First-order rigidity on Cayley trees

C. Moukarzel*
Höchstleistungsrechenzentrum, Forschungzentrum Jülich, D-52425 Jülich, Germany

P. M. Duxbury
Department of Physics/Astronomy, Michigan State University, East Lansing, Michigan 48824-1116

P. L. Leath
Department of Physics/Astronomy, Rutgers University, Piscataway, New Jersey 08855-849
(Received 19 August 1996)

Tree models for rigidity percolation, in systems with only central forces, are introduced and solved. A probability vector describes the propagation of rigidity outward from a rigid border. All components of this ''vector order parameter'' are singular at the *same* rigidity threshold p_c. The infinite-cluster probability P_∞ is usually first order at p_c, except in those cases which are equivalent to connectivity percolation. In many cases, $P_\infty \sim \Delta P_\infty + (p-p_c)^{1/2}$, indicating critical fluctuations superimposed on the first-order jump (ΔP_∞). Our tree models for rigidity are in qualitative disagreement with ''constraint-counting'' mean-field theories. In an important subclass of tree models ''bootstrap'' percolation and rigidity percolation are equivalent. [S1063-651X(97)16005-6]

PACS number(s): 61.43.Bn, 46.30.Cn, 05.70.Fh

I. INTRODUCTION

Soon after the resurgence of interest in percolation phenomena, the elastic constants of depleted materials were studied. Although early work suggested [1] that the conductivity and elasticity exponents were the same, it was soon realized that the elasticity exponents were usually different [2], and in particular one must draw a distinction between the elasticity of systems which have only ''central forces'' [3] and those which also have ''bond-bending'' forces. If a system has bond-bending forces, the percolation geometry is in many ways similar to that of the connectivity percolation problem. Of interest in this paper is the fact that when a system is supported by only central forces, the percolation geometry is *very different* from that occurring in connectivity percolation. We illustrate this difference by developing and solving models for rigidity percolation on Cayley trees, and by comparing those models with the analogous results for connectivity percolation on trees [4]. Many of the concepts we develop using tree models can be extended to regular lattices, as will be elaborated upon in the paper.

There have been several groups of scientists and engineers interested in the ability of central-force structures to transmit stress. Besides its intrinsic interest, this problem is relevant to the analysis of engineering structures, glasses, granular materials, and gels [5]. The straightforward way to study this problem is to construct particular models which have only central forces, and to study the types of structures which support stress. In the physics community, the standard model has been lattices composed of Hookian springs. Direct solution of the force equations for these lattices has provided quite variable estimates of the percolation threshold, and considerable controversy about the critical exponents [6–9]. In the mathematics community, there has been a long history of attempts to relate the connectivity of a ''graph'' to its ability to support stress [10–13]. The majority of physicists were unaware, till recently [5,15], that there is a rigorous theorem which relates connectivity to rigidity but only for *graphs in the plane*. Of more practical importance is the fact that there are *fast algorithms* [13,14] by which this theorem can be used to actually find the infinite cluster [15] and stressed backbone [5] of some graphs (e.g. the triangular lattice with central forces). These results are relevant to random lattices [5], which are in many cases of most practical interest.

There are two different types of mean-field theory available for the rigidity transition. The first, based on an approximate ''constraint counting,'' predicts a second-order transition in the ''number of floppy (flexible) modes'' [16], and has been extensively applied to the rigidity of glasses and gels. However it was realized in that paper, and recently quantitatively confirmed [15], that the number of floppy modes per site does not approach zero at the percolation point. There is also a recent phenomenological field theory [17] which predicts first-order rigidity, but the connection between the model parameters and the lattice parameters relevant to rigidity are not clear in that analysis. The tree models developed here provide a more complete mean-field theory for the rigidity transition. We find the infinite-cluster probability shows a first-order jump, and, superimposed on this first-order jump, there is a continuous singularity on approach to the transition from above. We also find that one subclass of our tree models is equivalent to tree models for

*Permanent address: Instituto de Física, Universidade Federal Fluminense, CEP 24210-340 Niteroi RJ, Brazil.

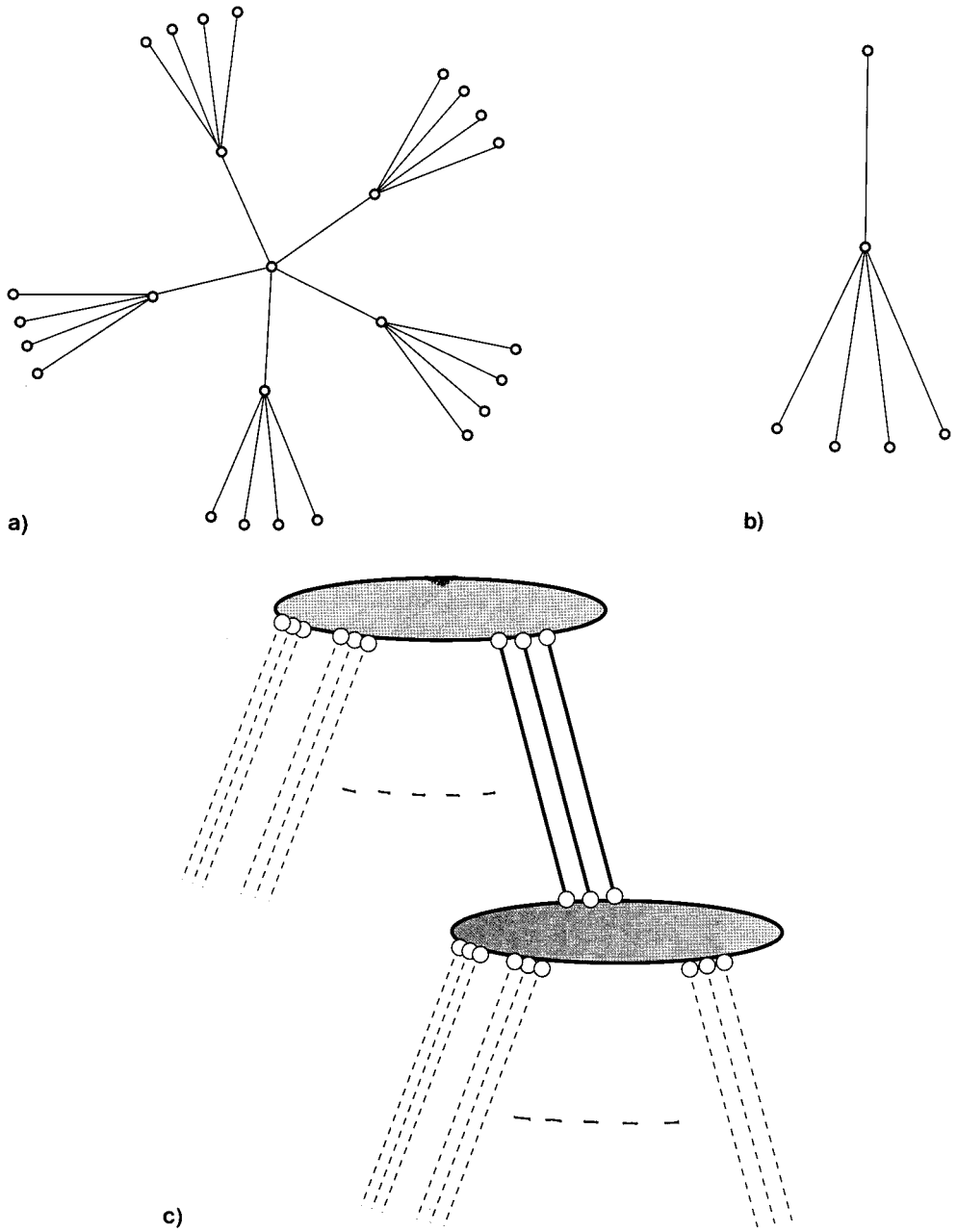

FIG. 1. The geometry of trees. (a) A $z=5$, $b=1$ tree. (b) One branch of the tree of (a). (c) One branch of a $b=3$ tree.

bootstrap percolation [18], although on regular lattices they are not equivalent. The paper is arranged as follows. In Sec. II, we introduce the tree geometry and the vector probability (order parameter) used to describe the transmission of rigidity from a rigid border. Section III contains a detailed analysis of tree models for both site and bond dilution. In Sec. IV we discuss the ''house of cards'' mechanism for first-order rigidity and discuss the failings of the traditional constraint-counting mean-field theories in the light of the tree results. We also calculate the number of floppy modes, and show that even the second derivative is nonsingular on trees. This is not too surprising, since surface bonds dominate if averages over the whole tree, as is done in the floppy mode calculation. Section V contains a brief summary and conclusion.

II. GEOMETRY AND DEFINITION OF VARIABLES

The structure of the tree models we consider is illustrated in Fig. 1. Following normal convention, we define z to be the number of branches of the tree [for example, in Fig. 1(a) $z=5$]. In Fig. 1(a), each site of the tree is connected by only one bond to a neighboring site. In general we may have b bonds or bars connecting neighboring sites [for example, in Fig. 1(c), $b=3$]. The b bonds are assumed to be nonparallel, so that each bond represents an independent constraint. Thus two variables in our analysis are z and b. A third important variable g is the number of degrees of freedom per site, and is discussed in the next paragraph. The feature of the tree geometry which makes the analysis tractable is that we can calculate the probability of rigidity along separate branches

of trees *independently*, and then combine the branches of the tree to form the final Cayley tree. For example one ''branch'' of the tree of Fig. 1(a) is presented in Fig. 1(b). We use the letter P, with various subscripts, for the site probabilities of the entire tree [e.g., Fig. 1(a)], while we use T, with various subscripts, to denote the site probabilities of the branches of the trees. The qualitative behavior of T and P are the same, and we concentrate for the most part on the analysis of T.

Each node [the sites in Figs. 1(a) and 1(b), and the ellipses in Fig. 1(c)] represents a ''joint'' (pointlike node) or ''body'' (see below) on a lattice or ''graph,'' and is assigned a certain number of ''degrees of freedom.'' In connectivity percolation each node is either connected or disconnected, so it has only one possible ''degree of freedom,'' i.e., if a site is disconnected it has one degree of freedom, while if it is connected it has no degrees of freedom. If we consider a lattice of joints connected by central-force springs, then each free joint has two translational degrees of freedom in two dimensions, and three degrees of freedom in three dimensions. However, when we make rigid clusters, they are rigid ''bodies'' so they also have rotational degrees of freedom. For example, a *body* in two dimensions has three degrees of freedom (two translations and one rotation), while a body in three dimensions has six degrees of freedom (three translations and three rotations). In general, we allow each site to have g degrees of freedom. Some practically important values for g are as follows:

$$g=1 \quad \text{for connectivity percolation,} \tag{1a}$$

$$g=d \quad \text{for a joint,} \tag{1b}$$

$$g=d(d+1)/2 \quad \text{for a body.} \tag{1c}$$

Here d is the spatial dimension. We consider growing a tree outward from a rigid boundary. A bond (bond percolation) or site (site percolation) is present on the tree with probability p. If there is a finite asymptotic probability that the center node of the tree is rigidly connected to the boundary, then we are above the rigidity percolation threshold p_c, whereas, if the probability of being on the infinite cluster dies out as the number of levels becomes very large, we are below p_c. The behavior on crossing p_c depends on whether the transition is first or second order, as will be discussed further below. In the case of connectivity percolation, there is only one degree of freedom per node, and we only have to keep track of the probability that connectivity is transmitted away from the boundary. In the case of rigidity percolation it is necessary to consider a larger set of site probabilities. In fact, each site may have $0,1,2,\ldots g$ degrees of freedom with respect to the boundary, so we define the probabilities $P_0,\ldots,P_g$ to be the probabilities that a site has between 0 and g degrees of freedom (DOF's) with respect to the boundary (a similar definition applies to the branch probabilities T). *For example*, if $g=3$, (1) P_3 (or T_3) is the probability that a node has three DOF's with respect to the border. (2) P_2 (or T_2) is the probability that a node has two DOF's with respect to the border. (3) P_1 (or T_1) is the probability that a node has one DOF with respect to the border. (4) P_0 (or T_0) is the probability that a node has no DOF with respect to the border.

The vectors P and T act as order parameters for the rigidity percolation problem on trees. However, it is also possible to define these quantities on regular lattices, and it is likely that an algorithm could be developed based on these probabilities. In fact for the case of a ''diode response,'' a transfer matrix could be used—this would be a ''directed rigidity percolation,'' and might be appropriate for granular media, where contacts only support compressive forces.

In many physical problems, it is important to distinguish between the probability that a site is in the backbone and *overconstrained* or stressed, P_B, and the probability that a backbone site is *isostatic* or *rigidly connected* but not stressed (which has probability P_D). In particular, we have previously defined [5] $P_\infty=P_0=P_D+P_B$ to be the ''infinite rigid cluster'' probability. This is closely analogous to the infinite cluster in connectivity percolation [19]. In this analogy, the overconstrained or ''stressed'' bonds are analogous to the ''backbone'' in connectivity percolation, and rigidly attached but unstressed (isostatic) regions are analogous to the dangling ends in connectivity percolation. Just as the dangling ends in connectivity percolation carry no current, the isostatic regions in rigidity percolation carry no stress. However, for trees we found it clearer to first concentrate on P_∞, since there are important subtleties associated with the definition of the appropriate boundary conditions for the stressed backbone, so in this paper we do not discuss P_B.

III. DILUTED CAYLEY TREES

Consider Cayley trees of coordination number z as shown in Fig. 1. In general our parameters are g (the number of degrees of freedom per node), z (the coordination number-actually we shall usually use $\alpha=z-1$), b (the number of bonds connecting each neighboring pair of nodes on undiluted trees), and p (the probability that a site or bond is present). We first do the calculations for a branch of the trees [see Fig. 1(b) for a $b=1$ case], and then join the branches together to obtain P_∞. To illustrate the method we first perform the case $b=1$, as illustrated in Figs. 1(a) and 1(b) with site dilution.

A. Site diluted trees with $b=1$

On any tree, rigidity can only be transmitted to higher levels of the tree if there are enough new constraints or bonds present to offset the number of degrees of freedom of a newly added node. For connectivity percolation only one bond is needed. If a node is added to a $g=2$ tree, two bonds are needed to offset the two degrees of freedom of the added node. In general, if a node with g degrees of freedom is added, generic rigidity is transmitted to the next level of the tree provided the node is occupied, *and* provided at least g of the lower-level nodes to which the added node is connected are rigid. We define the probability that a node is rigid to be T_0, and T_k to be the probability that a node have k remaining degrees of freedom. The branch probabilities T_k with $k=0,1,\ldots,g$ are then given by

$$T_0 = p\sum_{l=g}^{\alpha} \binom{\alpha}{l} (T_0)^l (1-T_0)^{\alpha-l},$$

$$T_1 = p\binom{\alpha}{g-1} (T_0)^{g-1}(1-T_0)^{\alpha-g+1},$$

$$\cdots \tag{2}$$

$$T_l = p\binom{\alpha}{g-l} (T_0)^{g-l}(1-T_0)^{\alpha-g+1} \quad \text{for } 1 \leqslant l \leqslant g,$$

$$T_g = 1 - \sum_{l=0}^{g-1} T_l .$$

The left-hand side of Eqs. (2) refer to a node at one higher level than the nodes on the right-hand side. Since we are looking for asymptotic probabilities a long way from the rigid boundary, we expect the probabilities T_l to approach steady state values upon iteration of Eqs. (2). Expressions similar to Eqs. (2) are found when the transition is made from the branch probabilities T_l [see Fig. 1(c)] to the tree probabilities P_l [see Fig. 1(a)], except that we now combine z branches instead of $z-1$ branches. Thus we find, for example,

$$P_0 = p\sum_{l=g}^{z} \binom{z}{l} T_0^l (1-T_0)^{z-l}. \tag{3}$$

In fact once we have solved the first of Eqs. (2), and have found T_0, all of the other components of P and T follow. In particular, if T_0 is first order at a particular p_c, then all of the other components of T and P are first order at the *same* p_c. Thus we concentrate on the behavior of T_0.

It is interesting to note that Eq. (3) is *the same* as Eq. (2) of [18], which treats *bootstrap percolation* on trees (with the change of variables $R = 1-P$, $g = m$, and $l = z-m$). In bootstrap percolation one considers that ferromagnetic order is propagated *only* if each site has at least m ferromagnetic neighbours. If we start with a ferromagnetic border, it is clear that Eq. (3), with the above change of variables, describes the propagation of ferromagnetic order outward from the border. The correspondence between bootstrap percolation and rigidity percolation is *not exact* on regular lattices. Now we solve Eqs. (2) for some simple cases.

Connectivity percolation ($g=1$)

In this case the first of Eqs. (2) reduces to that found previously [4]. For example, for $\alpha = 3$,

$$T_0 = p(3T_0(1-T_0)^2 + 3T_0^2(1-T_0) + T_0^3), \tag{4}$$

which yields the trivial solution $T_0 = 0$, and the nontrivial solution

$$T_0 = \frac{3-\sqrt{(4/p-3)}}{2}. \tag{5}$$

The percolation threshold p_c occurs when the nontrivial solution (5) approaches zero, and this occurs at $p_c = \frac{1}{3}$. Near p_c, T_0 approaches zero linearly, so the transition is *second order* and the order-parameter exponent $\beta = 1$.

In order for the problem to lie in the ''rigidity percolation'' class, there must be at least two degrees of freedom per node, i.e., $g \geqslant 2$. However, when $b=1$, if $\alpha = z-1 = 2$, then $p_c = 1$, as all bonds must be present in order to transmit rigidity. Thus the simplest nontrivial cases are $g=2$, $\alpha=3$, and $b=1$, which we now treat.

Rigidity transition for $g=2$, $\alpha=3$, and $b=1$

From the first of Eqs. (2), we have

$$T_0 = p[T_0^3 + 3T_0^2(1-T_0)]. \tag{6}$$

Of course, there is always the trivial solution $T_0 = 0$. In addition, Eq. (6) implies

$$T_0 = \frac{3 \pm \sqrt{(9-8/p)}}{4}. \tag{7}$$

To ensure that $T_0 = 1$ when $p = 1$, take the positive root. The interesting feature here is that the argument of the square root is negative for $p < p_c = \frac{8}{9}$, so this root becomes unphysical (imaginary) at $p = \frac{8}{9}$. For $p < p_c$, the only remaining physical (real) root is $T_0 = 0$, so this implies that there is a first-order jump in T_0 at $p_c = \frac{8}{9}$. The magnitude of this jump $\Delta T_0 = \frac{3}{4}$. Note also that on approach to p_c from above, we find [18]

$$T_0 - \tfrac{3}{4} \sim (p-p_c)^{1/2}, \tag{8}$$

which illustrates critical fluctations in addition to the first-order jump in T_0. This interesting behavior seems usual for both bootstrap percolation and for rigidity percolation, and it does not usually happen in ordinary thermodynamic phase transitions. From the second of Eqs. (2), we have

$$T_1 = 3pT_0(1-T_0)^2, \tag{9}$$

which has two solutions $T_1 = 0$ and the result found by substituting Eq. (7) for T_0 into Eq. (9). There is thus a first-order jump in T_1 *at the same* p_c as that found for T_0. The size of this jump $\Delta T_1 = \frac{1}{8}$. Note that T_1 is *zero* at $p = 1$, so T_1 *rises* from zero as p decreases, and peaks at $p = p_c$. Since $T_2 = 1 - T_0 - T_1$, all components of the vector order parameter are first order, and all of them have a singular correction near p_c as a consequence of Eq. (8).

Order of the transition for general g and α, with $b=1$

In the first of Eqs. (2), there is always the trivial solution $T_0 = 0$. After removing that, the following equations holds:

$$1 = p\sum_{k=g}^{\alpha} \binom{\alpha}{k} T_0^{k-1}(1-T_0)^{\alpha-k}. \tag{10}$$

If $g=1$ (connectivity percolation), there is always a term independent of T_0 on the right-hand side of this equation, and this allows a real solution for arbitrarily small T_0, and hence the transition is second order. However, if $g \geqslant 2$, the constant term on the right-hand side is absent and the equation cannot be satisfied for an arbitrarily small real T_0. Thus

there must be a first-order jump in T_0 for any $z>g\geq 2$. It is possible to solve Eq. (10) to find p_c explicitly in the case $g=\alpha-1$, in which case the first-order jump has magnitude $\Delta T_0=1-1/(\alpha-1)^2$ [18]. However, in general we resort to numerical methods. Before describing the numerical results, we first introduce a matrix method which allows us to treat general g, b, and α.

B. Site-diluted Cayley trees for arbitrary g, α, and b

It is possible to generalize the Bethe lattices described above to cases where more than one bond connects neighboring nodes. In the case of site dilution, removing a site removes all of the b bonds that enter that site from a neighbor. In contrast, bond dilution removes one bond at a time and must be treated differently (see, the later discussion in this section). Returning to the site-dilution case, we note that, if $b\geq g$, generic rigidity is transmitted across the tree as soon as connectivity percolation occurs. This is because any one connection between two nodes with $b\geq g$ ensures transmission of rigidity to the newly added node, provided of course that the prior node is also rigid with respect to the boundary. Thus, if $b\geq g$, there are only two possible states for each node: rigidly connected to boundary and not connected at all to the boundary, and the model is ''trivially'' in the connectivity percolation class. In contrast, if there are fewer than g bonds connecting two nodes, more interesting node states are possible, and we must again consider the full set $T_0,\ldots,T_g$, which allows the possibility of partial transmission of rigidity. We now develop a matrix method to treat the nontrivial cases $1\leq b<g$.

Consider adding a site to a branch of coordination α. We label the sites at the previous level $i=1,\ldots,\alpha$ [for example, we label the lower ellipse in Fig. 1(c) $i=1$.] Each of these nodes may have $l_i=0,1,\ldots,g$ degrees of freedom with respect to the border [for example, the lower ellipse in Fig. 1(c) has l_1 degrees of freedom with respect to the border].

We start by adding a ''free body'' to the tree, so it has g degrees of freedom with respect to the boundary. However, when we add the new higher-level body to the tree, we also add αb bonds. But not all of the bonds that are added are ''useful'' in reducing the number of degrees of freedom of the newly added body with respect to the border. For example, if a lower level node already has g degrees of freedom with respect to the border, no matter how many bonds connect it to the higher-level body, it does not produces any constraint on the newly added body with respect to the boundary. Therefore we must define the ''number of useful bonds'' u, which lies along any sub-branch. If a lower-level body has zero degrees of freedom with respect to the border, then every bond is ''useful.'' If the lower-level body has one degree of freedom with respect to the border, then the first bond that is added is used to ''cancel'' this degree of freedom, and does not constrain the newly added node, so that only $b-1$ of the bonds are useful. In general if a body has i degrees of freedom, only $u=b-i$ of the added bonds are useful in producing constraint in the higher-level body. Thus the probability Q_u that a sub-branch has u useful bonds is given by (note that since we are considering $1\leq b<g$, $Q_g=0$)

$$Q_u=\begin{cases} T_{b-u} & \text{for } u=1,\ldots b \\ 1-\sum_{v=1}^{b} T_{b-v} & \text{for } u=0. \end{cases} \tag{11}$$

Now each sub-branch adds u_i constraints to the newly added body, so the total number of constraints on the newly added body is $\Sigma_{i=1}^{\alpha}u_i$. Thus the probability that the new node has k degrees of freedom is,

$$T_0=p\sum_{l_1=0}^{g}\sum_{l_2=0}^{g}\cdots\sum_{l_\alpha=0}^{g}T_{l_1}T_{l_2},\ldots,T_{l_\alpha}\theta\left(g-\sum_{i=1}^{\alpha}u_i\right),$$

$$T_{k=1,\ldots,g-1}=p\sum_{l_1=0}^{g}\sum_{l_2=0}^{g}\cdots\sum_{l_\alpha=0}^{g}T_{l_1}T_{l_2},\ldots,T_{l_\alpha}\times\delta\left(g-k-\sum_{i=1}^{\alpha}u_i\right), \tag{12}$$

$$T_g=1-\sum_{l=0}^{g-1}T_l$$

where θ and δ are the step function and delta function, respectively.

For numerical purposes, a more convenient way of representing these equations is to add the α sub-branches one at a time using a matrix method. We define the vector $\mathbf{T}^L=(T_0^L,T_1^L,T_2^L,\ldots,T_g^L)$ to denote the probability that the newly added body be in one of its possible constraint ''states'' after the additions of L sub-branches ($L=1,2,\ldots,\alpha$). If we have a free node it has g degrees of freedom so before the addition of any sub-branches, $\mathbf{T}^0=(0,0,0,\ldots,1)$. We then have the recurrence relations

$$T_0^{L+1}=T_0^L+T_1^L(T_0+T_1+\cdots+T_{b-1})+\cdots+T_b^LT_0 \tag{13}$$

and, for $l=1,2,\ldots,g$,

$$T_l^{L+1}=T_l^L(T_b+T_{b+1}+\cdots+T_g)+T_{l+1}^LT_{b-1}+T_{l+2}^LT_{b-2}+\cdots+T_{l+b}^LT_0. \tag{14}$$

Equations (13) and (14) may be put into matrix form, so that

$$\mathbf{T}^{L+1}=\widetilde{M}\mathbf{T}^L=(\widetilde{M})^\alpha\mathbf{T}^0, \tag{15}$$

with

$$\widetilde{M}=\begin{pmatrix} 1 & \beta_1 & \beta_2 & \cdots & \beta_b & 0 & \cdots & 0 \\ 0 & \Gamma & T_{b-1} & \cdots & T_1 & T_0 & \cdots & 0 \\ 0 & 0 & \Gamma & T_{b-1} & \cdots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \Gamma & T_{b-1} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \Gamma \end{pmatrix},$$

where

$$\Gamma=\sum_{l=b}^{g}T_l \tag{16}$$

and

$$\beta_k = \sum_{l=0}^{b-k} T_l . \tag{17}$$

Finally, we must include the possibility that the site is present or absent, so the probability vector obeys

$$\mathbf{T} = p(\tilde{M})^{\alpha}\mathbf{T}^0 + (1-p)\mathbf{T}^0. \tag{18}$$

As before, the left-hand side of Eq. (18) is the probability vector at the next level of the tree in terms of the probabilities at the lower levels (which are in the matrix M).

A little algebra shows that Eq. (18) reproduces the $b=1$ equations [Eqs. (2)], as it must. We illustrate the matrix method with a special case ($b \neq 1$) which is analytically solvable.

Nontrivial solvable case, $\alpha=2$, $g=3$, $b=2$

For α, $b=2$ and $g=3$, Eq. (18) yields

$$\begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix} = p \begin{pmatrix} 1 & T_0+T_1 & T_0 & 0 \\ 0 & T_2+T_3 & T_1 & T_0 \\ 0 & 0 & T_2+T_3 & T_1 \\ 0 & 0 & 0 & T_2+T_3 \end{pmatrix}^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + (1-p) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The first two of these equations yield

$$T_0 = p(T_0^2 + 2T_1T_0) \tag{19}$$

and

$$T_1 = p(2T_0X + T_1^2), \tag{20}$$

where $X = T_2 + T_3$. Since the sum of the T's is 1, we have $X = 1 - T_0 - T_1$, and this, with Eqs. (19) and (20), yields

$$3T_0^2 - 4(2 - 1/p)T_0 + 1/p^2 = 0. \tag{21}$$

Solving for T_0 yields,

$$T_0 = \frac{(4p-2) + 2\sqrt{(2p-1)^2 - \frac{3}{4}}}{3p}. \tag{22}$$

Then the argument of the square root becomes negative for $p < p_c$, given by $p_c = (1+\sqrt{3}/2)/2 \sim 0.933$, so that $\Delta T_0 = 0.619$.

Numerical results for general b, g, α

The results of iterating the matrix Eqs. (18) are presented in Figs. 2–4. Figure 2(a) illustrates that, for $g \leqslant b$, the problem reduces to the connectivity percolation case. The transition is second order, and only two components of the vector T (T_0 and T_g) are non-zero. In contrast, when $b\alpha > g > b$ [see Fig. 2(b)], all of the components of T can be nonzero, although all of them are singular at the same percolation point. This figure also illustrates that the rigidity transition is first order even though sometimes it is only *weakly* first order.

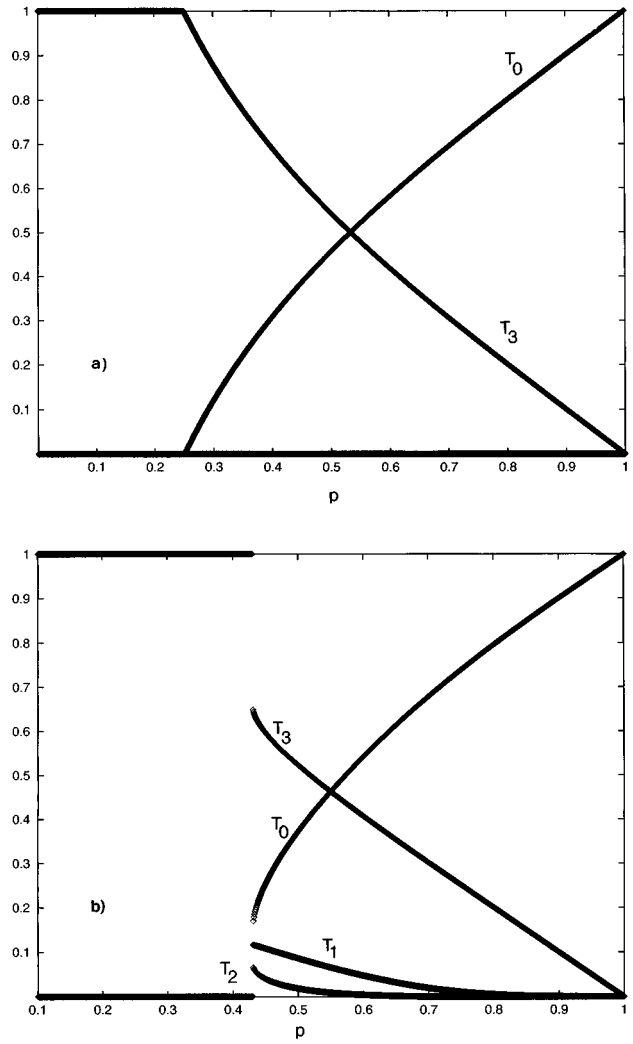

FIG. 2. Rigidity percolation of site-diluted trees: (a) $\alpha=4$, $g=3$, and $b=3$. The infinite cluster probability of one branch (T_0) and the probability T_3 are plotted. In this case the behavior is the same as connectivity percolation, so $p_c = 1/\alpha$ and the transition is second-order, with $\beta=1$. (b) $\alpha=5$, $g=3$, and $b=2$. T_0, T_1, T_2, and T_3 are plotted. All are first order and singular at the same p_c.

In Fig. 3, we illustrate the dependence of rigidity percolation on the coordination number α. In the case we choose here, $g=2$ and $b=1$, the transition is always strongly first order. The behavior near $p=1$ is typical of site dilution on any lattice, because the leading term in the probability that a site is not rigid with respect to the boundary, is just the probability that the site is absent, i.e., $1-p$. As α increases, the point at which T_0 breaks away from $1-p$ tends to $p=0$, as intuitively expected.

If we start from a rigid border, it is evident upon direct iteration using the matrix method that the transmission of rigidity depends on α and the *ratio* b/g. In the limit b/g

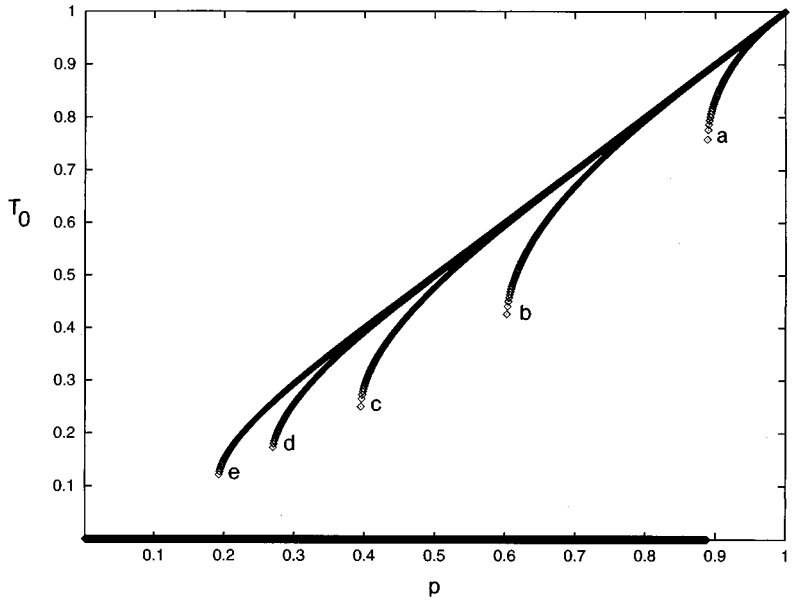

FIG. 3. T_0 for $g=2$ and $b=1$ and for $\alpha=3$ (a), $\alpha=5$ (b), $\alpha=8$ (c), $\alpha=12$ (d), and $\alpha=17$ (e). The transition is always first order.

$=1$, we have connectivity percolation, while, if $b/g\to 0$, the transition is first order and $p_c=1$. Using trees, we are able to probe various values of b/g and we present results for $p_c(\alpha,b/g)$ in Fig. 4. It is seen that for all cases, $p_c \sim G(b/g)/\alpha$ for $\alpha\to\infty$, where the function G is independent of α. We also find that for any $b/g<1$, the transition is first order, and the size of the first-order jumps ΔT_0 and p_c itself increases smoothly as b/g decreases.

From the site-dilution problem, we conclude that the rigidity transition is always first order, except in cases where it trivially reduces to connectivity percolation. However, there is a square-root singularity superimposed on the first-order jump in T_0 for all $b/g<1$. However, on site-diluted lattices with $b<g$, the only rigid clusters are those which are attached to the rigid border. In contrast, in bond percolation it is possible to have *internal* rigid clusters, and the cases $b>g$ are nontrivial. Thus we now describe calculations for the transmission of rigidity on bond-diluted trees.

C. Bond-diluted Cayley trees

As for the site-diluted case, we define the vector $\mathbf{T}=(T_0,T_1,T_2,\ldots,T_g)$. Now, we assume there is a total of b possible bonds between two nodes, and each is present with probability p, so the probability s_k that k bonds are actually present is

$$s_k=\binom{b}{k}p^k(1-p)^{b-k} \quad \text{for} \quad k\leqslant b. \tag{23}$$

Since each node has g degrees of freedom, at most g *independent bonds* can connect two nodes. If k bonds (with $k>g$) connect two nodes, $k-g$ of them will be redundant, and the two nodes will form part of a cluster that is internally rigid. Any number of bonds in excess of g does not add to the number of independent constraints. Therefore the probability q_k that k *independent* bonds are present between two nodes is, for general g and b,

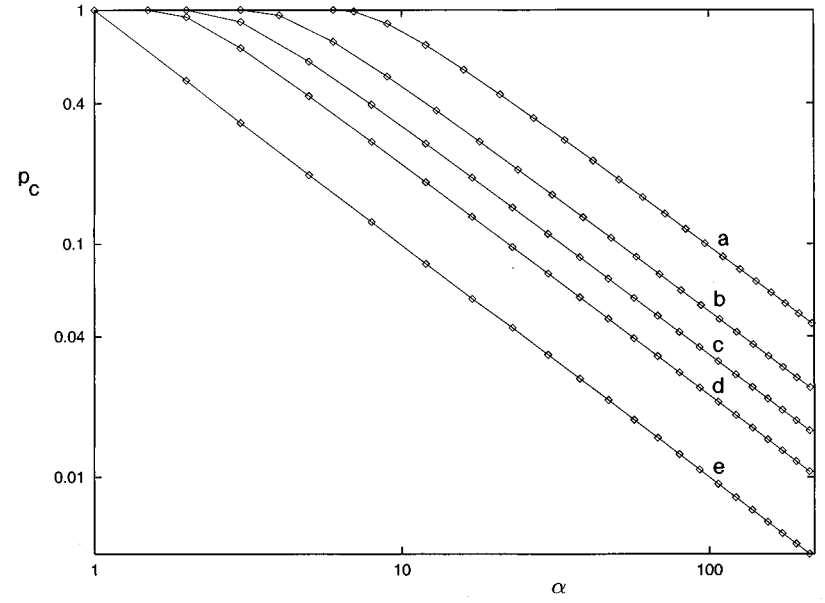

FIG. 4. p_c as a function of b/g and α. The data are for $b/g=\frac{1}{6}$(a), $b/g=\frac{1}{3}$(b), $b/g=\frac{1}{2}$(c), $b/g=\frac{2}{3}$(d), and $b/g=1$ (e).

$$q_k=\begin{cases} s_k & \text{for } k<g \\ \sum_{j=g}^{b} s_j & \text{for } k=g \\ 0 & \text{for } k>g,b. \end{cases} \tag{24}$$

As in the site-dilution case, these k bonds are not all ''useful'' in transmitting constraint from the boundary *unless* the sub-branch along which they lie is at least partially constrained. In particular, if the lower-level node has i degrees of freedom with respect to the boundary, only $k-i$ of the bonds connecting that node to the newly added node actually impose constraint. Clearly if $k\leqslant i$, the branch imposes no constraint (with respect to the boundary) on the newly added node. We thus define the *useful* bonds $u=k-i$, because they are able to propagate constraint outward from the boundary. The probability Q_u for a branch to have u useful bonds on it is then given by

$$Q_u=\begin{cases} \sum_{i=0}^{g-u} T_i q_{i+u} & \text{for } u=1,\ldots,g \\ 1-\sum_{v=1}^{g} Q_v & \text{for } u=0. \end{cases} \tag{25}$$

Now taking α such sub-branches, the total number U of useful bars is

$$U=\sum_{k=1}^{\alpha} u_k. \tag{26}$$

If $U\geqslant g$, then the new node body will be rigid. Otherwise it will have $k=g-U$ degrees of freedom. Formally we then write

$$T_f=\sum_{u_1=0}^{g}\sum_{u_2=0}^{g}\cdots\sum_{u_\alpha=0}^{g} Q_{u_1}Q_{u_2},\ldots,Q_{u_\alpha}\Phi(f,u_1,u_2,\ldots,u_\alpha), \tag{27}$$

where

$$\Phi(f,g,\alpha,U)=\begin{cases} \delta(U-(g-f\ )) & \text{for } 0<f\leqslant g \\ \theta(U-g) & \text{for } f=0, \end{cases} \tag{28}$$

where, as in the site case, we used the step function and the Kronecker δ to ensure that the constraint counting is correct.

As for the site-diluted case, we can write the Eqs. (27) in matrix form:

$$\mathbf{T}^{L+1}=\widetilde{M}\mathbf{T}^{L}, \tag{29}$$

with

$$\widetilde{M}=\begin{pmatrix} 1 & (Q_1+Q_2+\cdots+Q_g) & (Q_2+Q_3+\cdots+Q_g) & \cdots & Q_g \\ 0 & Q_0 & Q_1 & \cdots & Q_{g-1} \\ 0 & 0 & Q_0 & \cdots & Q_{g-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q_0 \end{pmatrix}.$$

Again starting from a bare node with $\mathbf{T}^0=\{0,0,\ldots,0,1\}$, and after connecting α legs, we obtain the desired probabilities as

$$\mathbf{T}=(\widetilde{M})^{\alpha}\mathbf{T}^0 \tag{30}$$

To illustrate the matrix method for the bond case, we again do a solvable case with $b\neq 1$.

Nontrivial solvable case $\alpha=2$, $g=3$, $b=2$

From Eqs. (23) and (24), we have,

$$(q_0,q_1,q_2,q_3)=[(1-p)^2,2p(1-p),p^2,0]. \tag{31}$$

Then from Eq. (25), we have,

$$(Q_0,Q_1,Q_2,Q_3)=[1-(p^2+2p(1-p))T_0-p^2T_1,2p(1-p)T_0+p^2T_1,p^2T_0,0]. \tag{32}$$

Using these expressions in the matrix equation (29), we have

$$\begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix}=\begin{pmatrix} 1 & (2p-p^2)T_0+p^2T_1 & p^2T_0 & 0 \\ 0 & 1-(2p-p^2)T_0-p^2T_1 & 2p(1-p)T_0+p^2T_1 & p^2T_0 \\ 0 & 0 & 1-(2p-p^2)T_0-p^2T_1 & 2p(1-p)T_0+p^2T_1 \\ 0 & 0 & 0 & 1-(2p-p^2)T_0-p^2T_1 \end{pmatrix}^2\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

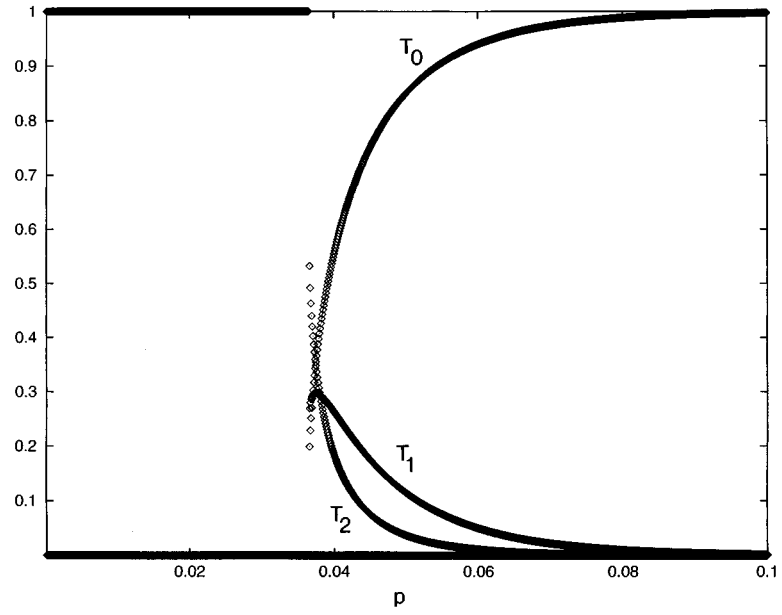

FIG. 5. Rigidity percolation for a *bond-diluted* tree with $\alpha=2$, $g=2$, and $b=40$. The transition is close to second order, and there is an interesting nonmonotonic behavior in T_1.

From the first of these equations, we find

$$T_0=p^3T_0\{(4-3p)T_0+2pT_1\}, \tag{33}$$

while the second implies

$$T_1=2p^2T_0[1-(2p-p^2)T_0-p^2T_1] +[2p(1-p)T_0+p^2T_1]^2. \tag{34}$$

Substituting $T_1=1/2p^4-(4-3p)T_0/2p$ from Eq. (33) into Eq. (34) yields a quadratic equation in T_0. Solution of this equation gives the nontrivial solution

$$T_0=\frac{\frac{2-3p}{p}+2p^2+\sqrt{\left(\frac{2-3p}{p}+2p^2\right)^2-\frac{3}{4}}}{1.5p^4}. \tag{35}$$

This again becomes imaginary at the rigidity threshold, which we find to be $p_c=0.919$, and the first order jump in T_0 is, $\Delta T_0=0.810$.

Numerical results for general b, g, α

First we note that, for $b=1$, the site dilution and bond dilution are the same, provided we make the transformation $p_{\text{site}}\rightarrow p_{\text{bond}}$ and $T_{\text{site}}=p_{\text{bond}}T_{\text{bond}}$ thus we focus attention on $b\geqslant 2$.

We present numerical results for bond-diluted trees in Figs. 5 and 6. In Fig. 5, we show that even when $b\gg g$ and many internal rigid clusters can exist on the trees, the rigidity transition remains first order. In fact, we have not found any values of g or b for which the bond-diluted trees are second order, except the trivial case $g=1$. However the rigidity transition is weakly first order for large b/g. A second interesting feature of Fig. 5 is the nonmonotic behavior of T_1. Nevertheless on all of the trees we studied, the rigidity transition

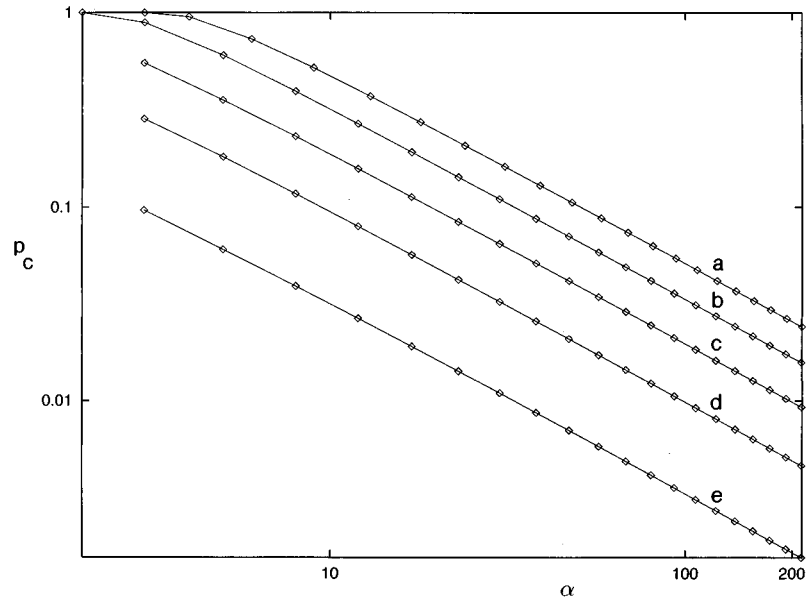

FIG. 6. p_c for bond-diluted trees. The data are for $g=3$, $b=1$ (a); $g=2$, $b=1$ (b); $g=6$, $b=5$ (c); $g=6$, $b=10$ (d); and $g=2$, $b=10$ (e).

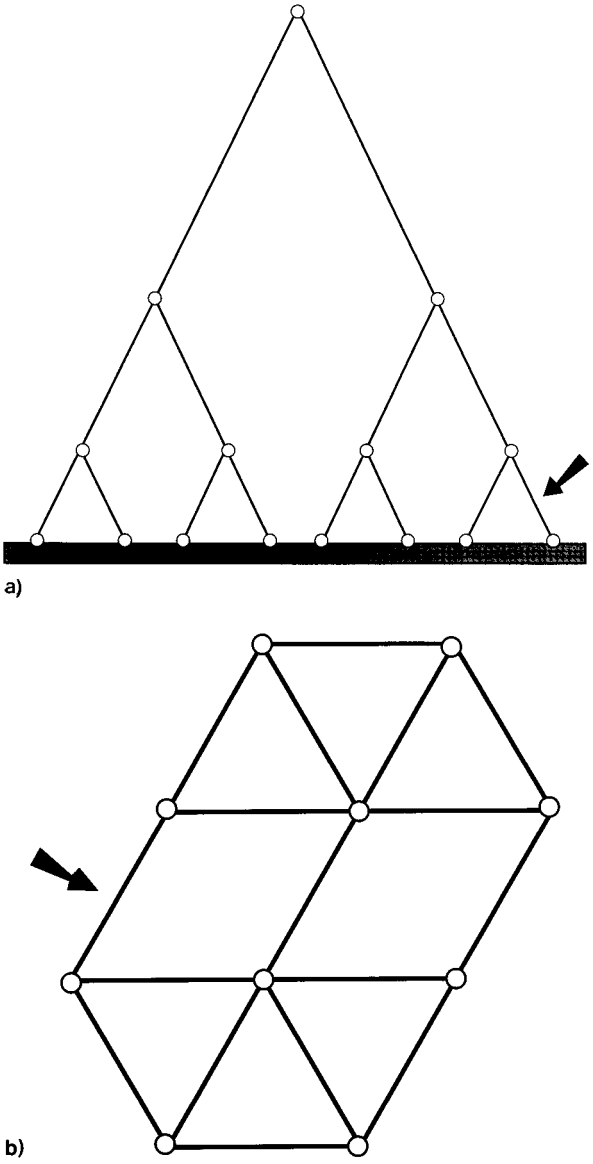

FIG. 7. The effect of removing a bond on the cluster size distribution. (a) Removing the arrowed bond from this rigid cluster leads to six separate rigid clusters. (b) Removing the arrowed bond from this connected cluster leads to four separate rigid clusters.

is unique and first order. As in Eq. (8), there appears to be a singular behavior superimposed on the first-order jump in T_0. On the bond-diluted trees, the percolation threshold depends on all three parameters g, b, and α, nevertheless there is a simple behavior in the large α limit (see Fig. 6), so that $p_c \sim G(g,b)/\alpha$ for $\alpha \to \infty$.

IV. ''HOUSE OF CARDS'' MECHANISM AND COMPARISON WITH OTHER WORK

A. ''House of cards'' mechanism for first-order rigidity

The mechanism for the first-order rigidity transition is illustrated in Fig. 7(a) for an $\alpha=2$, $g=2$, $b=1$ tree, and in Fig. 7(b) for the bond-diluted triangular lattice. In these figures, we presented a rigid cluster, and indicated a bond which we then remove. On removal of the arrowed bond, both of the rigid clusters ''break'' up into more than two rigid subclusters. In Fig. 7(a), removal of the arrowed bond leads to six rigid subclusters, while, in Fig. 7(b), removal of the arrowed bond leads to the formation of four rigid subclusters. In both cases we refer to clusters of mutually rigid *bonds*. In contrast, in connectivity percolation, removal of a ''cutting'' or red bond leads to the breakup of the system into *two subclusters*. On large rigid clusters, the removal of a ''cutting'' or red bond usually leads to formation of many subclusters, and this ''cluster collapse,'' like a ''house of cards,'' provides a mechanism for a first-order rigidity transition. However, it does not *insure* a first-order transition, as it depends on *how many* clusters are formed when a cutting bond is removed. In reverse, the phenomenon of cluster collapse is ''cluster freezing'' in which there is a sudden jump in the average cluster size as many clusters suddenly become mutually rigid (for example, by replacing the arrowed bonds in Fig. 8). It is likely that these ideas can be used to develop scaling arguments for the amount of cluster collapse required for there to be a first-order rigidity transition, and we are currently working in that direction.

B. Comparison with constraint counting methods

For simplicity, first consider bond percolation, for which the argument is simplest. On a *regular lattice*, there are N nodes of coordination z, with each node having g degrees of freedom, and with b bonds connecting each pair of nodes. Now we dilute the bonds of the network, with p the probability that any one bond is present. Then, ''on average,'' the number of degrees of freedom, fN, that remain at dilution p is [16]

$$fN = Ng - pbzN/2 + B, \tag{36}$$

where the factor of $\frac{1}{2}$is due to the fact that each bar is shared between two nodes. B is the number of bonds that are ''redundant'' in that they are in regions of the lattice which would be rigid even if they were removed. The mean-field approximation reduces to assuming that $B=0$, so that $f=g-pbz/2$, and thus f approaches zero at $p_c=2g/bz$. This counting procedure is slightly modified on trees, as the border is rigid, so that every bond which is next to but lower than a node in the tree contributes to the rigidity of that node (the bonds are not ''shared'' as on a regular lattice).

In this case, the constraint counting is

$$fN = Ng - pb\alpha N + B. \tag{37}$$

Thus we have the same expression as in Eq. (36), with the replacement $\alpha(\text{tree}) \sim z/2$ (regular lattice). If we again assume that $B=0$, we find $p_c(B=0)=g/(b\alpha)$. This estimate is grossly in error when compared with the actual results for trees (see Fig. 6). Clearly the stronger the first-order transition, the greater in error the constraint-counting mean-field theory becomes.

C. Global constraint counting

It has been observed that, in two dimensions [15], although the number of floppy modes is always continuous, the second derivative of that quantity is singular. This is based on counting the number of degrees of freedom *in the whole lattice*. If we perform a similar calculation on trees,

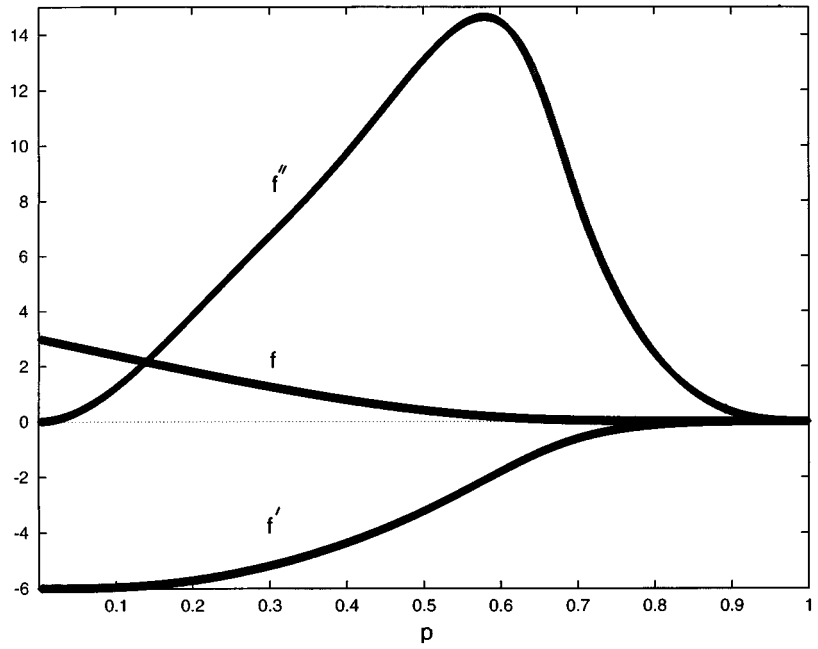

FIG. 8. Floppy modes on a bond-diluted tree with $\alpha=6$, $g=3$, and $b=1$. The number of floppy modes per site (f) is continuous as are its first (f') and second (f'') derivatives.

the surface bonds dominate, nevertheless the results are interesting. Thus we performed a calculation which keeps track of the number of redundant bonds on the trees for all levels going outwards from a rigid boundary. We performed the calculation for bond-diluted lattices with $b=1$. In that case, the number of redundant bonds l levels away from the boundary is given by

$$B_t=\alpha^{L-l}\sum_{k=g}^{\alpha}(k-g)\binom{\alpha}{k}(pT_0^{l-1})^k(1-pT_0^{l-1})^{\alpha-k}, \tag{38}$$

where L is the total number of levels in the tree. The total number of redundant bonds in the tree is

$$B=\sum_{l=1}^{L}B_l. \tag{39}$$

From global constraint counting, we then have

$$f=g-p\alpha+B/N_s. \tag{40}$$

$N_s=\alpha^L/(\alpha-1)$ is the number of sites on the L level tree. Results for f, $\partial f/\partial p$, and $\partial^2 f\partial p^2$ are presented in Fig. 8. It is clear from these calculations that there is no singular behavior in the second derivative of f on trees. However, there is a peak in the second derivative, but at a value of p considerably less than p_c.

V. CONCLUSIONS

We have shown that it is straightforward to develop and analyze tree models for the transmission of rigidity from a rigid border. In order to analyze these models we must, in general, consider the transmission of ''partial'' rigidity, as partially rigid structures may lead to rigidity higher up the tree. Some of our main conclusions are the following:

(1) Except for some ''trivial'' cases which are equivalent to connectivity percolation, the rigidity transition is *first order* on trees. However there is a square-root singularity superimposed upon the first-order transition occuring in the infinite-cluster probability. For example, this is explicitly demonstrated in Eq. (8).

(2) A constraint-counting mean-field theory which ignores redundant bonds is qualitatively incorrect for trees. This method does not describe correctly the nature of the rigidity transition. It can also grossly underestimate p_c, especially if the transition is strongly first order.

(3) We defined a vector order parameter which describes the number of degrees of freedom two points have with respect to each other. Although there is the possibility of multiple phase transitions with such a vector order parameter, we find that there is only one transition on trees.

(4) The number of floppy modes and its first and second derivatives are nonsingular, probably due to the dominance of surface bonds on trees.

(5) Bootstrap percolation and rigidity percolation are exactly the same on $b=1$ trees, but different on regular lattices. It is not clear, at least to these authors, to which case (if either), the current field theory of Obukhov applies [17].

Taken together with numerical results in two and three dimensions [5,20,21], there is now quite strong evidence that the rigidity transition on random lattices is often *first order*, in contrast to the large number of earlier papers which have assumed the opposite. Although the the evidence is strong that the infinite-cluster probability is usually first order, the triangular lattice data [5] suggest that the *stressed backbone* is second order. Thus the elastic constants may be second order, while the infinite cluster is first order. We suggest that the singular behavior superimposed on the first-order transition in P_∞ is in fact a reflection of the second-order character of the backbone [20]. It is interesting to note that the superposition of a first-order jump and a critical behavior occurs in some exactly solvable ''vertex models'' of ferroelectrics [22]. It is also possible that on some lattices, the partial-rigidity probabilities $P_1,P_2\ldots P_g$ may be singular at different disorder thresholds. To our knowledge, there has been no

numerical study of that possibility on regular lattices yet. Finally, it is important to emphasize that the work using exact constraint counting is correct for *random lattices*, while the earlier work was for regular lattices. It is still an open question as to whether these two cases are qualitatively different.

ACKNOWLEDGMENTS

Two of us (C.M. and P.M.D.) thank the DOE under Contract No. DE-FG02-90ER45418 and the PRF for financial support. P.M.D. thanks Mike Thorpe for useful discussions concerning floppy modes.

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