterion of Eq. (16) by means of the conserved quantities referring to Eqs. (4). In terms of the energy Q and the momentum P and the parameters κ and v of the family of solitary wave solutions it becomes much simpler:

$$\frac{\partial Q}{\partial \kappa} \frac{\partial P}{\partial v} - \frac{\partial Q}{\partial v} \frac{\partial P}{\partial \kappa} = 0 \,. \tag{18}$$

The meaning of the above relation is that the vector function $(Q(\kappa,v),P(\kappa,v))$ is not invertible at a critical point. In this work the relation of Eq. (18) derives from the fact that each solitary wave parameter is connected to a fundamental symmetry of the evolution equations, namely, the velocity v to translational invariance and the wave number κ to a phase invariance. The fundamental symmetries in turn are reflected in corresponding bound states of the linearized problem. This may be compared with the case of vectorial second harmonic interaction where also a two-parameter family of solitary wave solutions exists. The manifold in parameter space where the solitary waves destabilize can be expressed in a similar way on the basis of conserved quantities [15].

We describe now the family of solitary wave solutions in the space of velocity and average frequency. We find that the entire space between the straight lines $\partial\beta_1/\partial\Omega=-\Omega$ (group velocity of fundamental harmonic as a function of Ω) and $\partial\beta_2/\partial\Omega=-2\sigma\Omega$ (group velocity of second harmonic as a function of 2Ω) is filled with solitary wave solutions. Here the linear dispersion relations of Eqs. (4) ($\gamma=0$),

$$\overline{\beta_1}(\overline{\Omega})=-\overline{\Omega}^2/2\ ,\quad \overline{\beta_2}(\overline{\Omega})=-\sigma\overline{\Omega}^2/2-\beta\ , \tag{19}$$

were used. There is no general restriction due to the walk-off. The allowed range of velocities even increases with increasing average frequencies.

Figure 7 shows the energy Q of the solitary waves versus the parameter v for various Ω and $\sigma>1/2$. For $\sigma<1/2$ the behavior is similar. The Schrödinger limit (the phase mismatch is positive) corresponds to wide low-energy pulses traveling with the velocity of the fundamental wave ($v=-\Omega$). If the energy is increased the pulses become narrower changing their velocity, but do not approach the velocity of the second harmonic. The residual part of the velocity space is filled with solitary wave solutions corresponding to a negative phase mismatch. Above a certain threshold two different solutions exist for a given energy. One is stable, the other is unstable. If the energy is increased the width diverges and all the energy is carried by the second harmonic. Thus the pulse travels with the velocity of the second harmonic ($v=-2\sigma\Omega$). It follows that the velocity domain where solitary waves exist is given by $\Delta v=(2\sigma-1)\Omega$.

Comparing the results for different average frequencies we find that the energy threshold decreases if the absolute value of Ω is increased [Fig. 8(a)] and thus if the walk-off is increased. The same is true for the corresponding total peak power [Fig. 8(b)]. The reason for this unexpected behavior is that a deviation from the carrier frequency changes the propagation constants for both field components. This leads to a new effective phase mismatch $\beta_{\rm eff}$, which, with respect to Eqs. (4), is given by $\beta_{\rm eff}=2\beta_1(\Omega)-\beta_2(2\Omega)=(2\sigma-1)\Omega^2+\beta$. The rapid decrease of the energy is found where the effective phase mismatch changes its sign. If $\beta_{\rm eff}$ is kept constant the phase mismatch β (at ω_0) of the underlying dispersion relation varies according to

$$\beta=\frac{\beta_{\rm eff}[(2-\alpha)(2\sigma-1)+\delta^2]}{(2-\alpha)(2\sigma-1)+\delta^2-[(\overline{\omega}(2\sigma-1)-\delta)]^2}\ . \tag{20}$$

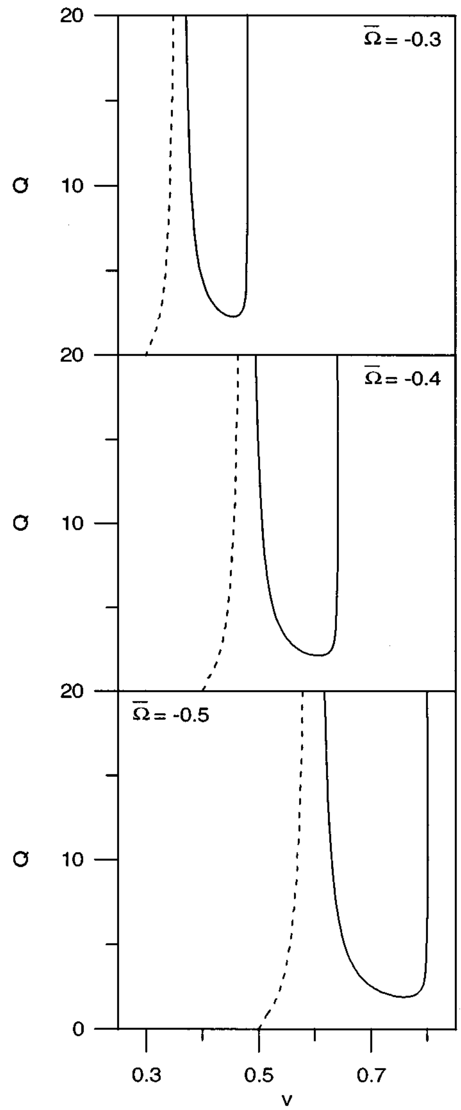

FIG. 7. Energy Q of solitary waves vs velocity v for different values of the average frequency Ω for $\gamma=0$ and $\sigma=0.8$. Dashed lines refer to $\beta=1$ and solid lines to $\beta=-1$.

Figure 9 displays the energy threshold of Fig. 8 versus the velocity difference Δv keeping $\beta_{\rm eff}$ fixed. Now the energy increases for sufficiently large absolute values of the velocity difference, as expected. But for small velocity differences a decrease of the energy is observed. Thus the walk-off can be favorable.

V. CONCLUSIONS

To conclude, we identified a two-parameter family of solitary wave solutions. The results describe the experimental fact of walk-off. A finite group-velocity difference between the fundamental and second harmonics does not prevent the formation of solitary waves, but induces a chirp. Solitary waves are found to move with velocities between those of the fundamental and second harmonics, where the respective velocity is power dependent. We derived the critical manifold in parameter space where the solitary waves destabilize. The domain of instability was

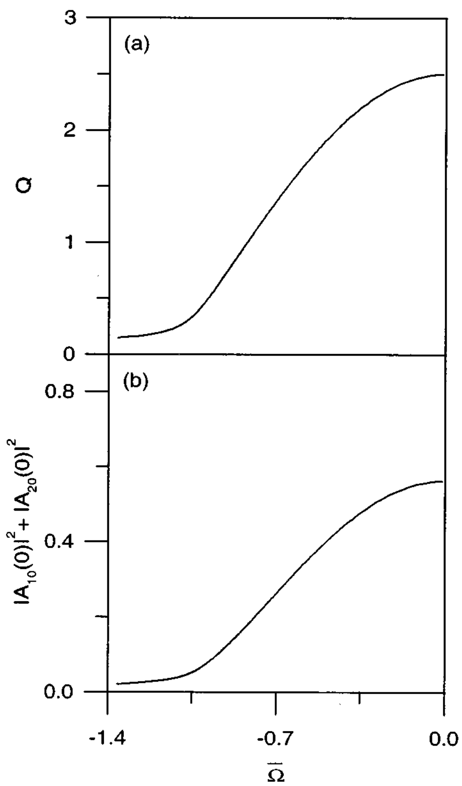

FIG. 8. Energy (a) and total peak power (b) of solitary waves vs the average frequency $\bar{\Omega}$ at the local minima of Fig. 7.

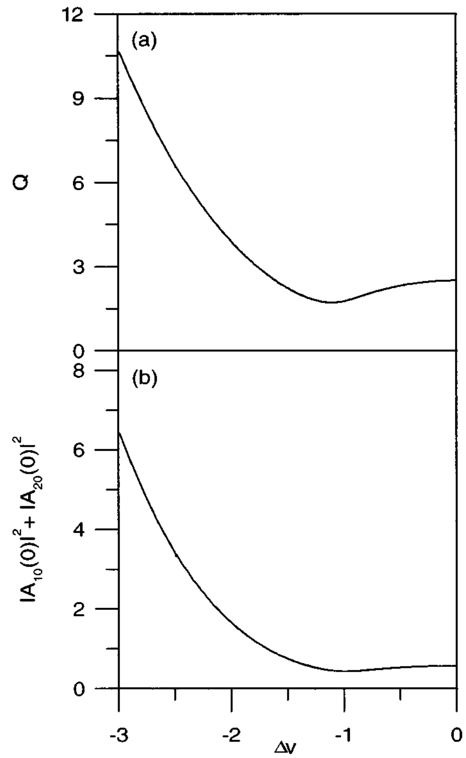

FIG. 9. Energy (a) and total peak power (b) of solitary waves versus velocity difference Δv at the local minima of Fig. 7 for $\beta_{\mathrm{eff}} = -1$.

found to be rather narrow. The stability behavior was confirmed by a beam propagation method. Using different scalings the solutions are adequately described in terms of their energy and momentum, which are conserved during propagation. It was found that the energy required to excite a solitary wave can be reduced by a small walk-off. But eventually the energy diverges with increasing walk-off.

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