Solving the Langevin equation with stochastic algebraically correlated noise

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The long time tail in the velocity and force autocorrelation function has been found recently in molecular dynamics simulations of peripheral collisions of ions. Simulation of those slowly decaying correlations in the stochastic transport theory requires the development of new methods of generating stochastic force of arbitrarily long correlation times. In this paper we propose a Markovian process, the multidimensional kangaroo process, which permits the description of various algebraically correlated stochastic processes.
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I. INTRODUCTION

Dynamics of a classical many-body system can be investigated using either the molecular dynamics approach or the kinetic rate equations. Both approaches have their semiclassical counterparts and can be modified to incorporate also the Pauli exclusion principle for fermions. In the latter case, one considers, for example, different variants of the Boltzmann or Boltzmann-Langevin equations, whereas in the former case the ''quantal'' version of the molecular dynamics, the so-called antisymmetrized molecular dynamics [1] has been proposed. Chaotic properties of atomic nuclei have been discussed in the framework of the classical molecular dynamics (CMD). For central collisions where fusion processes dominates, it has been demonstrated [2,3] that both the velocity autocorrelation function $C(t)=\langle \mathbf{v}(t_0)\mathbf{v}(t_0+t)\rangle$ and the force autocorrelation function $\widetilde{C}(t)=\langle \mathbf{F}(t_0)\mathbf{F}(t)\rangle$ decay exponentially in time. The equilibration time is short, allowing the statistical properties of the compound nucleus to show up at the early stage of the reaction. On the contrary, in the peripheral collisions of ions, the algebraic, long time tail $\sim t^{-\gamma}$ ($\gamma=1$) was found in both the velocity and force autocorrelation functions [2,3]. Moreover, the survival probability is given by a power law [4].

The Fourier transform of $C(t)$ gives the power spectrum $S(\omega)$. For the peripheral collisions [2], $C(t)\sim t^{-1}$ and hence $S(\omega)\sim|\ln\omega|$. The mean square displacement in configurational space is in this case [2] $\sigma^2(t)\equiv\langle[\mathbf{r}(t)-\mathbf{r}(t_0)]^2\rangle\propto t\ \ln(t/t_0)-t+t_0$. The diffusion is anomalously enhanced (superdiffusion) and the diffusion rate $D\equiv\lim_{t\to\infty}\sigma^2(t)/t$ diverges logarithmically; i.e., the dissipation rate does not stabilize, as would be the case for a normal diffusion ($D=$const). The same dependence holds also for the mean-square displacement in the velocity space.

The logarithmic power spectrum and the enhanced diffusion have been found for the periodic Lorentz gas (PLG) of hard disks (the extended Sinai billiard) [5]. From the point of view of transport phenomena, many physical systems can be reduced to a simple lattice of periodic potentials. Besides the CMD in the orbiting regime, the dynamics of electrons in crystals moving in a magnetic field or the ballistic-electron dynamics in lateral superlattices are other examples that can be modeled in terms of periodic two-dimensional (2D) lattices [6,7]. The similarity of the diffusive behavior for systems as different as the CMD and the PLG follows from the fact that the power-law tail of the velocity autocorrelation function is due to the existence of long free paths. This behavior is *universal* and insensitive to the details of the potential, in particular to its short distance features. Such universality allows one to describe phenomena involving long free paths in the framework of the Langevin equation with algebraically correlated noise [3]. Inclusion of effects connected with the antisymmetrization of the wave function for fermions does not modify this picture qualitatively. The nonlocality of the Pauli potential destroys cantori in the phase space and the diffusion process, for sufficiently large lattice spacing, is dominated by long free paths and hence its power spectrum is logarithmic at small frequency limit [8]. This finding makes the purely classical description more reliable.

The relevance of the Langevin approach for the description of an induced fission process has been realized a long time ago [9]. The slow collective motion with its high mass parameter is treated as a Brownian particle, whereas the fast nucleonic degrees of freedom form the heat bath. In generalized Brownian motion theory [10], the Hamilton equations can be rewritten in the form of the Langevin equation by making use of the projection operator technique. The total force acting on a Brownian particle is divided into a systematic part and a random part. The slowly varying part describes the evolution of macroscopic variables. The fast varying part leads to the fluctuations around the most probable path. In the conventional Langevin approach, it is usually assumed that the time evolution of the fast varying random part is stochastic and the time rate of change is much faster than that of the systematic part. Consequently, it is assumed that the correlation function of the random part is δ correlated or decays exponentially. (For a recent review of stochastic theories with the colored noise see Ref. [11].)

A hypothesis of the rapid decay of the force correlations holds for central collisions and the CMD yields in this case fast decaying correlations in both velocity and acceleration (force) [2]. However, for peripheral reactions and/or strongly elongated shapes, the correlations decay algebraically [2]. Such slowly decaying correlations are known in various phenomena including the chemical reactions in solutions [12], ligand's migration in biomolecules [13], atomic diffusion through a periodic lattice [14], Stark broadening [15], and many others. This regime is certainly beyond the standard Langevin approach and requires the consideration of the colored noise of arbitrarily long correlation time. The first important step in this direction was the theory of line shapes and relaxation in magnetic resonance systems through the study of the so-called Kubo oscillator [16]. More recently, the Kubo-Anderson process [17] with the slowly decaying noise correlation function, the so-called kangaroo process (KP) [18], was used to explain the noise-induced Stark broadening [15].

Recently, we have proposed a method that extends the Langevin approach for phenomena with either exponentially or algebraically decaying force correlations [2]. In these studies, we have investigated a two-dimensional Langevin equation, describing stochastic trajectories $\mathbf{r}(t)$:

$$\frac{d\mathbf{r}}{dt}=\mathbf{v},$$
$$\mu\frac{d\mathbf{v}}{dt}(t)=-\beta\mathbf{v}(t)-\frac{\partial V(r)}{\partial\mathbf{r}}+\mathbf{F}(t), \tag{1}$$

where the spherically symmetric potential $V(r\equiv|\mathbf{r}|)$ generates a conservative force, β is the friction constant, and μ stands for the mass of the system. The external noise (stochastic force) $\mathbf{F}(t)$ has algebraically decaying correlations

$$\langle\mathbf{F}(0)\mathbf{F}(t)\rangle\sim 1/t,$$
$$\langle\mathbf{F}(t)\rangle=0. \tag{2}$$

These conditions do not determine the noise uniquely. In our earlier studies, we have proposed simulating $\mathbf{F}(t)$ by deterministic time series of the particle velocity in the PLG. In the present work, we investigate the possibility of simulating algebraically correlated noise $\mathbf{F}(t)$ by a Markov process, and for this purpose we shall study the generalized KP. In spite of fundamental differences, there are some similarities between these two realizations of the algebraic noise. For both processess, algebraic correlations involved are due to the existence of ''long free paths,'' i.e., the value of the stochastic process (velocity of the particle in the case of the PLG) remains constant for long time intervals.

The main goal of this work is to investigate and compare the Langevin processes for these two different ways of generating the stochastic force. For that purpose we shall compare the most relevant physical quantities such as the energy spectrum of the particles and their escape time distribution obtained with the different generators of the external noise $\mathbf{F}(t)$. In Sec. II we shall recall the most essential features of the PLG process that can be used to generate both algebraic and exponentially correlated deterministic noise [3]. Sec. III is devoted to the discussion of the KP. In Sec. III B we discuss the multidimensional, norm-conserving generalization of this stochastic process, which can be directly compared with the norm-conserving PLG process. We perform this comparison in Sec. IV, solving the Langevin equation (1) for particles escaping from the spherically symmetric potential well. Finally, the most important results of this work are summarized and concluded in Sec. V.

II. PERIODIC LORENTZ GAS AND THE CORRELATIONS FOR OPEN AND CLOSED HORIZONS

Before discussing the KP and its generalization, we want to remind the reader of the most essential properties of the PLG (or the extended Sinai billiard). As stated before, the PLG was used to generate event by event the erratic chaotic force acting on Brownian particles [3]. The PLG consists of a single point particle moving in a two-dimensional periodic array of fixed circular scatterers of radius R [19]. The lattice spacing is assumed to be equal to two, then the separation between disk borders $l=2-2R$. The point mass is scattered elastically from scatterers and the particle velocity has a unit length. The particle is reflected upon hitting an arc of hard disks or meets the periodic boundary condition when it passes between hard disks, crossing a straight line linking their centers. The phase space is spanned by the arc length s with $0<s<L=2\pi R+4l$ and by the tangential momentum p, which is related to the reflection angle ϕ: $p=\cos\phi$. The orbit consists of the succession of pairs $\{s_n(s_0,p_0), p_n(s_0,p_0)\}$ corresponding to the nth bounce when the initial condition was $\{s_0,p_0\}$. This dynamics is a mapping M of the phase space $\{s,p\}$ onto itself [20]:

$$\begin{pmatrix} s_{n+1}\\ p_{n+1}\end{pmatrix}=M\begin{pmatrix} s_n\\ p_n\end{pmatrix}. \tag{3}$$

The sequence of iterates (3) is uniquely determined as a function of the initial value. The separation between disks completely determines the behavior of the system. If $R>1$ (''the high-density regime'' of the PLG) then the disks overlap and the particle is trapped in a region bounded by four arcs of circles. This situation corresponds to the closed horizon as the particle trajectory is bound. If $R<1$ (''the low-density regime'' of the PLG) the particle sees an infinite horizon and may move to an arbitrarily long distance between subsequent collisions; i.e., the length of free paths is unbounded.

The PLG, for both $R<1$ and $R\geqslant 1$, belongs to the category of so-called K systems [19], for which the nearby trajectories diverge exponentially and the metric entropy is positive. This system is known to be ergodic in two dimensions and numerical experiments in higher dimensions also indicate its ergodicity [21]. Despite this, the PLG exhibits long-time correlations that are typically associated with the existence of tori in the phase space. For $R<1$, there exist families of trajectories that do not collide with hard scatterers and correspond to a regular motion. The existence these families is a reason for the long-time tails in the correlation functions: the velocity autocorrelation function changes from a stretched exponential decay [21] for the closed horizon situation to algebraic decay [22] $[C(t)\sim t^{-1}]$ for the infinite horizon situation. Consequently, the self-generated diffusion process changes from an ordinary diffusion process $[D(t)=\text{const}]$ to a superdiffusive process $[D(t)\sim\ln t]$ when the horizon for a wandering particle is opened. In the latter case, the distribution of free path lengths is algebraic: $S(s) \sim s^{-3}$ for large s [21], independent of the dimensionality of the billiard.

At around $R=1$, many quantities, including the probability density of free path length $S(s)$ and the velocity autocorrelation function $C(t)$, exhibit a critical behavior that resembles a second-order phase transition. In particular, the correlation length diverges and the length scale disappears. One can also define the order parameter

$$\lim_{t\to\infty}\frac{\langle r^2(t)\rangle}{t\ \ln t},$$

which is zero for $R>1$ and changes to a finite value D_0 for $R<1$.

III. THE KANGAROO PROCESS

The stepwise constant random function $m(t)$ is called a Kubo-Anderson process if the jumping times t_i ($i=-\infty,\dots,+\infty$) are uniformly and independently distributed with density ν in the interval $(-\infty,+\infty)$, and $m(t)$ is a constant $m(t)=m_i$ in the interval $t_i \leqslant t < t_{i+1}$. $m(t)$ is the stationary Markov process with the probability density $\hat{P}(m)$. Assuming $\langle m\rangle=0$, one obtains for the covariance of this process:

$$\tilde{\Gamma}(|t-t'|)\equiv\langle m(t)m(t')\rangle=\langle m^2\rangle\exp(-\nu|t-t'|). \tag{4}$$

Both the probability density $\hat{P}(m)$ and the correlation time $T_{\rm corr}=\nu^{-1}$ for the Kubo-Anderson process may be chosen arbitrarily. However, the functional form of the covariance is always exponential.

The study of the problem of stochastic Stark broadening [15], where the covariance is proportional to $1/t$ and is not integrable, has led to modifying the Kubo-Anderson process by requiring that the frequency of jumping times $\nu(m)$ is a function of the value of the process itself. This process has been called the ‘‘kangaroo process’’ (KP). The KP is a stepwise stationary Markov process, whose transition probability depends only on time differences. This probability is given for infinitesimal time intervals Δt by

$$P_{\rm KP}(m,\Delta t|m',0)=\{1-\nu(m')\Delta t\}\delta(m-m') + \hat{Q}(m)\nu(m')\Delta t, \tag{5}$$

where $\hat{Q}(m)$ is a given probability density to be specified below. $P_{\rm KP}dm$ is the probability that the KP at time Δt is between m and $m+dm$, knowing that it was equal to m' at time $t=0$. The first term on the right-hand side of Eq. (5) is the probability that no jump occurred in the time interval $(0,\Delta t)$. The term $\nu(m')\Delta t$ is the probability that one jump occurred. Immediately after such a jump, the probability density of m becomes $\hat{Q}(m)$. The Focker-Planck equation for the KP reads [23,18]

$$\frac{\partial}{\partial t}\hat{P}(m,t) = \lim_{\substack{\Delta t\to 0\\(\Delta t\geqslant 0)}}\left\{\int P_{\rm KP}(m,\Delta t|m',0)\hat{P}(m',t)dm'-\hat{P}(m,t)\right\}\times(\Delta t)^{-1} \tag{6}$$

$$=-\nu(m)\hat{P}(m,t)+\hat{Q}(m)\int \nu(m')\hat{P}(m',t)dm'. \tag{7}$$

The stationary probability density $\hat{P}(m)$ of $m(t)$ is related to $\hat{Q}(m)$ by

$$\hat{Q}(m)=\frac{\nu(m)\hat{P}(m)}{\int \nu(m')\hat{P}(m')dm'}=\frac{\nu(m)\hat{P}(m)}{\langle\nu\rangle}. \tag{8}$$

The calculation of the covariance $\tilde{\Gamma}(t)$ of the KP requires the summation of a series to take into account the possible occurrence of an arbitrary number of jumps between 0 and t. For that let us calculate the Laplace transform of $\tilde{\Gamma}(t)$:

$$\tilde{\Gamma}(z)=\int_0^\infty \exp(izt)\tilde{\Gamma}(t)dt, \tag{9}$$

which will allow one to relate $\nu(m)$ and $\hat{P}(m)$ for a given $\tilde{\Gamma}(t)$. It becomes [18]

$$\tilde{\Gamma}(z)=\left\langle\frac{m^2}{\nu(m)-iz}\right\rangle_S - \left(iz\left\langle\frac{\nu(m)}{\nu(m)-iz}\right\rangle_S\right)^{-1}\left(\left\langle\frac{m}{\nu(m)-iz}\right\rangle_S\right)^2, \tag{10}$$

where $\langle\ \rangle_S$ denotes averaging over the stationary probability distribution $\hat{P}(m)$. If $\hat{P}(m)$ and $\nu(m)$ are even functions, then

$$\left\langle\frac{m}{\nu(m)-iz}\right\rangle_S=0$$

or, equivalently,

$$\langle m\ \exp[-\nu(m)t]\rangle_S=0,$$

and Eq. (10) simplifies to

$$\tilde{\Gamma}(z)=\left\langle\frac{m^2}{\nu(m)-iz}\right\rangle_S. \tag{11}$$

The covariance of the KP is then

$$\tilde{\Gamma}(t)=\int_{-\infty}^{+\infty} m^2\hat{P}(m)\exp[-\nu(m)|t|]dm, \tag{12}$$

i.e., the ordinary variance of $\hat{P}(m)$ conditioned by the probability $\exp[-\nu(m)|t|]$ that no jump occurs between 0 and t. Given $\hat{P}(m)$ and the covariance $\tilde{\Gamma}(t)$, the jumping frequency $\nu(m)$ can be obtained as follows. Let us assume that $\nu(m)$ is a monotonic increasing function of $|m|$ such that $\nu(\infty)=\infty$. Then, taking ν as a new integration variable, one obtains

$$\tilde{\Gamma}(t)=2\int_{\nu(0)}^{+\infty} m^2\hat{P}(m)\frac{dm}{d\nu}\exp(-\nu|t|)d\nu. \tag{13}$$

Calculation of $\nu(m)$ requires then the inversion of the Laplace transformation and the solution of a simple differential equation. For some probability distributions $\hat{P}(m)$, $\nu(\infty)$ can be finite. In this case the covariance $\tilde{\Gamma}(t)$ is properly reproduced by the above procedure asymptotically, i.e., in the limit of large t.

It is always possible to construct the KP with an arbitrary probability distribution $\hat{P}(m)$ and a quite arbitrary covariance $\tilde{\Gamma}$. For the exponential correlations,

$$\tilde{\Gamma}(|t-t'|)\equiv\langle m(t)m(t')\rangle\sim\exp(-\nu_0|t-t'|), \tag{14}$$

we have

$$\nu(m)=\nu_0=\text{const.} \tag{15}$$

For the most interesting, algebraic correlations,

$$\tilde{\Gamma}(|t-t'|)\sim\frac{\Gamma(1/\kappa)}{|t-t'|^{1/\kappa}} \quad (\kappa>0), \tag{16}$$

singular for $t=t'$, we have

$$\nu(m)=\left(2\int_0^{|m|} m'^2\hat{P}(m')dm'\right)^{\kappa}. \tag{17}$$

Γ in Eq. (16) is the gamma function and $\hat{P}(m)$ is an even function.

A. One-dimensional kangaroo process

In the following, we shall assume that both $\hat{P}(m)$ and $\nu(m)$ are even functions. For $\tilde{\Gamma}(t)=1/t$, the frequency $\nu(m)$ is

$$\nu(m)=2\int_0^{|m|} m'^2\hat{P}(m')dm'. \tag{18}$$

The ‘‘free path’’ length can be defined as

$$s=1/\nu. \tag{19}$$

Knowing $\hat{P}(m)$ we want to determine the probability density of jump frequency $R(\nu)$, as well as the free path distribution $S(s)$. Since $\hat{P}(m)d|m|=R(\nu)d\nu$, then

$$R(\nu)=\hat{P}(m)\left(\frac{d\nu}{d|m|}\right)^{-1}.$$

From Eq. (18) we have

$$\frac{d\nu}{d|m|}=2m^2\hat{P}(m),$$

and therefore

$$R(\nu)=\frac{1}{2m^2}. \tag{20}$$

The free paths distribution is then

$$S(s)=R(\nu)\left|\frac{d\nu}{ds}\right|=\frac{R(\nu(s))}{s^2}. \tag{21}$$

In order to see whether and how details of the chosen probability density $\hat{P}(m)$ influence the properties of the KP, in particular the free path distribution, let us now consider a few simple examples. First, let us take

$$\hat{P}(m)=\begin{cases}1 & \text{for } |m|<1\\ 0 & \text{for } |m|>1.\end{cases} \tag{22}$$

The jump frequency in this case is $\nu(m)=\frac{2}{3}|m|^3$, and the frequency distribution is $R(\nu)\sim\nu^{-2/3}$. Consequently, the free path distribution becomes

$$S(s)\sim s^{-4/3}. \tag{23}$$

Now, let us consider a general algebraic distribution:

$$\hat{P}(m)=\begin{cases}0 \quad \text{for } |m|<\varepsilon\\ (\alpha-1)\varepsilon^{\alpha-1}|m|^{-\alpha} \quad \text{for } |m|>\varepsilon,\end{cases} \tag{24}$$

where $\alpha>1$ to ensure a correct normalization. We consider two cases: (i) $\alpha\neq 3$ and (ii) $\alpha=3$. In case (i), the jump frequency is

$$\nu(m)=\frac{2(\alpha-1)}{3-\alpha}\varepsilon^{\alpha-1}(|m|^{3-\alpha}-\varepsilon^{3-\alpha}) \tag{25}$$

and the frequency distribution is

$$R(\nu)=\frac{1}{2\varepsilon^2}\left(1+\frac{3-\alpha}{2(\alpha-1)}\varepsilon^{-2}\nu\right)^{2/(\alpha-3)}. \tag{26}$$

Consequently, the free path distribution becomes

$$S(s)=\frac{1}{2s^2\varepsilon^2}\left(1+\frac{3-\alpha}{2(\alpha-1)}\varepsilon^{-2}s^{-1}\right)^{2/(\alpha-3)}. \tag{27}$$

Therefore, in the limit of long paths, $S(s)\sim s^{-2}$. This is the fastest decaying free path distribution that can be obtained with the one-dimensional KP. For $\alpha>3$, Eq. (27) implies a low-s cutoff in the free path length:

$$s\geqslant s_{\min}=\frac{\alpha-3}{2(\alpha-1)}\varepsilon^{-2}.$$

In the second case (ii), the jump frequency is

$$\nu(m)=4\varepsilon^2\ln\frac{|m|}{\varepsilon} \tag{28}$$

and the frequency distribution is given by the Poisson distribution

$$R(\nu)=\frac{1}{2\varepsilon^2}\exp\left(-\frac{\nu}{2\varepsilon^2}\right). \tag{29}$$

Consequently, the free path distribution becomes

$$S(s)=\frac{1}{2\varepsilon^2 s^2}\exp\left(-\frac{1}{2\varepsilon^2 s}\right). \tag{30}$$

Also in this case, in the limit of long paths, $S(s)\sim s^{-2}$.

In d dimensions with *independent* and the same kangaroo processes in all d directions, the path-length distribution $S(s)$ becomes $S_d(s)=[S_1(s)]^d$, where $S_1(s)$ stands for the one-dimensional free path distribution. One should stress, however, that the norm for such a process, $|\mathbf{m}|=(\Sigma_i m_i^2)^{1/2}$, is not conserved during the evolution. Moreover, $|\mathbf{m}|$ does not have any specific and physically motivated distribution, which is a serious drawback of the $(d\times 1)$-dimensional kangaroo processes.

B. Multidimensional generalization of the kangaroo process

In this section, we will present the multidimensional generalization of the KP and discuss in details the two-dimensional case. The value of the process is now a vector $\mathbf{m}=[m_1,m_2]$ with coordinates m_1,m_2 and a constant norm: $|\mathbf{m}|=1$. Hence, the KP takes random values on a unit circle and the coordinates $m_1=\cos\phi$ and $m_2=\sin\phi$, as well as the frequency ν, are expressed in terms of a single random angle ϕ $(0\leqslant\phi<2\pi)$. Let us denote the probability distribution of this process by $\hat{P}_\Phi(\phi)$. The covariance of the KP in this case becomes

$$\widetilde{\Gamma}(t)=\int_{\nu(0)}^{\infty}\hat{P}_\Phi(\phi)\frac{d\phi}{d\nu}\exp(-\nu|t|)d\nu. \tag{31}$$

Analogously to the one-dimensional case, the above formula leads to

$$\hat{P}_\Phi(\phi)\left(\frac{d\nu}{d\phi}\right)^{-1}=1,$$

which resolves itself into

$$\nu(\phi)=\int_0^{\phi}\hat{P}_\Phi(\phi')d\phi'. \tag{32}$$

Since $R(\nu)d\nu=\hat{P}_\Phi(\phi)d\phi$, we come to the conclusion that $R(\nu)=\mathrm{const}$, independent of the form of $\hat{P}_\Phi(\phi)$. Obviously, the free path distribution becomes in this case

$$S(s)\sim s^{-2}. \tag{33}$$

Since the probability distribution must be an even function, $\hat{P}_\Phi(\phi)$ satisfies the condition $\hat{P}_\Phi(\phi)=\hat{P}_\Phi(\phi+\pi)$. According to Eq. (32), the frequency becomes zero for $\phi=0$, which means the infinitely long path. A particular form of $\hat{P}_\Phi(\phi)$ is unimportant if one looks at the probability distributions $R(\nu)$ and $S(s)$. However, $\hat{P}_\Phi(\phi)$ influences the asymptotic angular distribution of the processes, an important quantity for many applications. As a simple example of highly nonisotropic processes, it could be enough to sample ϕ uniformly only in the interval $[0,\pi/2]$ and then to randomly choose signs of m_1 and m_2 coordinates. The long paths correspond then to $\phi_0\approx 0$ and $\phi_1\approx\pi$, being restricted to one direction. A more isotropic distribution can be obtained by dividing the full angle on sectors of size $\Delta\phi=\pi/n$, where the integer n may be arbitrarily large. One defines $\hat{P}_\Phi(\phi)$ only in the interval $[0,\pi/n]$ and then chooses the sector itself, with the uniform probability. In this case, the long paths will be found at around $\phi_k=2\pi k/n$ $[k=0,1,\ldots,(n-1)]$. If $n=4$ then the free path distribution is similar to that in the PLG with the infinite horizon for which long paths are found at $\phi\approx 0$, $\pi/2$, π, and $3/2\pi$. The precise value of n depends on the physical problem considered and, in particular, on the geometry involved in this problem. For example, in the fission-fusion dynamics of an atomic nucleus with a preferred direction of the collective process specified by the elongation parameter, one expects that the long path distribution should be strongly nonisotropic and, hence, the case $n=2$ is more realistic.

The above stochastic process can be easily generalized to still higher dimensions. For example, in three dimensions one has two angles, θ and ϕ, given by the probability distributions $\hat{P}_\Theta(\theta)$ and $\hat{P}_\Phi(\phi)$, respectively. If $\nu=\nu(\theta)$, independent of the angle ϕ, then the covariance of the KP becomes

$$\widetilde{\Gamma}(t)=\int \hat{P}_\Theta(\theta)\hat{P}_\Phi(\phi)\exp[-\nu(\theta)|t|]\sin\theta d\theta d\phi, \tag{34}$$

where the integration is performed over both angles. Since the probability distribution must be an even function, therefore, $\hat{P}_\Theta(\theta)=\hat{P}_\Theta(\pi-\theta)$ and $\hat{P}_\Phi(\phi)=\hat{P}_\Phi(\pi+\phi)$. For a normalized probability $\hat{P}_\Phi(\phi)$ in (34), we obtain

$$\hat{P}_\Theta(\theta)\sin\theta(d\theta/d\nu)=1$$

and

$$\nu(\theta)=\int_0^{\theta}\hat{P}_\Theta(\theta')\sin\theta' d\theta'. \tag{35}$$

Since $R(\nu)d\nu=\hat{P}_\Theta(\theta)\sin\theta d\theta$, the free path distribution is $S(s)\sim s^{-2}$, as in the above two-dimensional KP. The same holds for an arbitrary number of dimensions providing the frequency ν of the KP depends only on one angle.

The multidimensional generalized KP can be also easily applied to generate stochastic processes with any algebraic covariance $\widetilde{\Gamma}(t)\sim|t|^{-1/\kappa}$ $(\kappa>0)$. In two dimensions, the frequency ν of the stochastic process becomes

$$\nu=\left(\int_0^{\phi}\hat{P}_\Phi(\phi')d\phi'\right)^{\kappa} \tag{36}$$

and

$$\frac{d\nu}{d\phi}=\kappa\left(\int_0^{\phi}\hat{P}_\Phi(\phi')d\phi'\right)^{\kappa-1}\hat{P}_\Phi(\phi).$$

Since $R(\nu)d\nu = \hat{P}_\Phi(\phi)d\phi$, then

$$R(\nu) = \frac{1}{\kappa}\left(\int_0^{\phi} \hat{P}_\Phi(\phi')d\phi'\right)^{1-\kappa} = \frac{1}{\kappa}\nu^{(1-\kappa)/\kappa}. \quad (37)$$

The free path distribution can now be easily found:

$$S(s) \sim s^{-(1+1/\kappa)}. \quad (38)$$

For the exponential covariance (14), the multidimensional norm-conserving generalization of the KP is trivial because ν in this case is constant. Hence, we have succeeded in formulating the Markovian multidimensional process, which can approximate the multidimensional extended Sinai billiard (the PLG) both in the situation of the closed horizon when the correlations are exponential as well as in the situation of the open horizon when correlations are algebraic.

IV. LANGEVIN PROBLEM FOR THE MARKOVIAN STOCHASTIC FORCE

Let us now consider a dissipative system that consists of many particles. Each particle in the system obeys an intrinsic damping that is independent of the fluctuation term. The stochastic force $\mathbf{F}(t)$ acting on a particle in the dissipative system is $\mathbf{F}(t) = \varepsilon \mathbf{m}$, where $\mathbf{m} = [m_1, m_2]$ is the value of the KP and ε is a constant force amplitude [24].

We shall consider the motion of Brownian particle in the circular attractive potential defined as

$$V(|\mathbf{r}|) = \begin{cases} V_0\left[1 - \left(\dfrac{\mathbf{r}}{r_B}\right)^2\right] & \text{for } |\mathbf{r}| \le r_B \\ 0 \quad \text{for } |\mathbf{r}| > r_B, \end{cases} \quad (39)$$

where V_0 and r_B are the depth and the radius of the potential, respectively. Inside the potential, the motion of the particle is given by the Langevin equation (1) with the correlated stochastic force $\mathbf{F}(t)$. Otherwise the particle is free. Initially $(t=0)$, the particle rests at the bottom of the well $(|\mathbf{r}|=0)$. At later times, the stochastic force $\mathbf{F}(t)$ accelerates the particle that may eventually escape from the well. At each jump in the two-dimensional KP, the direction of the vector $\mathbf{m}$ changes what corresponds to the update of $\mathbf{F}(t)$. Otherwise the value of the force remains constant. The length of the vector $\mathbf{m}$ is $|\mathbf{m}| = 1$ and remains constant.

The quantities of interest are the energy distribution of escaping particles $P(E)$ and the survival-time distribution $N(t)$. The energy distribution of the Langevin particles escaping from the potential well is shown in Fig. 1 both in the case when the stochastic force is generated by the adjoint PLG (the short-dashed line) and in the case when the generalized KP is applied for this purpose. In the latter case, we consider the two-dimensional generalization of the KP (Sec. III B) where $\hat{P}_\Phi(\phi)$ is uniformly distributed within the angle interval of size π/n. The calculations were made with (i) $n=2$ (the long-dashed line) for which the long paths are close to $0,\pi$ and with (ii) $n=4$ (the solid line) for which the long paths are near the 0, $\pi/2$, π, and $3/2\pi$ directions, as in the PLG. Thus these two cases are essentially the same stochastic processes but because they are defined on the intervals of different sizes, they correspond to different degrees of isotropy.

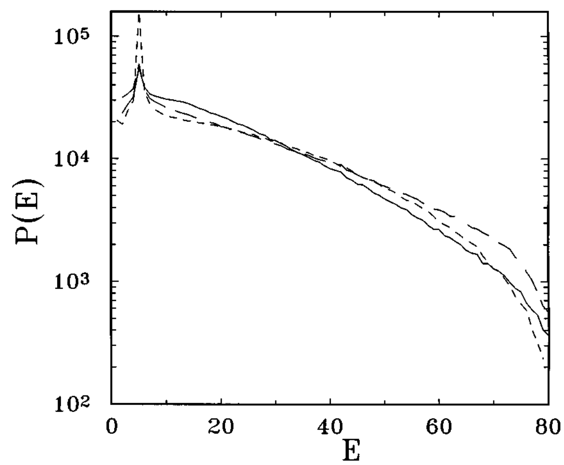

FIG. 1. Asymptotic energy distribution of the particles escaping from the circular attractive potential of depth $V_0 = -40$ and radius $r_B = 50$ for the stochastic force with the covariance proportional to $1/t$. The constant of intrinsic friction is $\gamma = 0.02$. The short-dashed line shows the results for the stochastic force generated by the deterministic, chaotic rule of the PLG. The long-dashed and solid lines exhibit results for the generalized KP in two dimensions with $n=2$ and $n=4$, respectively (see Sec. III B). In these cases, the probability distributions depending on a random angle ϕ are defined on a circle of radius $|\mathbf{m}| = 1$ and $\hat{P}_\Phi(\phi)$ is sampled uniformly within the angle interval of size π/n.

In spite of important differences in the definition of the stochastic force generator, the three curves exhibit similar features such as, for example, the appearance of the peak for ''prerandomized'' particles [3], which is a characteristic feature of the Langevin approach with long-time correlated noise and corresponds to the ''long free path,'' i.e., the long time interval (small ν) between the subsequent changes of $\mathbf{m}$ in the stochastic force generator. The Brownian particle escapes as soon as the long free path ends. The second very important qualitative similarity between Markovian (the generalized KP) and non-Markovian (the PLG) generators corresponds to the Gaussian shape of the energy tail for randomized particles, which is a benchmark of the algebraic $(\sim 1/t)$ velocity and force autocorrelation functions. The details of this Gaussian bump $P(E) \sim \exp(-E^2/2\sigma^2)$ as quantified by the width parameter σ are obviously different for those different generators and equal $\sigma = 30.25$ for the non-Markovian generator, and $\sigma = 47.6, 43.4$ for $n = 2,4$ for the Markovian generators, respectively. We have checked that the width of the Gaussian bump remains almost unchanged when increasing n above $n=4$.

As stated above, the existence of the peak for prerandomized particles is related to the existence of long free paths. Its sharpness is due to the assumed norm conservation in the generalized KP. The qualitative features would remain the same if we would allow for an independent variation of the norm $|\mathbf{m}|$ from a given distribution, say Gaussian.

The escape from the potential requires acceleration by the stochastic force to climb the well. Frequent changes of the applied force reduce the mean acceleration. This is the case for the exponential correlations of $\mathbf{F}(t)$. For the algebraic correlations, in the absence of the external potential $V(|\mathbf{r}|)$ or in the case of small amplitude $\mathbf{F}(t)$, all trajectories generated either by the PLG or by the KP, both long and short ones, provide a sufficient change of the momentum of the Brownian particle to allow its escape. All those particles that appear sufficiently close to the absorbing barrier at $|\mathbf{r}|=r_B$ may escape. The survival probability of the system is thus proportional to the number of particles $N(t)$ still inside the potential well at time t. The time variation of this number depends only on the phase-space density of particles, which is just proportional to the number of particles, $\dot{N}(t)\sim N(t)$; i.e., $N(t)\sim \exp(-at)$, where a is a constant. This result coincides with that for the exponential correlations of $\mathbf{F}(t)$.

The situation is quite different for a large amplitude random force. The constant random force and the conservative force from the potential act like the gravitation and elastic forces in the problem of the oscillatory motion of a string. The friction force slows down the Brownian particle, which finally stops at the point where the constant random force compensates *exactly* the conservative force [25]. Since the Brownian particle on such trajectories cannot escape, their weight in the ensemble increases in time. The Brownian particle is at a standstill or moves in a quasiperiodic orbit as long as the value of the stochastic force remains constant. In this case, the balance of forces ensures that the Brownian particle remains inside the absorbing barrier. When the long path finishes, the balance of forces changes and the Brownian particle escapes immediately. Therefore, the particle remains inside the potential well until time t if the long path in the adjoined generator is longer than t. Hence, for the Markovian generator (the generalized KP) for which the path length is independent of the length of the previous path, the decay probability for the Brownian particle is for large times proportional to the path length distribution $S(s)$. Therefore, the survival probability until time t is

$$N(t)\sim \int_t^\infty S(s)ds, \tag{40}$$

and the path length distribution is directly related to the survival time distribution inside the potential well.

In general, the surviving probability of the Brownian particle may depend on the stationary probability distribution of the KP, $\hat{P}(m)$, as well as the dimensionality of the system. For instance, independent KP's in d directions yield $N(t)\sim \int_t^\infty S_d(s)ds=\int_t^\infty [S_1(s)]^d ds$, where $S_1(s)$ stands for the one-dimensional free path distribution.

A multidimensional, norm-conserving generalization of the KP, as discussed in Sec. III B, yields $S(s)\sim s^{-2}$, independently of the stationary probability distribution $\hat{P}$, and independently of the degree of isotropy in the long path distribution. Hence, in this case, $N(t)\sim t^{-1}$, as for the non-Markovian generator based on the PLG with the open horizon. Figure 2 shows the number of surviving Brownian particles inside the spherically symmetric potential both in the case when the stochastic force is generated by the adjoint PLG (the short-dashed line) and in the case when it is generated by the generalized KP in two dimensions with $n=4$ (the solid line). The line $1/t$ is shown with the long-dashed line for comparison.

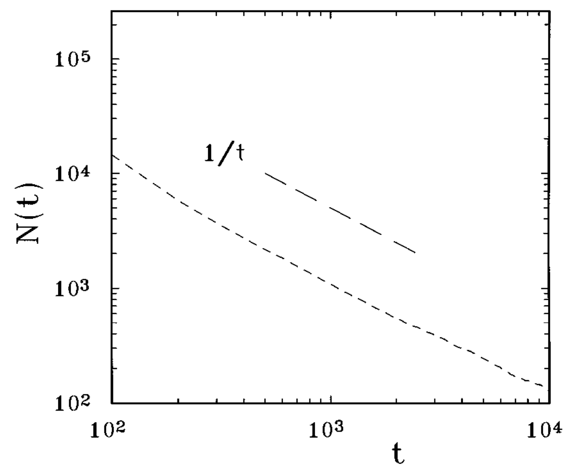

FIG. 2. Variation of the particle number inside the circular attractive potential of depth $V_0=-50$ and radius $r_B=50$. The absorbing barrier is at $|\mathbf{r}|=r_B$. The intrinsic friction constant is $\gamma=0.02$. The short-dashed line has been obtained for the deterministic chaotic force generated by the PLG with the open horizon. The solid line corresponds to the calculations performed with the two-dimensional generalized KP with $n=4$ (see Sec. III B). In both cases, the stochastic force has the covariance proportional to $1/t$. The line $1/t$ is shown with the long-dashed line for comparison. For more details see the description in the text.

V. CONCLUSIONS

The Langevin approach provides a useful framework in which complicated multidimensional Hamiltonian problems can be changed into low-dimensional dissipative problems, allowing one to separate slow, ‘‘collective’’ degrees of freedom from remaining fast variables. In this case, the collective motion is treated as a Brownian particle embedded in a heat bath rendering fluctuations around the most probable ‘‘macroscopic’’ collective path. Technically, the influence of this ‘‘environment’’ of fast variables on the slow variables is taken into account by introducing a stochastic force. Properties of such a force, in particular its autocorrelation function, must be properly adjusted to fit the ‘‘phenomenological’’ data (e.g., the CMD data) for a given choice of macroscopic (collective) degrees of freedom. Recent CMD studies for the peripheral collisions of ions showed that the local force acting on the ‘‘elementary’’ particle in the CMD is correlated algebraically, $\widetilde{C}(t)\sim t^{-1}$, and is associated with the presence of long free paths. This universal behavior can be described in the framework of the Langevin formalism including algebraically correlated stochastic force. In our earlier studies [3], we designed a generator of such force applying time series of point particle velocity in the two-dimensional PLG. This generating process is deterministic and chaotic. In the case of the open horizon, the velocity autocorrelation function of the particle in the PLG is proportional to $1/t$. Thus the generator has the desired correlation properties but its practical implementation may be cumbersome. Hence, in this work we have studied Markovian generators of the stochastic force that are based on the KP. For the same covariance $\tilde{\Gamma}(|t-t'|) \equiv \langle m(t) m'(t') \rangle \sim 1/|t-t'|$, different possible realizations characterized by different probability distributions $\hat{P}(m)$ are possible. For example, for a simple probability distribution (23) with $\alpha = 3$, one can find the Poisson form of the frequency distribution $R(\nu)$. This particular kind of KP process may be important for practical applications. We have proposed a special, multidimensional generalization of the KP, conserving the norm and having the covariance $\tilde{\Gamma}(t) \sim t^{-1}$ as the PLG process for the open horizon case. We have also found that the path length distribution equals $S(s) \sim s^{-2}$ in the generalized KP, independent of the dimensionality of the problem. Moreover, $S(s)$ appears to be insensitive to the particular choice of the stationary probability distribution of KP. The path length distribution for the non-Markovian PLG is also independent of the dimensionality but in that case $S(s) \sim s^{-3}$. This difference is, however, not essential for the properties of the Brownian particles. In particular, both the survival probability for the Brownian particle to remain inside the potential as well as the asymptotic energy distribution of particles are qualitatively the same and can be made almost identical by an appropriate change of the geometry of the PLG, i.e., by changing the radii R of the circular scatterers. These results remain unchanged if one allows variations of $|\mathbf{m}|$ (or $|\mathbf{u}|$ in the case of the PLG) of the stochastic process. Hence, we have constructed a reliable and simple generator of the long-time correlated stochastic process that in the particular case of the $1/t$ covariance is equivalent to the deterministic $1/t$-correlated process in the PLG. The advantage of the Markovian generator lies in its flexibility to describe physical situations with a different degree of isotropy in the distribution of the long free path. One should also stress that for both the Markovian and non-Markovian generators, the long free paths are responsible for the appearance of the algebraic covariance of the process.

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[24] In studies using the two-dimensional PLG as a generator of the ''deterministic'' random force acting on the Brownian particle, $\mathbf{F}(t) = \varepsilon \mathbf{u}$, where $\mathbf{u}$ ($|\mathbf{u}| = 1$) is the velocity of particle in the PLG and ε is the force amplitude.

[25] In the absence of friction, the Brownian particle evolves on a quasiperiodic orbit.