Method of constructing exactly solvable chaos

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We present a systematic method of constructing rational mappings as ergodic transformations with nonuniform invariant measures on the unit interval $I=[0,1]$. As a result, we obtain a two-parameter family of rational mappings that have a special property in that their invariant measures can be explicitly written in terms of algebraic functions of parameters and a dynamical variable. Furthermore, it is shown here that this family is the most generalized class of rational mappings possessing the property of exactly solvable chaos on I, including the Ulam–von Neumann map $y=4x(1-x)$. Based on the present method, we can produce a series of rational mappings resembling the asymmetric shape of the experimentally obtained first return maps of the Beloussof-Zhabotinski chemical reaction, and we can match some rational functions with other experimentally obtained first return maps in a systematic manner. [S1063-651X(97)09405-1]

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Characterizing invariant measures for explicit nonlinear dynamical systems is a fundamental problem which connects dynamical theory with statistics and statistical mechanics. In some cases, it would be desirable to characterize ergodic invariant measures for simple chaotic dynamical systems. However, in the cases of chaotic dynamical systems, such attempts to obtain explicit invariant measures have rarely been made. One well-known exception is the logistic map $Y=4X(1-X)\equiv f_0(X)$ on $I=[0,1]$ given by Ulam and von Neumann in the late 1940s [1].

The Ulam–von Neumann dynamical system $x_i=f_0(x_{i-1})$ has an ergodic measure $\mu(dx)=dx/\pi\sqrt{x(1-x)}$ such that the time averages of a function $Q(x)$ can be explicitly computed by the formula

$$\lim_{N\to\infty}\sum_{i=0}^{N-1}\frac{1}{N}Q(x_i)=\int_0^1\frac{Q(x)dx}{\pi\sqrt{x(1-x)}}$$

for almost all initial conditions $x_0\in I$. The first attempt to generalize the Ulam–von Neumann map within a set of rational functions was made by Katsura and Fukuda in 1985 [2]. The Katsura and Fukuda model is written as

$$Y=\frac{4X(1-X)(1-lX)}{(1-lX^2)^2}\equiv f_l(X), \tag{1}$$

for $0\leqslant l<1$. Clearly, the Ulam–von Neumann map can be regarded as a special case of Katsura–Fukuda systems where the parameter l is set to 0. The author show [3] that the Katsura–Fukuda mappings (1) also have ergordic measures which can be written explicitly as

$$\mu(dx)=\rho(x)dx=\frac{dx}{2K(l)\sqrt{x(1-x)(1-lx)}}, \tag{2}$$

where $K(l)$ is the elliptic integral of the first kind ($g=1$) given by $K(l)=\int_0^1 du/\sqrt{(1-u^2)(1-lu^2)}$. It is known [2] that the Katsura-Fukuda systems and the Ulam–von Neumann system also have explicit solutions in terms of the Jacobi sn elliptic function, as

$$x_n=\mathrm{sn}^2(K(l)2^n\theta_0), \quad \theta_0\in I, \tag{3}$$

where $\sqrt{l}$ corresponds to the modulus of Jacobi elliptic functions. The validity of the formulas of the general solutions (3) is easily checked using the duplication formula [4] of the Jacobi sn elliptic function

$$\mathrm{sn}(2u)=\frac{2\,\mathrm{sn}(u)\sqrt{[1-\mathrm{sn}^2(u)][1-l\,\mathrm{sn}^2(u)]}}{[1-l\,\mathrm{sn}^4(u)]}. \tag{4}$$

Because the Ulam–von Neumann system and Katsura-Fukuda systems have *not only* exact solutions (3), *but also* the explicitly written ergodic invariant measures (2), we call a dynamical system *exactly solvable chaos* if it has both of these properties.

Thus, it is of great interest to investigate whether we can generalize the Ulam–von Neumann system and the Katsura-Fukuda systems further within a set of rational functions, maintaining the property of exactly solvable chaos. The main purpose of the present paper is to show that by using the addition formulas of elliptic functions we can construct a two-parameter family of rational mappings of exactly solvable chaos, and, at the same time, that there is a certain limitation to generalizing this family within a set of rational functions.

Our results reported here [5] concern the following rational transformations

$$Y=f_{l,m}(X)=\frac{4X(1-X)(1-lX)(1-mX)}{1+AX^2+BX^3+CX^4}\in I, \tag{5}$$

where $A=-2(l+m+lm)$, $B=8lm$, $C=l^2+m^2-2lm-2l^2m-2lm^2+l^2m^2$, and $X\in I$. The parameters l and m are arbitrary real numbers satisfying the condition $-\infty<m\leqslant l<1$. Figure 1(a) shows various shapes of the proposed mappings (5). Surprisingly, some rational maps in Eq. (5) strongly resemble the asymmetric shape of the experimentally obtained first return maps of the Beloussof-Zhabotinski chemical reaction. Here, we will prove the following statement. The two-parameter family of rational mappings (5) is also exactly solvable chaos, such that the

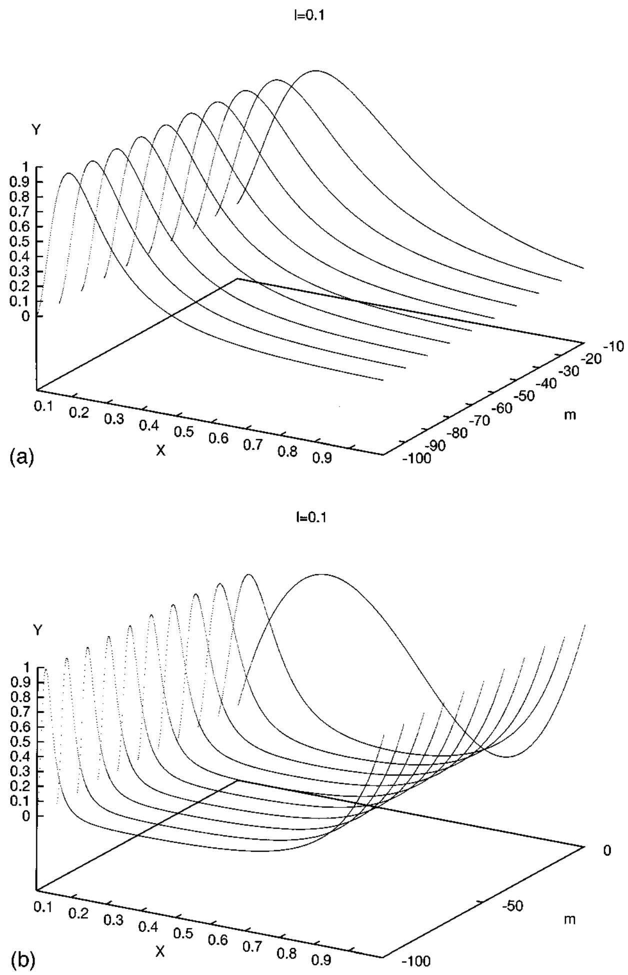

FIG. 1. (a) The two-parameter family of exactly solvable chaos mappings (generalized Ulam–von Neumann maps) (5) for $(l,m)=(0.1,-10),(0.1,-20),\ldots,(0.1,-100)$. The asymmetric shapes of this class of mappings are very similar to the first return maps of the Beloussof-Zhabotinski chemical reaction. (b) The two-parameter family of exactly solvable chaos mappings (generalized cubic maps) (25) for $(l,m)=(0.1,-10),(0.1,-20),\ldots,(0.1,-100)$.

dynamical systems $x_{i+1}=f_{l,m}(x_i)$ Eq. (5) have ergodic invariant measures explicitly given by

$$\mu(dx)=\rho(x)dx=\frac{dx}{2K(l,m)\sqrt{x(1-x)(1-lx)(1-mx)}}, \tag{6}$$

where K is given by the integrals

$$K(l,m)=\int_0^1 \frac{du}{\sqrt{(1-u^2)(1-lu^2)(1-mu^2)}}, \tag{7}$$

and it has general solutions explicitly given by

$$x_n=s^2[K(l,m)2^n\theta_0], \quad \theta_0\in I. \tag{8}$$

We prove this by explicitly computing the duplication formula of the following *degenerated* hyperelliptic function $s(x)$ defined by

$$s^{-1}(x)=\int_0^x \frac{du}{\sqrt{(1-u^2)(1-lu^2)(1-mu^2)}}. \tag{9}$$

Here, *degenerated* means that this hyperelliptic integral of the right-hand side of Eq. (9) can be reduced to a certain elliptic integral by a rational change of variables. Although the reduction of Abelian integrals of genus $g\geqslant 2$ to elliptic functions was intensively studied in the 19th century by Jacobi, Weierstrass, Königsberger, Kovalevskaya, and others, it was only in the 1980s that the theory of the reduction was successfully applied to physics for obtaining, for example, explicit periodic solutions in terms of elliptic functions for the Korteweg–de Vries equation and the sine-Gordon equation [6]. Let us consider reduction of the hyperelliptic integrals (9) as

$$s^{-1}(x)=\int_0^x \frac{du}{\sqrt{(1-u^2)(1-lu^2)(1-mu^2)}} = \int_0^{x^2} \frac{dv}{2\sqrt{v(1-v)(1-lv)(1-mv)}}, \tag{10}$$

where $u^2=v$. Thus, we can write the degenerated hyperelliptic function $s(x)$ in terms of the Weierstrass elliptic functions. The Weierstrass elliptic function $\wp(u)$ of $u\in C$ is defined by

$$\wp(u)=\frac{1}{u^2}+\sum_{j,k}{}' \left\{\frac{1}{(u-2j\omega_1-2k\omega_2)^2}-\frac{1}{(2j\omega_1+2k\omega_2)^2}\right\}, \tag{11}$$

where the symbol Σ' means that the summation is made over all combinations of integers j and k, except for the combination $j=k=0$, and $2\omega_1$ and $2\omega_2$ are periods of the function $\wp(u)$ [4]. The Weierstrass elliptic function $\wp(u)$ satisfies the differential equation

$$\left(\frac{d\wp(x)}{dx}\right)^2=4\wp^3(x)-g_2\wp(x)-g_3, \tag{12}$$

with the invariants

$$g_2(\omega_1,\omega_2)=60\sum_{j,k}{}' \frac{1}{(j\omega_1+k\omega_2)^4}$$

and

$$g_3(\omega_1,\omega_2)=140\sum_{j,k}{}' \frac{1}{(j\omega_1+k\omega_2)^6}$$

[4]. Let e_1,e_2, and e_3 be the roots of the equation $4z^3-g_2z-g_3=0$; that is,

$$e_1+e_2+e_3=0, \quad e_1e_2+e_2e_3+e_3e_1=-\frac{g_2}{4},$$

$$e_1e_2e_3=\frac{g_3}{4}. \tag{13}$$

The *discriminant* Δ of the function $\wp(u)$ is given by $\Delta = g_2^3 - 27g_3^2$. If $\Delta > 0$, all roots e_1, e_2, and e_3 of the equation $4z^3 - g_2 z - g_3 = 0$ are *real*. Thus, we can assume that $e_1 > e_2 > e_3$. In the case that $\Delta > 0$, it is known that the periods ω_1 and ω_2 are written simply as

$$\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}, \quad \omega_2 = i\int_{-\infty}^{e_3} \frac{dz}{\sqrt{g_3 + g_2 z - 4z^3}}. \tag{14}$$

Using the transformation of the variable as

$$v = -\frac{(1-l)(1-m)}{y - \dfrac{2l+2m-3lm-1}{3}} + 1,$$

we can rewrite $s^{-1}(x)$ in Eq. (10) as

$$\int_0^{x^2} \frac{dv}{2\sqrt{v(1-v)(1-lv)(1-mv)}} = \int_{(2-l-m)/3}^{[(2l+2m-3lm-1)/3]+[(1-l)(1-m)]/(1-x^2)} dy \times \frac{1}{\sqrt{4y^3 - g_2 y - g_3}}, \tag{15}$$

where

$$g_2 = \frac{4(1-l+l^2-m+m^2-lm)}{3}$$

and

$$g_3 = \frac{4(2-l-m)(2l-m-1)(2m-l-1)}{27}.$$

We note here that $4y^3 - g_2 y - g_3$ can be factored as

$$4y^3 - g_2 y - g_3 = 4\left(y - \frac{2-l-m}{3}\right)\left(y - \frac{2l-m-1}{3}\right) \times \left(y - \frac{2m-l-1}{3}\right). \tag{16}$$

We set

$$e_1 = \frac{2-l-m}{3} > e_2 = \frac{2l-m-1}{3} > e_3 = \frac{2m-l-1}{3}.$$

Thus, using the integral representation of the period ω_1 (14) and the differential equation (12) for the Weierstrass elliptic function, $s(x)$ can be written explicitly in terms of the Weierstrass elliptic function as

$$s^2(x) = 1 - \frac{(1-l)(1-m)}{\wp(\omega_1 - x) - \dfrac{2l+2m-3lm-1}{3}}. \tag{17}$$

The function $s^2(x)$ also has the same periods ω_1 and ω_2 computed using formula (14). It is noted here that because

$$\Delta \equiv g_2^3 - 27g_2^2 = 16(1-l)^2(1-m)^2(l-m)^2 > 0, \tag{18}$$

for $-\infty < m < l < 1$, the period $2\omega_1$ is always *real*, while the period $2\omega_2$ is always *pure imaginary*. Using the addition formula,

$$\wp(z+y) = \frac{1}{4}\left\{\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right\}^2 - \wp(z) - \wp(y), \tag{19}$$

and the duplication formula,

$$\wp(2z) = \frac{1}{4}\left\{\frac{\wp''(z)}{\wp'(z)}\right\}^2 - 2\wp(z), \tag{20}$$

for the Weierstrass elliptic function $\wp(u)$ [4], we finally obtain the *explicit* duplication formula of $s(x)$ as

$$s^2(2x) = \frac{4s^2(x)[1-s^2(x)][1-ls^2(x)][1-ms^2(x)]}{1 + As^4(x) + Bs^6(x) + Cs^8(x)}, \tag{21}$$

where $A = -2(l+m+lm), B = 8lm$, and $C = l^2 + m^2 - 2lm - 2l^2 m - 2lm^2 + l^2 m^2$. If we set $X = s^2(x)$ and $Y = s^2(2x)$, we obtain system (5) as $Y = f_{l,m}(X)$.

Using the relations

$$s^2(\omega_1 2\theta) = f_{l,m}[s^2(\omega_1 \theta)],$$

$$s^2[\omega_1(2-2\theta)] = f_{l,m}[s^2(\omega_1 \theta)], \tag{22}$$

for $\theta \in [0,1]$ and by defining the homeomorphism of $[0,1]$ into itself given by $\phi_{l,m}(x) = (1/\omega_1)s^{-1}(\sqrt{x})$, we derive the tent map $\tilde{f}(x) = \phi_{l,m} \circ f_{l,m} \circ \phi_{l,m}^{-1}$ as

$$\tilde{f}(x) = 2x \quad \text{for} \quad x \in \left[0, \frac{1}{2}\right],$$

$$\tilde{f}(x) = 2 - 2x \quad \text{for} \quad x \in \left[\frac{1}{2}, 1\right]. \tag{23}$$

Because this tent map (23) is clearly ergodic and preserves the Lebesgue measure, the map $f_{l,m}$ preserves the measures

$$\mu(dx) = \frac{d\phi_{l,m}}{dx} dx = \frac{dx}{2K(l,m)\sqrt{x(1-x)(1-lx)(1-mx)}}. \tag{24}$$

This measure (24) is absolutely continuous with respect to the Lebesgue measure, which implies that the Kolmogorov-Sinai entropy $h(\mu)$ is equivalent to the Lyapunov exponent of ln2 from the Pesin identity [7], and that the measure (24) is a physical one in the sense that it is the Sinai-Ruelle-Bowen measure such that for almost all initial conditions x_0, the time averages $\lim_{N\to\infty}(1/N)\Sigma_{i=0}^{N-1}\delta(x-x_i)$ reproduce the invariant measure $\mu(dx)$ [7].

In the same way, we can construct *generalized cubic maps* $f_{l,m}^{(3)}$ from the triplication formula $s^2(3x) = f_{l,m}^{(3)}[s^2(x)]$ as

$$Y=f_{l,m}^{(3)}(X)=\frac{X\left(-3+4X+\sum_{i=1}^{4}A_iX^i\right)^2}{1+\sum_{i=2}^{9}B_iX^i},\qquad(25)$$

where $A_1,\ldots,A_4$ and $B_2,\ldots,B_9$ are polynomial functions in the parameters l and m which vanish for $l=m=0$ [8]. The generalized cubic map $f_{l,m}^{(3)}$ has the same invariant measures (24) because the relation $\widetilde{f}^{(3)}(x)=\phi_{l,m}\circ f_{l,m}^{(3)}\circ\phi_{l,m}^{-1}$ holds for the piecewise-linear map $\widetilde{f}^{(3)}(x)=3x$ for $0\leqslant x\leqslant\frac{1}{3}$, $-3x+2$ for $\frac{1}{3}\leqslant x\leqslant\frac{2}{3}$ and $3x-2$ for $\frac{2}{3}\leqslant x\leqslant 1$. If we set $l=m=0$, this rational mapping is reduced to the cubic map $Y=X(3-4X)^2$ as a special case of Chebyshev maps obtained by Adler and Rivlin [9]. Thus, we can obtain *generalized Chebyshev maps* as rational functions $f_{l,m}^{(p)}$ from the addition formulas $s^2(px)=f_{l,m}^{(p)}[s^2(x)]$, which have the same invariant measures (24). The shapes of the generalized cubic maps are depicted in Fig. 1(b). Based on the case of Beloussof-Zhabotinski map, we predict here that some rational mappings (25), such as that shown in Fig. 1(b), can resemble the first return maps experimentally constructed from some unknown chaotic phenomena.

Are there more generalized rational mappings that possess the properties of exactly solvable chaos, such as Eq. (5)? The exact solvability of the present rational mappings (5) and (25) is due to the fact that $s^2(x)$ in Eq. (17) is a rational function of the Weierstrass elliptic function $\wp(u)$ having the *addition formula* and the real period ω_1. For an arbitrary set of parameters $e_1, e_2(<e_1)$, and $e_3(=-e_1-e_2)$ which determine the Weierstrass elliptic function $\wp(u)$ with the real period, there exists a set of parameters l and m of $f_{l,m}^{(p)}$ given by $l=1-(e_1-e_2)$ and $m=1-(2e_1-2e_2)<l$; i.e., the mapping $h:(e_1,e_2)\mapsto(l,m)$ is a *bijection*. In other words, every element of the family $\{f_{l,m}^{(p)}\}$ has one-to-one correspondence to an element of the set of Weierstrass elliptic functions $\{\wp(u)\}$ with real periods. Furthermore, any elliptic function $w(u)$ can be expressed in terms of Weierstrassian elliptic functions $\wp(u)$ and $\wp(u)'$ with the same periods, the expression being rational in $\wp(u)$ and linear in $\wp'(u)$ [4,10]. Since it is known [11] from the *Weierstrassian theorem* that the class of analytic functions $h(u)$ having the algebraic addition formulas [12] including the duplication formulas is exactly the class of algebraic functions of elliptic functions $\wp(u)$, which of course includes the class of algebraic functions of sine functions, we can say that this two-parameter family of dynamical systems $Y=f_{l,m}^{(p)}(X)$ essentially forms a maximal class in representing exactly solvable chaos induced by one-dimensional rational mappings.

In conclusion, we present a method of constructing ergodic transformations related to rational functions with explicit nonuniform invariant measures using the addition formulas of elliptic functions. As a result, we systematically generalize the Ulam–von Neumann (logistic) map and Chebyshev maps into two-parameter families of rational mappings. We also showed that constructing the more generalized family of rational mappings possessing explicit ergodic invariant measures has a certain limitation due to the Weierstrassian argument concerning the addition formulas for general analytic functions. As for the applications, all of the constructive results given here for ergodic invariant measures by rational mappings can be directly used as nonlinear random number generators for the Monte Carlo methods.

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$A_1=4(l+m)$, $A_2=-6(l+m+lm)$, $A_3=12lm$,

$A_4=l^2+m^2-2lm-2l^2m-2lm^2+l^2m^2$,

$B_2=-12(l+m+lm)$,

$B_3=8(l+m+l^2+m^2+l^2m+lm^2+15lm)$,

$B_4=6(5l^2+5m^2-26lm-26l^2m-26lm^2+5l^2m^2)$,

$B_5=24(-2l^2-2m^2-2l^3-2m^3+4lm+7l^2m+7lm^2)$
$+24(4l^3m+4lm^3+7l^2m^2-2l^3m^2-2l^2m^3)$,

$B_6=4(4l^2+4m^2+4l^4+4m^4+17l^3+17m^3-8lm)+4(-17l^2m$
$-17lm^2-17l^3m-17lm^3-8l^4m-8lm^4)+4(4l^2m^4+4l^4m^2$
$-17l^3m^2-17l^2m^3+17l^3m^3-54l^2m^2)$,

$B_7=24(-l^3-m^3-l^4-m^4+l^2m+lm^2-l^3m-lm^3)$
$+24(l^4m+lm^4+4l^2m^2+4l^3m^2+4l^2m^3)$
$+24(l^4m^2+l^2m^4-l^3m^3-l^4m^3-l^3m^4)$,

$B_8=3(3l^4+3m^4+4l^3m+4lm^3+4l^4m+4lm^4-14l^2m^2)$
$+3(-4l^3m^2-4l^2m^3-4l^3m^3-14l^4m^2$
$-14l^2m^4+4l^4m^3+4l^3m^4+3l^4m^4)$,

$B_9=8(-l^4m-lm^4+l^3m^2+l^2m^3+l^4m^2+l^2m^4$
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the Weierstrass elliptic function $\wp(u)$ with the same periods as $G(w,\wp)=0$, G being a rational function of two variables.

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