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Kinetic theory of fluidized granular matter

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In this paper, we present a statistical treatment of fluidized, elastic granular matter and a kinetic equation that describes the evolution of macroscopic properties of such matter. The present kinetic theory recognizes that the effects of excluded volume become dominant in the dynamic evolution of an assembly of granules and accordingly takes them into account in the formulation. On the basis of the equilibrium solution of the kinetic equation, a thermodynamics-like mathematical structure is constructed for the Boltzmann entropy of granular matter. The meaning of temperature in this mathematical structure is fixed by the shear rate. The equilibrium solution is shown to yield a density distribution comparable with the experimental data of Clement *et al.* [Europhys. Lett. **16**, 133 (1991)]. The shear viscosity of granular matter is shown to increase with the packing fraction. This behavior is in qualitative agreement with experimental result by Hanes *et al.* [J. Fluid Mech. **150**, 357 (1985)]. The viscosity also increases with the shear rate since the ''temperature'' increases with the shear rate in the case of granular matter. Consequently, the granular matter is shown to be dilatant, as is experimentally known. [S1063-651X(97)01504-3]

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I. INTRODUCTION

Granular matter has unusual properties and behavior [1,2] that defy easy comprehension from the conventional way of thinking and theories of continuum matter known to us through our studies of condensed matter up until now. The subject has been studied in engineering [3–6] and has been lately drawing attention in physics [7–9]. There have been some kinetic theories [10–14] proposed for such matter along the lines of the classical Chapman-Enskog theory in gases [15], but they appear to be only a first step toward theories of a deeper, perhaps more appropriate, nature. In a recent experimental article and computer simulations on granular matter, Clement and Rajchenbach [16] and Gallas, Herrmann, and Sokolowski [17], respectively, report that a density distribution of granules subjected to an acoustic perturbation at the bottom of its pile exhibits a Fermi-like distribution. Although this feature appears to be what is intuitively expected and intriguing, it requires some thought with regard to the underlying dynamical and kinetic theoretic principles. It is the aim of the present paper to present a viewpoint on the kinetic theory of granular matter which is sufficiently fluidized to treat it as if it is a fluid. To make the treatment as simple as possible without sacrificing the essential aspects of granular matter, we assume that the granules are elastic, since the assumption can still describe dissipation in the system. The inelasticity of collisions between granules is important, but not of primary importance in a statistical description of granular matter, since a similar effect can be achieved by means of momentum correlations, as will be shown later. A truly particulate, statistical treatment of the subject should take into account the internal states of granules and their evolution at the kinetic theory level, but such a treatment would be rather involved in formalism. Here, we set aside the question of internal states of granules by treating them as an elastic substance and pay attention to their translational motions only. Nevertheless, we believe that the theory presented in this work still has sufficient validity as a starting point for further investigations on granular matter. The present theory is quite different from the statistical treatment given recently by Bernu, Delyon, and Mazighi [18] on granular matter and those in Refs. [9–14].

In Sec. II, we examine granular matter for its features, distinct from the ordinary molecular fluids, with a specific aim of guiding us in developing a kinetic theory of fluidized, elastic granular matter. Based on what we have found on granular matter in Sec. II, a kinetic theory is then developed for the fluidized granular matter in Sec. III. The balance equations can be derived for mass, momentum, and energy—the conservation laws—from the kinetic equation if the mass, momentum, and internal energy densities are statistically defined appropriately. Such definitions are given in Sec. IV, but the explicit forms for the balance equations are referred to in the literature, since they are in the same forms as those for the ordinary continuum matter. However, in applying them in flow problems the existence of a basic, finite length scale of the granules must be carefully taken into account. The *H* theorem is also briefly discussed together with the equilibrium solution of the kinetic equation in the same section. The equilibrium solution is used to construct a thermodynamic analogy in Sec. V, since thermodynamics of granular matter is unavailable at present. In Sec. VI, the parameters appearing in the equilibrium solution are corresponded to observables such as shear rate and so on. We thereby construct an analogy of thermodynamics by means of the equilibrium solution of the kinetic equation. The calculated density distribution is compared with experiment in Sec. VII. The viscosity of granular matter is calculated in Sec. VIII. It shows the dilatancy of the granular matter and a qualitatively correct density dependence reminiscent of experiment. The concluding remarks are given in Sec. IX.

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II. GRANULAR MATTER IN CONTRAST TO ORDINARY MOLECULAR FLUIDS

To acquire a clear picture of how distinct the behavior of granular matter is from the ordinary molecular fluids, we compare characteristics of the two kinds of matter in question. First of all, for a given volume there is a difference of many orders of magnitude in number between the ordinary molecular fluids and granular matter, since there are about 10^{19} molecules in 1 cm^3 of a molecular fluid whereas there can be, for example, only 10^4 (of course, depending on the size of the grain) in the case of granular matter. This implies that considerable caution must be exercised when a statistical theory is applied to granular matter, since the characteristically small number density of granules can give rise to sizable fluctuations in the mean values calculated. Closely related to this feature is the fact that granular particles are macroscopic in size and mass. This also gives rise to the question under what condition can granular matter be treated as a continuum and calls into question the validity of continuum equations such as the mass, momentum, and energy balance equations. In general, since grains are massive, the gravitational force cannot be left out in general when granular matter is considered for its dynamic properties, whereas the gravitational force plays an insignificant role in the case of molecular fluids except, perhaps, for interfacial phenomena. Unlike molecular fluids, attractive forces between granular particles are negligible and play an insignificant role in granular dynamics. Dynamics in granular matter is dominated by the hard, repulsive forces and the excluded-volume effects come into play significantly in the case of granular matter, whereas in the case of molecular fluids it becomes important only when the density of the latter is close to the close-packed fluid value. In the case of molecular fluids, thermodynamics provides a well-defined continuum description of their behavior and such a continuum description can be furnished by a molecular theory with the help of statistical mechanics and, in particular, kinetic theory. In fact, in the case of molecular fluids, these two lines of theory are mutually complementary in the sense that, whereas the thermodynamic theory is given molecular foundations by the statistical theory of the system, the latter is elevated to a physical theory from a purely mathematical probability theory with the help of thermodynamics. If we wish to develop a statistical theory of granular matter, it is therefore important to have a phenomenological thermodynamic theory of the system firmly founded on the principles of thermodynamics and, if such a phenomenological theory is absent, on an analogy that is well thought through at least from the viewpoint of thermodynamic principles. Since it is not clear at present what the thermodynamics of processes in granular matter should be like, this question should be resolved before we attempt to formulate statistical and kinetic theories for granular matter. We will return to this question later.

The consideration made earlier on granular matter suggests that the notion of finite size and excluded volume associated with granules play a crucial role. This of course is well recognized in the phenomenological studies [1–6,8], statistical treatments [10–12,15,18], and computer simulations [9,17], but the subtle aspects of excluded volume do not seem to have been fully and clearly elucidated at the dynamical level when the granular assembly is in motion. We believe that the excluded-volume effect and, especially, the momentum correlations arising therefrom should be better accounted for than has been so far in the kinetic-theory treatments mentioned. For definiteness of our discussion we consider an assembly of N spherical, elastic granular particles of diameter σ and mass m in this work. The total volume of N such granules then is

$$V_0 = N v_0, \tag{1}$$

where v_0 is the volume of the granule

$$v_0 = \frac{4\pi}{3}\left(\frac{\sigma}{2}\right)^3. \tag{2}$$

Note that V_0, however, is not the actual volume of the system. If the actual volume of the assembly in a configuration is denoted by V, then the number density n at the given configuration of the granular assembly is given by

$$n = N/V. \tag{3}$$

The packing fraction of the assembly may be defined by

$$\phi = \tfrac{1}{6} n \pi \sigma^3. \tag{4}$$

Unlike molecular fluids, the number density of granular matter sensitively depends on its configuration, since it can assume an unusual metastable form (e.g., arches and voids) with a volume larger than a close-packed volume, which is the minimum volume the system can assume. The close-packed volume of the granular matter will be denoted by V_c, which yields the volume per particle $v_c = V_c/N$ that may be written as

$$v_c = \tfrac{1}{6}\pi R_c^3. \tag{5}$$

Here R_c, defined by Eq. (5), is the diameter of imaginary spheres making up the volume V_c. The ratio R_c/σ is larger than unity. The volume of configuration space of N granular particles in the close-packed configuration is therefore estimated to be

$$\Gamma_{\mathrm{conf}} = (\tfrac{1}{6}\pi R_c^3)^N. \tag{6}$$

The diameter σ of the granule or R_c sets the basic length scale in the description of the granular assembly of interest here. All the distances in the theory of granular matter considered here can be reckoned in this scale. The difference in the configuration space volumes V_c and V_0 is

$$\Delta\Gamma_{\mathrm{conf}} = [\tfrac{1}{6}\pi(R_c^3 - \sigma^3)]^N, \tag{7}$$

which ranges in the interval

$$[\tfrac{1}{2}\pi\sigma^2\delta R_c]^N \leqslant \Delta\Gamma_{\mathrm{conf}} \leqslant [\tfrac{1}{2}\pi R_c^2 \delta R_c]^N. \tag{8}$$

Here $\delta R_c = R_c - \sigma$. Therefore, the minimum volume uncertainty of the excluded volume of the system, i.e., *the granular matter*, is given by

$$\Gamma_R \equiv \Delta_{R_c}^N = [\tfrac{1}{2}\pi\sigma^2\delta R_c]^N. \tag{9}$$

Left unperturbed, granular matter in a stable or metastable configuration (state) does not flow and the velocities of the particles are equal to zero. Only an external force such as shearing or other means of forcing produces a motion of the assembly in part or as a whole. If the frequency of the perturbation is ω and the characteristic distance associated with it is amplitude A, then the momentum of a granule of mass m associated with the motion is

$$p_A = \epsilon m A \omega, \tag{10}$$

where ϵ is a dimensionless proportionality constant. If we denote the velocity of the particle by $\mathbf{c}$ and the motion is in the direction of the gravitational field, then at equilibrium

$$mgA = \tfrac{1}{2} mc^2,$$

where g is the gravitational acceleration. That is, the characteristic distance in this instance is given by

$$A = \frac{c^2}{2g}. \tag{11}$$

Bagnold [2] found that the mean velocity, or more precisely the speed of a granular assembly subjected to shearing is proportional to shear rate γ. Therefore, since the characteristic frequency ω is proportional to γ in the case of shearing, we may take

$$c = a'\gamma = a\omega. \tag{12}$$

Here a is a constant parameter with the dimension of distance which may be set $a = \sqrt{2\epsilon}A$. By combining Eqs. (10)–(12), we obtain

$$p_A = \Theta_g m A \omega, \tag{13}$$

where

$$\Theta_g = \frac{A\omega^2}{g}. \tag{14}$$

This parameter Θ_g characterizes the motion of granular particles subjected to the gravitational field and an external perturbation that displaces the particles from their positions in a static configuration. Therefore, the volume of the momentum space of N granular particles going through such a mean motion is given by

$$\Delta_p^N = (\Theta_g m A \omega)^{3N} \tag{15}$$

and the corresponding phase volume is then given by

$$\Gamma_c \equiv \Delta_c^N = (\Delta_p \Delta_{R_c})^N = [\tfrac{1}{2}\pi \sigma_A^3 \Theta_g^3 (mA\omega)^3]^N, \tag{16}$$

where

$$\sigma_A = \Omega_c^{1/3}\sigma,$$

$$\Omega_c = \frac{\delta R_c}{\sigma} = \frac{R_c}{\sigma} - 1. \tag{17}$$

Therefore, it is possible to interpret σ_A as the effective diameter of the granule swelled by the factor $\Omega_c^{1/3}$ associated with the perturbation applied to the granular system. This Γ_c is the minimum phase volume associated with the granular assembly, which under the gravitational field is perturbed by an external force of frequency ω and amplitude A in a close-packed configuration. This phase volume must be properly taken into account in the development of a kinetic theory of fluidized granular matter, since it must serve as distance and momentum scales for the phase volume of the system in the statistical description of the granular matter of interest. The Δ_c is essentially the phase volume which accommodates a granular particle of the mean momentum p_A. This basic phase volume per granule Δ_c is not negligible compared with the actual phase volume in the case of granular particles. This point is crucial in determining the behavior of an assembly of granular particles.

Let us denote the phase volume element of the granular assembly around the phase point, or simply the phase, $(\mathbf{r}^{(N)},\mathbf{p}^{(N)})$ by $d\Gamma = d\mathbf{r}^{(N)} d\mathbf{p}^{(N)}$, where $\mathbf{r}^{(N)}$ and $\mathbf{p}^{(N)}$ collectively denote the center-of-mass position and momentum vectors of granular particles in a fixed coordinate system. Then, the statistical description of such a system can be valid only if this phase volume of the system is much larger than the basic phase volume Γ_c presented earlier. This will be generally the case if the granular matter is sufficiently fluidized. Thus, we will assume that the phase volume of the system contains a large number of the basic phase volume Γ_c mentioned, and the former is reckoned in the scale of the latter, Γ_c. The number of basic phase volumes in unit phase volume is then equal to $1/\Gamma_c$. And such a number is an element that must be carefully taken into consideration when the collision term in the kinetic equation is calculated on the basis of granular collision dynamics.

III. KINETIC EQUATION FOR FLUIDIZED GRANULAR MATTER

We have now come to realize that a granular assembly should behave qualitatively differently from the behavior of an isolated pair of granular particles, which should strictly obey the laws of motion within the framework of Newtonian mechanics. It is because of the very fact that the particles have a sizable excluded volume, which causes strong correlations of momenta, that there are severe limitations on the collective motion of the particles in a congested assembly of granular particles. In the case of granular matter, because of the relatively large volume fraction of the excluded volume the aforementioned limitations become a dominant factor for collective motions of granular particles. We believe that the momentum correlations take precedence over the inelasticity of collisions in the hierarchy of dominant effects in granular matter. We now take this feature into account in formulating a kinetic theory of such matter. A cogent picture for the kind of difference in the behaviors of an isolated pair of granules and an assembly of them can be seen in the example used by Jaeger, Nagel, and Behringer in their recent article [19]. These authors compare, say, the motions of a bead with a sack of many beads that falls on a glass plate. A single bead falling on the glass plate repeatedly bounces off the plate before it comes to rest, whereas the sack falls dead on the plate and remains there. The basic difference is attributed to many inelastic collisions that the beads in the sack go through, but it is basically the excluded-volume effect of beads in the sack that frustrates the bouncings of the first beads hitting the plate which are predicted to occur by the Newtonian laws of motion. In the case of the beads in the sack—a congested assembly of beads—the individual motions of beads are highly correlated and constrained, unlike isolated beads. In this connection, it must be pointed out that the friction forces on contact are indispensable in the molecular-dynamics simulations of granules by Gallas, Herrmann, and Sokolowski [17], and that they are a kind of momentum correlation that springs into action when the particles come into contact. We thus see that a congested assembly of granules puts certain statistical constraints on the dynamical evolution that isolated individual particles would have followed according to the Newtonian mechanical laws, and that the normal Newtonian mechanical evolution has been thereby frustrated. That is, the statistical evolution of such an assembly, therefore, cannot be inferred to be classical in the sense of the Boltzmann kinetic theory. This is a crucial point we would like to stress in this work.

We are interested in the probability of finding a granular particle with momentum $\mathbf{p}$ at position $\mathbf{r}$ in phase volume element $d\mathbf{r}\, d\mathbf{p}$ at time t and the spatiotemporal evolution of the probability distribution function. That is, the object of interest is the single distribution function $f(\mathbf{v},\mathbf{r};t)$ of velocity $\mathbf{v}=\mathbf{p}/m$ at time t. If this distribution function is integrated over the entire phase volume, it is normalized to the total number of particle N:

$$\int d\mathbf{r}\, d\mathbf{v}\, \overline{f(\mathbf{v},\mathbf{r};t)}=N. \tag{18}$$

Therefore, when integrated over the momentum space, it gives

$$\int d\mathbf{v}\,\overline{f(\mathbf{v},\mathbf{r};t)}=n, \tag{19}$$

where $n=N/V$ is the number density. If we count the number of particles flowing in and out of the elementary phase volume $d\mathbf{r}\, d\mathbf{v}$ at $\mathbf{r}$ and $\mathbf{v}$ in time interval dt, it is given by

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}+\mathbf{F}\cdot\boldsymbol{\nabla}_v\right)\overline{f(\mathbf{v},\mathbf{r};t)}d\mathbf{r}\, d\mathbf{v}\, dt. \tag{20}$$

Here $\mathbf{F}$ is the external force per mass and $\boldsymbol{\nabla}_v=\partial/\partial\mathbf{v}$. This is the change in number density due to the kinematic flow of particles in the elementary phase volume mentioned. It is important to note that the distribution function is taken to be uniform in this elementary phase volume, and this phase volume must be clearly larger than Γ_c. This means that the distribution function is coarse grained in a scale comparable to or larger than that of Δ_c. The kinematic change in number density must be balanced by the change in number density due to collisions of particles within the phase volume. If we denote the collisional rate of change in number density by $R'[f]$, then the number density change in $d\mathbf{r}\, d\mathbf{v}$ in time duration dt is

$$R'[\overline{f}]d\mathbf{r}\, d\mathbf{v}\, dt. \tag{21}$$

On equating Eqs. (20) and (21), there follows the evolution equation for $f(\mathbf{v},\mathbf{r};\mathbf{t})$:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}+\mathbf{F}\cdot\boldsymbol{\nabla}_v\right)\overline{f(\mathbf{v},\mathbf{r};t)}=R'[\overline{f}]. \tag{22}$$

We now calculate the collisional contribution. It is useful for this purpose to recognize that the collision integral in the kinetic (e.g., Boltzmann) equation consists of two basic components. One is the collision dynamical part, which accounts for the collision event that is imagined to occur between isolated particles (e.g., a pair of particles). This collision event is governed by Newtonian laws of motion for a few particles and, in the present theory, two particles. The other is the statistical part that gives the statistical weight for the occurrence of particles having the values of the dynamical variables predicted by the Newtonian laws. It is important to keep a clear distinction between these two factors in making up the collision integral in the kinetic equation. The statistical weighting factor is not altering the collision dynamics itself governed by the Newtonian mechanical laws, but the statistical probability of the mechanically predicted collision event deviates from the classical, Boltzmann kinetic-theory form, which is usually taken in the kinetic-theory treatments [20].

First, let us define by vector $\mathbf{k}$ the unit vector along the apse line of the two hard spheres in contact that is parallel to the vector connecting the centers of mass of the two particles. The relative velocity vector of the particles is denoted by $\mathbf{g}_{12}=\mathbf{v}_1-\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of the two particles 1 and 2 in collision. Then the volume swept by the two particles in collision in time dt is

$$\sigma^2(\mathbf{g}_{12}\cdot\mathbf{k})dt.$$

We will henceforth drop the subscript 1 from the quantities for particle 1. The initial velocities of the colliding pair of particles are denoted by $\mathbf{v}$ and $\mathbf{v}_2$, where $\mathbf{v}\equiv\mathbf{v}_1$. We also take into account the aforementioned coarse graining of distribution functions so that the distribution functions are assumed to be uniform in the collision volume. The population of the colliding pair of particles is $f(\mathbf{v},\mathbf{r};t)f(\mathbf{v}_2,\mathbf{r};t)\chi(\mathbf{r},\mathbf{r}-\mathbf{k}\sigma)$, where $\chi(\mathbf{r},\mathbf{r}-\mathbf{k}\sigma)$ is the spatial correlation function for the pair of granular particles in collision. In the case of a granular assembly that has a large excluded volume relative to the actual volume of the system and thus a relatively large, elementary-phase volume occupied by a particle, the population of the particles in the final state of collision *that is statistically realizable* is

$$C_g^*\equiv[1-\Delta_c\overline{f^*}(\mathbf{v},\mathbf{r};t)][1-\Delta_c\overline{f^*}(\mathbf{v}_2,\mathbf{r};t)]. \tag{23}$$

We elaborate on the presence of the C_g^* factor in the collisional rate of change. In the case of the collision of an isolated pair of granular particles, the collision process is deterministically described by the Newtonian laws of motion and the particles end up in the final state as predicted by the classical mechanical laws. However, in a congested assembly of granular particles that are constantly in collision, some collisional events are not statistically realizable, although they are dynamically dictated by the Newtonian laws of motion, if the basic phase volumes into which the particles are destined to go are already occupied by other particles in the assembly. At this point, it is useful to recall the comparison of a bead and a sack of beads made by Jaeger, Nagel, and Behringer [19], which was mentioned earlier. Since $\Delta_c f^*(\mathbf{v},\mathbf{r};t)$ gives, on the average, the fraction of particles to be found in Δ_c, which are already at the state predicted for the collision of an isolated pair by the Newtonian laws and thus precludes other particles coming into Δ_c, the factor $[1-\Delta_c f^*(\mathbf{v},\mathbf{r};t)]$ gives the fraction of particles that are statistically allowed in the said state on completion of the said event of collision between the pair of particles 1 and 2. Since the other particle must be found at $\mathbf{r}-\mathbf{k}\sigma$ if one is at $\mathbf{r}$, the other factor for particle 2 is similarly $[1-\Delta_c f^*(\mathbf{v}_2,\mathbf{r}-\mathbf{k}\sigma;t)]$, but this factor is taken in the form $[1-\Delta_c f^*(\mathbf{v}_2,\mathbf{r};t)]$ for the consistency with the coarse graining of the distribution function over the collision volume. This factor, of course, has a physical meaning similar to the factor $[1-\Delta_c f^*(\mathbf{v},\mathbf{r};t)]$. Therefore, the factor C_g^* represents, on the average, the physical reality that not all collisional events in a congested assembly of granular particles can be realized, at the statistical level of description, in states as predicted by the Newtonian laws of motion for an isolated pair of particles and that there is an additional statistical constraint manifesting itself because of the interactions of mutual exclusion among the particles. This exclusion effect even appears in the momentum space in the form of a momentum correlation function, which C_g^* in effect represents, since the statistical evolution of the system is considered in the phase space. This factor C_g^*, a momentum correlation function, is absent in the ordinary classical fluids since Δ_c is $O(\hbar^3)$ for ordinary fluids and hence, for example, $\Delta_c f^*(\mathbf{v},\mathbf{r};t)\sim n\Delta_c = O(n\hbar^3)$, which is small at normal states. The factor C_g^* may also be compared to the friction force on contact that is indispensably used in molecular-dynamics simulations [17] of granular matter. As will be seen later, the end effects of the momentum correlation factors such as C_g^* and the friction forces used in molecular-dynamics simulations are the same in that they tend to keep the particles in the gravitational field piled up. We remark that this momentum-correlation effect is not taken into account in the existing kinetic theories of granular fluids cited earlier.

In the configuration space conjugate to the momentum space, the exclusion effect is relatively easy to comprehend if a spatial correlation function is made use of, and for this purpose we insert $\chi(\mathbf{r},\mathbf{r}-\mathbf{k}\sigma)$ in the collisional rate. This is the probability of finding a pair of particles separated by $\mathbf{k}\sigma$—namely, a pair correlation function—at the instant of collision and the excluded-volume effect is already built into it. Since the spatial correlation function must be a function of the relative distance between the particles, we find

$$\chi(\mathbf{r},\mathbf{r}\pm\mathbf{k}\sigma)=\chi(\sigma),$$

which is the pair correlation function evaluated at the contact point of the hard granular particles regardless of the momenta of the particles involved. This $\chi(\sigma)$ is also in conformation with the coarse graining of distribution functions over the collision volume. This distribution function can be, in principle, determined within the framework of kinetic theory if its own kinetic equation is formulated and solved, but in this work we will use the equilibrium-pair correlation function determined *ad hoc* from elsewhere, e.g., the equilibrium theory.

Taking the aforementioned factors into account, we obtain the forward collisional rate of change in the number of particles in the volume element $d\mathbf{k}\,d\mathbf{v}\,d\mathbf{r}\,d\mathbf{v}_2$ and the collision volume

$$(R'[\Delta_c\overline{f}]d\mathbf{r}\;d\mathbf{v}\;dt)_{\text{forward}}$$

$$=\sigma^2 m^3\int d\mathbf{v}_2\int d\mathbf{k}(\mathbf{g}_{12}\cdot\mathbf{k})\chi(\mathbf{r},\mathbf{r}-\mathbf{k}\sigma)\overline{f(\mathbf{v},\mathbf{r};t)}\,\overline{f(\mathbf{v}_2,\mathbf{r};t)}$$
$$\times[1-\Delta_c\overline{f^*}(\mathbf{v},\mathbf{r};t)][1-\Delta_c\overline{f^*}(\mathbf{v}_2,\mathbf{r};t)]d\mathbf{v}\;d\mathbf{r}\;dt,\qquad(24)$$

where the asterisk denotes the postcollision value. Note that $\chi(\mathbf{r},\mathbf{r}-\mathbf{k}\sigma)=\chi(\sigma)$ in Eq. (24).

It is possible to calculate the collisional rate for the reverse collision in a manner similar to Eq. (24), and we obtain

$$(R'[\Delta_c\overline{f}]d\mathbf{r}\;d\mathbf{v}\;dt)_{\text{reverse}}$$

$$=\sigma^2 m^3\int d\mathbf{v}_2\int d\mathbf{k}(\mathbf{g}_{12}\cdot\mathbf{k})\chi(\mathbf{r},\mathbf{r}+\mathbf{k}\sigma)\overline{f^*}(\mathbf{v},\mathbf{r};t)\overline{f^*}(\mathbf{v}_2,\mathbf{r};t)$$
$$\times[1-\Delta_c\overline{f(\mathbf{v},\mathbf{r};t)}][1-\Delta_c\overline{f(\mathbf{v}_2,\mathbf{r};t)}]d\mathbf{v}\;d\mathbf{r}\;dt,\qquad(25)$$

for which we have used the Liouville theorem for the phase volume. Recall $\chi(\mathbf{r},\,\mathbf{r}+\mathbf{k}\sigma)=\chi(\sigma)$ in Eq. (25). The restitutive coefficient, which appears in the existing kinetic theories [10–14] for granular matter, is set equal to unity in these collision terms, since the granules are assumed to be elastic. We reiterate that despite the fact that the collisions are elastic, the assembly is still dissipative because of random collisions between the particles. The net coarse-grained collisional rate of change in number is given by combining (24) and (25):

$$R'[\Delta_c\overline{f}]=\sigma^2 m^3\chi(\sigma)\int d\mathbf{v}_2\int d\mathbf{k}(\mathbf{g}_{12}\cdot\mathbf{k})$$
$$\times\{\overline{f^*}(\mathbf{v},\mathbf{r};t;t)\overline{f^*}(\mathbf{v}_2,\mathbf{r})[1-\Delta_c\overline{f(\mathbf{v},\mathbf{r};t)}]$$
$$\times[1-\Delta_c\overline{f(\mathbf{v}_2,\mathbf{r};t)}]-\overline{f(\mathbf{v},\mathbf{r};t)}\,\overline{f(\mathbf{v}_2,\mathbf{r};t)}$$
$$\times[1-\Delta_c\overline{f^*}(\mathbf{v},\mathbf{r};t)][1-\Delta_c\overline{f^*}(\mathbf{v}_2,\mathbf{r};t)]\}.\qquad(26)$$

This collision term will be used in this work. We remark that a fine-grained form of the collision integral does not give rise to a satisfactory kinetic equation and macroscopic continuum equations.

With the definition of a new distribution function

$$f=\Delta_c\overline{f},\qquad(27)$$

the kinetic equation for $f(\mathbf{r},\,\mathbf{v},\,t)$ may be written in the form

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}+\mathbf{F}\cdot\boldsymbol{\nabla}_v\right)f(\mathbf{v},\mathbf{r};t)=R[f],\qquad(28)$$

where $R[f]=\Delta_c^{-1}R'[\Delta_c\overline{f}]$. The kinetic equation (28) with the collision integral defined by Eq. (26) is the basis of the kinetic theory of fluidized, elastic granular matter presented below. To be more precise, this kinetic equation must be accompanied by a kinetic equation for configuration correlation function $\chi(\sigma)$ appearing in the collision integral, but the theory becomes much more complex if such a kinetic equation is added. Since we are interested in the essential feature that captures the kinetic evolution of nonequilibrium processes in fluidized, granular matter, we will confine the discussion to the level of precision by which we not only neglect inelastic collisions and set the restitutive coefficient equal to unity, but also take the equilibrium pair correlation function for $\chi(\sigma)$. For a more complete theory, we defer to future studies that more properly treat $\chi(\sigma)$ and take into account the internal degrees of freedom for the granules (e.g., inelasticity of granules).

IV. CONSERVATION LAWS, THE *H* THEOREM, AND EQUILIBRIUM SOLUTION OF THE KINETIC EQUATION

A. Conservation laws

Mean values for macroscopic observables are statistically calculated by the formula

$$\overline{A}(\mathbf{r},t)=\Delta_c^{-1}m^3\int d\mathbf{v}\, A(\mathbf{r},\mathbf{v})f(\mathbf{v},\mathbf{r};t)\equiv\langle A(\mathbf{r},\mathbf{v})f(\mathbf{v},\mathbf{r};t)\rangle. \tag{29}$$

With this definition for macroscopic variables, the kinetic equation (28) yields the balance equations for mass, momentum, and energy as well as other evolution (constitutive) equations. They can be readily derived by using the well-known procedure in kinetic theory. Since the procedure is well documented [20] and no particular consideration is required for the present problem, we will simply give the statistical definitions for the quantities involved and will refer to the explicit forms for the balance equations in Ref. [20].

The mass density ρ, momentum density $\rho\mathbf{u}$, and internal energy density $\rho\mathcal{E}$ are defined, respectively, by the statistical formulas

$$\rho=\langle mf\rangle, \tag{30}$$

$$\rho\mathbf{u}=\langle m\mathbf{v}f\rangle, \tag{31}$$

$$E=\rho\mathcal{E}=\langle \tfrac{1}{2}mC^2f\rangle, \tag{32}$$

where $\mathbf{C}=\mathbf{v}-\mathbf{u}$ denotes the peculiar velocity. These definitions and the kinetic equation give rise to the balance equations for the conserved variables, if the stress tensor $\mathbf{P}$ and the heat flux $\mathbf{Q}$ are defined, respectively, by the statistical formulas

$$\mathbf{P}=\langle m\mathbf{C}\mathbf{C}f\rangle, \tag{33}$$

$$\mathbf{Q}=\langle \tfrac{1}{2}mC^2\mathbf{C}f\rangle. \tag{34}$$

The stress tensor $\mathbf{P}$ and the heat flux $\mathbf{Q}$, in turn, obey their own evolution equations which can be derived from the statistical formulas with the help of the kinetic equation. Such evolution equations are in fact the constitutive equations for the stress and the heat flux. The derivations of the balance equations indicate that the conventional continuum mechanical equations hold for fluidized granular matter, if the kinetic equation (28) is assumed. They are in the same forms [20] as those for ordinary molecular fluids. It implies that granular matter, when sufficiently fluidized, acts as if it is a continuum matter obeying the conventional mass, momentum, and energy conservation laws.

B. *H* theorem and equilibrium solution of the kinetic equation

The stability of the equilibrium solution of the kinetic equation (28) can be examined in terms of a Lyapunov function, which in the case of kinetic equations appears as the Boltzmann H function, that is, the Boltzmann entropy S. It is defined by the statistical formula

$$S\equiv\rho\mathcal{S}=-k_B\langle f(\mathbf{r},\mathbf{v};t)\ln f+[1-f(\mathbf{r},\mathbf{v};t)]\ln(1-f)\rangle. \tag{35}$$

The derivative of this Boltzmann entropy satisfies the inequality

$$\frac{dS}{dt}\geqslant 0, \tag{36}$$

where the equality holds only at equilibrium. This inequality can be proved by the well-known procedure of the Boltzmann kinetic theory. We remark that the statistical form for S in (35) is strictly dictated by the collision term (26) of the kinetic equation (28). It is not arbitrarily chosen at all. If the form for S is not taken exactly as in (35), the H theorem, namely, the inequality (36), cannot be proved, given the collision term (26) for the kinetic equation.

If the kinetic equation is used to calculate the local form of the inequality (36), the Boltzmann entropy balance equation follows:

$$\rho d_t\mathcal{S}=-\boldsymbol{\nabla}\cdot\mathbf{J}_s(\mathbf{r},t)+\sigma_{\text{ent}}(\mathbf{r},t), \tag{37}$$

where the statistical formulas for the Boltzmann entropy flux $\mathbf{J}_s$ and the Boltzmann entropy production σ_{ent} are respectively given by

$$\mathbf{J}_s=-k_B\langle\mathbf{C}\{f(\mathbf{r},\mathbf{v};t)\ln f+[1-f(\mathbf{r},\mathbf{v};t)]\ln(1-f)\}\rangle, \tag{38}$$

$$\sigma_{\text{ent}}=k_B\langle\ln(f^{-1}-1)R[f]\rangle\geqslant 0. \tag{39}$$

The positivity of σ_{ent} is a local representation of the H theorem and σ_{ent} vanishes at equilibrium.

Thus, the equilibrium solution f_e of the kinetic equation is given by

$$\langle\ln(f_e^{-1}-1)R[f_e]\rangle=0. \tag{40}$$

On applying the procedure [20] of the kinetic theory of ordinary fluids, the solution of this equation can be easily obtained in a well-known form. That is, if the Hamiltonian of a particle subjected to the gravitational field is denoted by

$$H=\tfrac{1}{2}mC^2+mgz, \tag{41}$$

then the equilibrium solution is given by

$$f_e = [\exp\beta(H-\mu_e)+1]^{-1}, \tag{42}$$

where β is a constant and the parameter μ_e is closely related to the normalization factor of f_e. These parameters generally depend on **r**. The gravitational field is taken parallel to the z axis. The equilibrium solution has exactly the same form as the distribution function for Fermi-Dirac particles. The reason for this is the excluded volume effect that manifests itself even in the momentum space by statistically putting constraints on the population of particles undergoing collisional transitions and thus resulting in strong momentum correlations for the collision pairs in the congested granular particle assembly. The distribution function f_e is normalized to density:

$$n = \langle [\exp\beta(H-\mu_e)+1]^{-1}\rangle. \tag{43}$$

In the case of granular matter, the meaning of parameter β is more subtle than the case of Fermi-Dirac particles, since it is not clear whether there is a thermodynamic structure admissible from the viewpoint of the thermodynamic laws. In fact, the meaning of temperature must be altered in the case of granular matter. The reason for this statement is discussed in the following.

V. THERMODYNAMIC ANALOGY FOR GRANULAR MATTER

In the case of molecular fluids, the thermodynamic theory of processes [21] is well founded on the notion of cycles, the Carnot cycle being the prototype. As is well known, it was the Carnot cycle and its analysis [22,23] that gave rise to the mathematical structure of thermodynamics. The thermodynamics of fluidized granular matter, however, presents an unfamiliar and vexing situation, since it is not obvious how to construct a cycle with it as a working substance. In the case of granular matter, the notions of heat and temperature are not as well clarified or established as for molecular fluids, and it is not clear if one can simply apply to granular matter the thermodynamic theory valid for molecular fluids. To see this point more clearly, let us examine whether or not it is possible to construct a cycle similar to the Carnot cycle on the conventional notions of heat and temperature, which are used for the Carnot cycle with a molecular fluid as the working substance. In the case of the latter fluid, the temperature of the fluid rises and the volume expands on absorption of heat by the system, whereas injecting the same amount of heat into granular matter as supplied to the mass of a molecular fluid will not result in a noticeable change in it except for, perhaps, a slight expansion of the volume due to the thermal expansion of the granules, but the change is not due to random translational motions of the granules themselves, which are induced by the supplied heat and the raised temperature. Therefore, we see that the granular matter is translationally cold and random translational motions of the granules do not get excited by heat. This indicates that it is not possible to construct a thermodynamic theory of granular matter on the basis of the conventional notions of heat and temperature, even if the matter is fluidized by a mechanical external force, in the same manner as for molecular fluids. However, we might be able to resort to a thermodynamic analogy to acquire from the equilibrium solution of the kinetic equation a mathematical structure of thermodynamics for fluidized granular matter, that resembles that of thermodynamics of molecular fluids.

An example of such a thermodynamic analogy in physics and in the description of natural phenomena is the case of radiation [24]. It is helpful to recall that the thermodynamics of radiation was constructed on the basis of an analogy to the thermodynamics of matter under the assumption that there exists an equilibrium Gibbs relation for the radiation entropy and with the help of the black body in equilibrium with radiation and the Stefan-Boltzmann law [25] $E_{\text{rad}} = a_{\text{sb}}T^4$, where T is the temperature of the black body. We use this procedure [24] as a guide for constructing a thermodynamicslike phenomenological theory of fluidized granular matter under consideration and acquire the meaning of temperature from the experimental observation of the energy-shear rate relation [2].

Anticipating the result presented later, we may set in the canonical form (42)

$$\beta = 1/k_B T. \tag{44}$$

Here T is the translational temperature of granular matter that is mechanically fluidized, but not the internal temperature of the granules at rest. We remark that this ''temperature'' is not the conventional temperature we talked about in the case of ordinary molecular fluids.

Bagnold [2] observed on the basis of experimental studies that the mean velocity of granular matter is proportional to the shear rate and, consequently, the mean internal energy is related to the shear rate squared. In fact, we have already availed ourselves of this result in Sec. II, where the notion of basic phase volume is introduced. Thus, we may write the internal energy of fluidized granular matter in the form

$$E = E_0[1 + cg(\gamma)\gamma^2], \tag{45}$$

where c is a constant with the dimension of time squared, $g(\gamma)$ is a dimensionless function of shear rate γ which tends to unity as γ vanishes, and E_0 is the energy of the system in a stationary configuration at $\gamma=0$ (e.g., a stationary metastable configuration). Within the range of shear rate studied, Bagnold found $g(\gamma)$ is independent of the shear rate and thus can be set equal to unity. Following Bagnold, we will take $g(\gamma)=1$ in this work. However, this assumption concerning $g(\gamma)$ is not mandatory in the subsequent analysis and may be easily removed without altering the thermodynamic analogy presented for granular matter. We now state the following assumption about the existence of the Clausius entropy of fluidized granular matter that obeys the equilibrium Gibbs relation

$$dS_e = T^{-1}(dE + p\,dV). \tag{46}$$

Here p is the pressure and V is the conjugate volume. We emphasize that this relation will be established for granular matter not on the basis of the thermodynamic second law, which was originally stated only for cycles with a molecular fluid as the working substance, but by means of statistical mechanics under the thermodynamic-statistical mechanical correspondence used in the ensemble theory; this time the procedure is only reversed from that for ordinary molecular fluids. We elaborate on this in the following. We remark that we are not considering a system at rest, which was considered by Edwards and Oakeshott [7] for the thermodynamic formalism for a granular assembly.

It is now asserted that as in the Gibbs ensemble method [26] for ordinary molecular fluids, there is the following correspondence [27] between thermodynamic S_e in Eq. (46) and the equilibrium Boltzmann entropy computed from Eq. (35) with the equilibrium canonical form (42)

$$S_e|_{\mathrm{st}} \Rightarrow S_e|_{\mathrm{th}}, \tag{47}$$

where the subscripts st and th mean the statistically computed and the thermodynamic entropy, respectively, and, similarly, there hold the correspondences for E and p

$$E|_{\mathrm{st}} \Rightarrow E|_{\mathrm{th}}, \quad p|_{\mathrm{st}} \Rightarrow p|_{\mathrm{th}}. \tag{48}$$

We use Eqs. (45)–(48) to fix the meaning of parameter β and thus the meaning of T therewith. First, the meaning of β in the distribution function is fixed by comparing Eq. (45) with the statistically computed mean energy. Then the correspondence (48) is used to endow T in Eq. (46) with the ''thermodynamic'' meaning. It must be noted that this process is just the reverse of the one used for ordinary molecular fluids where the meaning of β is fixed in terms of T that is founded on the thermodynamic second law which provides the thermodynamic temperature scale. This reversal of the procedure is forced upon us because Eq. (46) is not given by the second law on the basis of analysis of a reversible cycle conventionally made in thermodynamics of molecular fluids. This time, it is the statistical theory, constructed by an analogy to the molecular statistical theory which is known to yield a structure of thermodynamics, and the correspondences, Eqs. (47) and (48), that is enabling us to construct a thermodynamics-like mathematical structure for fluidized, granular matter, which is analogous to the thermodynamics of molecular fluids.

VI. DETERMINATION OF PARAMETERS β AND μ_e

It is necessary, for the stated aim, to calculate the normalization condition for f_e and the mean internal energy for the granular matter subjected to the gravitational force when the granules have acquired kinetic energies by an external force. Since the integrals involved do not permit exact evaluations, they will be calculated to an approximation, first, in the limit where the parameter β is large and $\mu_e - mgz > 0$. In this limit the meaning of β is elucidated. With the meaning of β thus established, we then consider the case where β is moderate in value. It is necessary to evaluate the following integrals:

$$n(z) = \frac{2\pi}{\Delta_c}\left(\frac{2m}{\beta}\right)^{3/2}\int_0^\infty dt\,\frac{t^{1/2}}{\exp(t-\alpha)+1}, \tag{49}$$

$$E = \frac{2\pi}{\Delta_c}\left(\frac{2m}{\beta}\right)^{3/2}\beta^{-1}\int_0^\infty dt\,\frac{t^{3/2}}{\exp(t-\alpha)+1}, \tag{50}$$

where with $b = \beta mg$

$$\alpha = \beta(\mu_e - mgz) \equiv \overline{\mu_e} - bz. \tag{51}$$

These integrals can be treated by means of well-known procedures [28]. We summarize the results below.

A. The case of α positive and very large

The normalization integral for this case is given by the formula

$$n(z) = \frac{4\pi}{3\Delta_c}\left(\frac{2m}{\beta}\right)^{3/2}\alpha^{3/2}\left[1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4} + \cdots\right]. \tag{52}$$

By using this result, we find to the lowest order

$$\alpha = \overline{\mu_e} - bz \simeq \left(\frac{3\Delta_c n(z)}{4\pi}\right)^{2/3}\frac{\beta}{2m}. \tag{53}$$

The mean internal energy is given by the formula

$$E = \frac{4\pi}{5\Delta_c}\left(\frac{2m}{\beta}\right)^{3/2}\beta^{-1}\alpha^{5/2}\left[1 + \frac{5\pi^2}{8\alpha^2} - \frac{7\pi^4}{384\alpha^4} + \cdots\right]$$
$$\equiv \frac{4\pi}{5\Delta_c}\left(\frac{2m}{\beta}\right)^{3/2}\beta^{-1}\alpha^{5/2}[1 + \alpha^{-2}K(\alpha)]. \tag{54}$$

On truncation of the series, Eq. (54) may be approximated by

$$E \simeq \frac{4\pi}{5\Delta_c}\left(\frac{2m}{\beta}\right)^{3/2}\beta^{-1}\alpha^{5/2}\left(1 + \frac{5\pi^2}{8\alpha^2}\right). \tag{55}$$

The front factor in Eq. (54) is independent of β if Eq. (53) is used. With the definitions

$$E_0 = \frac{3n(z)^{5/3}}{10m}\left(\frac{3\Delta_c}{4\pi}\right)^{2/3} \tag{56}$$

and

$$\theta = \sqrt{2/5}(\pi k_B m)^{-1}\left(\frac{3\Delta_c n(z)}{4\pi}\right)^{2/3}, \tag{57}$$

we obtain the internal energy in the form

$$E = E_0(z)\left[1 + \left(\frac{T}{\theta}\right)^2\right], \tag{58}$$

for which Eq. (44) is used. By using Eq. (45) and the correspondence (48), we now identify T:

$$T = c\theta\gamma. \tag{59}$$

This identification of T with γ (more precisely, the absolute value of the shear rate) can be made more precise if a more precise phenomenological expression for E exists than Eq. (45) and if Eq. (54) is used without truncation. However, it would not change the basic conclusion that T is directly proportional to γ. Therefore, we will be content with the result, Eq. (59), obtained here. We remark that the constant c in (59) cannot be calculated by means of statistical mechanics, but is a parameter like the Boltzmann constant that must be chosen to correlate the mechanics of granular matter with its thermodynamics.

Since f_e, on substitution into the statistical formula for the Boltzmann entropy, yields the differential form for S_e

$$dS_e = k_B\beta(dE + p\,dV) \tag{60}$$

if p is defined statistically by the formula

$$p = 2\pi k_B T \Delta_c^{-3/2}\int_0^\infty dt\, t^{1/2}\ln(1+e^{\alpha-t}). \tag{61}$$

On making use of the correspondence (47), we find

$$T|_{\rm th} \Leftrightarrow (k_B\beta)^{-1}|_{\rm st}, \tag{62}$$

which by virtue of Eq. (44) implies that the thermodynamic temperature must be identified with T as in Eq. (59), namely,

$$T|_{\rm th} = c\,\theta\gamma. \tag{63}$$

In this manner the statistical theory gives rise to an interpretation of thermodynamic temperature and thus to a thermodynamic analogy, as characterized by Eq. (60) for the granular matter under consideration. We note the relation

$$p = \tfrac{2}{3}E(z). \tag{64}$$

This relationship can be used to compute the density dependence of p. If Eq. (54) is used in this expression for p, the equation of state is given by

$$p = \frac{8\pi}{15\Delta_c}\left(\frac{2m}{\beta}\right)^{3/2}\beta^{-1}\alpha^{5/2}[1+\alpha^{-2}K(\alpha)]$$
$$\approx \tfrac{2}{3}E_0(z)\left[1+\left(\frac{T}{\theta}\right)^2\right]. \tag{65}$$

This is the equation of state in the limit of $\alpha \to \infty$.

B. The case of α moderate in value

The density is given by the expansion

$$n(z) = n_0\sum_{l=0}^{\infty}\frac{a_l}{(1+e^{-\alpha})^{l+1}}, \tag{66}$$

where

$$n_0 = \frac{2}{\pi\sigma_A^3\Theta_g^3(mA\omega)^3}\left(\frac{2\pi m}{\beta}\right)^{3/2}, \tag{67}$$

$$a_l = \sum_{k=0}^{l}\binom{l}{k}\frac{(-1)^k}{(1+k)^{3/2}} > 0. \tag{68}$$

It is possible to solve Eq. (66) for α to calculate the normalization factor. The result is

$$\alpha = \ln\left[n(z)\Delta_c\left(\frac{\beta}{2\pi m}\right)^{3/2}\right] + \sum_{k\geq 1}c_k\xi^k, \tag{69}$$

where

$$\xi = n(z)\Delta_c(2\pi m k_B T)^{-3/2}, \tag{70}$$

$$c_k = \frac{1}{k!}\left\{\frac{d^k}{d\xi^k}\ln\sum_{j=1}^{\infty}\frac{\xi^{j-1}}{j!}\left[\frac{d^{j-1}}{d\lambda^{j-1}}\varphi^j(\lambda)\right]_{\lambda=0}\right\}_{\xi=0}, \tag{71}$$

with $\varphi(\lambda)$ defined by

$$\varphi^{-1} = \frac{d}{d\lambda}\sum_{l=1}^{\infty}\frac{(-1)^{l-1}\lambda^l}{l^{5/2}}.$$

The leading examples for c_k are

$$c_1 = 2^{-3/2}, \quad c_2 = \frac{3}{2}\left(\frac{1}{8} - \frac{2}{3^{5/2}}\right), \tag{72}$$

etc. If $a_l(1+e^{-\alpha})^{-l} < 1$, Eq. (66) may be approximated by

$$n(z) \simeq \frac{n_0}{1+e^{-\alpha}}. \tag{73}$$

By applying for the same procedure as for Eq. (69), we obtain the internal energy in the form

$$E = \tfrac{3}{2}n(z)\beta^{-1}\left(1+\sum_{k\geq 1}\frac{k}{k+1}c_k\xi^k\right), \tag{74}$$

which implies that the equation of state is given by the formula

$$p = n(z)\beta^{-1}\left(1+\sum_{k\geq 1}\frac{k}{k+1}c_k\xi^k\right). \tag{75}$$

These results suggest that the granular matter is not ideal and the virial coefficients are given by

$$B_k = \frac{k}{k+1}c_k\Lambda^{3k}, \tag{76}$$

where

$$\Lambda = \Delta_c^{1/3}\left(\frac{\beta}{2\pi m}\right)^{1/2}. \tag{77}$$

These coefficients B_k stem from the momentum correlations induced by the excluded volume inherent in the granular particles in a congested fluidized assembly. Perhaps here we may make an *ad hoc* correction to the equation of state by adding the contribution from the spatial correlation of hard spheres:

$$p = n(z)\beta^{-1}\left(1+\sum_{k\geq 1}B_k n^k + \frac{2\pi\sigma^3 n}{3}\chi(\sigma)\right). \tag{78}$$

Note that the last term in this equation is the hard-sphere contribution to the equation of state that arises from the spatial correlation of particles and the trace of the virial tensor.

It is possible to resume the series in Eq. (75) into a form similar to the Carnahan-Starling form [29] by applying the Padé approximant method as used in Ref. [30] and obtain the formula for the equation of state

$$p = n(z)\beta^{-1}\left\{1 + \frac{\xi[\frac{1}{2}\,c_1 + \frac{1}{16}\,(\frac{32}{3}\,c_2 - 3c_1)\xi]}{(1-\frac{1}{8}\,\xi)^3}\right\}$$

$$= n(z)\beta^{-1}\left[1 + \frac{\xi(0.177 - 0.070\xi)}{(1-\frac{1}{8}\,\xi)^3}\right]. \quad (79)$$

This equation of state will be useful for calculating the transport coefficients later in this work.

VII. COMPARISON WITH EXPERIMENT

A. Distribution of unperturbed granular matter

Having identified the parameters in f_e in terms of phenomenological quantities, we are now ready to examine the distribution function in more detail and in comparison with available experimental observations. Since T tends to zero according to the relation (59) as γ vanishes, the limit of vanishing shear rate corresponds to the absolute zero of temperature for molecular fluids. Thus, we may say that the granular matter is absolutely cold with respect to the translational motion when it is not perturbed by an external force such as shearing force or tapping. In this limit the distribution function becomes a step function:

$$f_e = \begin{cases} 1 & \text{if } \alpha \geqslant 0 \\ 0 & \text{if } \alpha < 0. \end{cases} \quad (80)$$

The value of μ_e determines the position z where the distribution function vanishes. This is the situation where granules are statically piled up in the gravitational field. The distribution function in Eq. (80), together with the density distribution already calculated in Eq. (52), indicates the behavior of the granular matter under the gravitational field in the limit of vanishing perturbation. Their behavior is physically reasonable and in agreement with experimental observation and everyday experience with granular matter. In a recent work of Goldhirsh and Tan [9], which was not in print before the original version of this work was submitted, a reduced velocity—in fact, radial speed—distribution function computed by a simulation method for a sheared granular fluid is shown to deviate from the normal Gaussian form. The velocity distribution function f_e obtained in the present work is consistent with their finding, since f_e and any reduced distribution function obtained from it is certainly not Gaussian. This aspect and other macroscopic properties of granular fluids predicted by the present theory will be discussed in a separate work in the future.

B. Density distribution at a nonvanishing perturbation

If a granular assembly is vertically shaken or tapped at a finite amplitude and frequency as was done by Clement and Rajchenbach [16], the particles in the assembly collectively move more or less vertically, although their velocities are not uniform but have a distribution. Such a velocity distribution is given by f_e. There are cases where some velocity components vanish owing to the setup of the experiment in hand. For example, if the x and y components of the velocity are virtually absent, as in some experiments, and furthermore, if the z component of the velocity in the up phase or the down

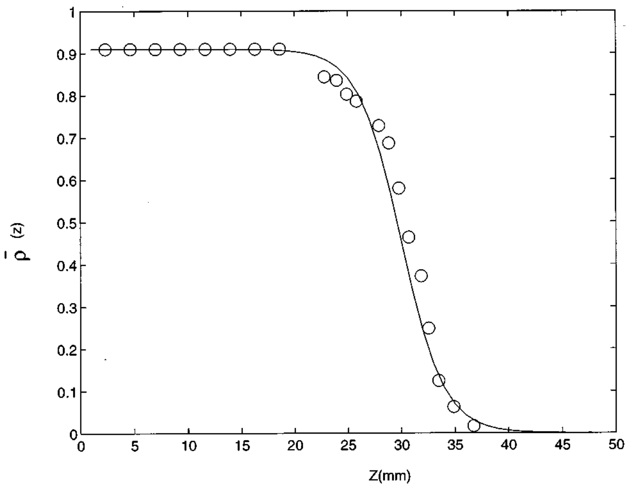

FIG. 1. Reduced density distribution with respect to z. The open circles are the data by Clement and Rajchenbach [16]. Here the static packing density is taken $\overline{\rho_0}$=0.91. Note that $\overline{\rho_0}$ is dimensionless.

phase has a narrow distribution, as seems to be the case with the experiment by Clement and Rajchenbach [16], then the distribution of the particles can be given by the density distribution function defined by the dimensionless density $\overline{\rho(z)}$:

$$\overline{\rho(z)} = \frac{\pi\sigma^3}{\Delta_c}\left(\frac{2\pi m}{\beta}\right)^{3/2}\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}ds \times \int_{-\infty}^{\infty}dt\,\frac{\delta(q)\,\delta(s)\,\delta(t-t_0)}{\exp[q^2+s^2+(t-t_0)^2+bz-\overline{\mu_e}]+1}, \quad (81)$$

where $q=\sqrt{m\beta/2}v_x$, $s=\sqrt{m\beta/2}v_y$, and $t=\sqrt{m\beta/2}v_z$ with $t_0=\sqrt{m\beta/2\pi}u$ (u=the collective velocity of granules). This gives rise to the density distribution

$$\overline{\rho(z)} = \frac{\overline{\rho_0}}{\exp(bz-\overline{\mu_e})+1}, \quad (82)$$

where

$$\overline{\rho_0} = \frac{\pi\sigma^3}{\Delta_c}\left(\frac{2\pi m}{\beta}\right)^{3/2}. \quad (83)$$

By choosing $b=0.5\ \text{mm}^{-1}$, $\overline{\rho_0}=0.91$, and $\overline{\mu_e}=15$, which implies $m\beta=50\ \text{m}^{-2}\,\text{s}^2$, we show that the distribution in Eq. (82) fits well the experimental data on the z dependence of density obtained by Clement and Rajchenbach [16] as shown in Fig. 1. The experimental values for A and ω are, respectively, $A=2.5$ mm and $\omega=20\ \text{s}^{-1}$. We have not made comparison with the simulation data of Gallas, Herrmann, and Sokolowski [17], since the simulation data agree with the experiment.

VIII. THEORY OF TRANSPORT PROCESSES IN GRANULAR MATTER

The theory of transport processes in granular matter can be developed in a completely parallel manner analogous to the procedures developed for ordinary fluids in Ref. [20]. We therefore will simply present the relevant results for granular matter. We will confine our discussion to linear transport processes in this paper. It is possible to show that the viscosity η and thermal conductivity κ are given by the formulas

$$\eta=\frac{2p^2\beta g}{R^{(11)}},\tag{84}$$

$$\kappa=\frac{(\hat{C}_pTp)^2\beta g}{R^{(33)}},\tag{85}$$

where $g=(m/2k_BT)^{1/2}/n^2\sigma^2$, $\hat{C}_p=5k_B/2m$, and $R^{(ii)}$ ($i=1,3$) are collision bracket integrals defined by the formula

$$R^{(ii)}=\tfrac{1}{4}[(h_1^{(i)}+h_2^{(i)}-h_1^{(i)*}-h_2^{(i)*})\times(h_1^{(i)}+h_2^{(i)}-h_1^{(i)*}-h_2^{(i)*})]_{12}.\tag{86}$$

Here various symbols are defined as

$$h_1^{(1)}=m[\mathbf{CC}-\tfrac{1}{3}C^2\boldsymbol{\delta}]_1,\tag{87}$$

$$h_1^{(3)}=m[\tfrac{1}{2}C^2\mathbf{C}-\hat{C}_pT\mathbf{C}]_1,\tag{88}$$

with the subscript 1 and 2 referring to particles and the square brackets in Eq. (86) stands for the integral

$$[AB]_{12}=\beta g\Delta_c^{-2}m^6\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\mathbf{k}(\mathbf{g}_{12}\cdot\mathbf{k})f_e(\mathbf{v}_1)f_e(\mathbf{v}_2)\times[1-f_e(\mathbf{v}_1)][1-f_e(\mathbf{v}_2)]AB.\tag{89}$$

These collision bracket integrals are explicitly evaluated to an approximation and presented below.

We will calculate the collision bracket integral in the case where α is moderate in value. By using the expansion

$$f_e(\mathbf{v})[1-f_e(\mathbf{v})]=\sum_{l=0}^{\infty}\frac{d}{d\alpha}\left[(1+e^{-\alpha})^{-1}e^{-t}\left(\frac{1-e^{-t}}{1+e^{-\alpha}}\right)^{l}\right],\tag{90}$$

where t is the reduced kinetic energy $t_i=\frac{1}{2}mC_i^2\beta$ ($i=1,2$) and α is as defined before. Let us now define the integrals

$$[AB]_{12}^{(c)}=\beta gn^2\left(\frac{m\beta}{2\pi}\right)^3\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\mathbf{k}(\mathbf{g}_{12}\cdot\mathbf{k})\times\exp(-t_1-t_2)AB,\tag{91}$$

$$[AB]_{12}^{(l_1l_2)}=\beta gn^2\left(\frac{m\beta}{2\pi}\right)^3\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\mathbf{k}(\mathbf{g}_{12}\cdot\mathbf{k})\times\exp(-t_1-t_2)(1-e^{-t_1})^{l_1}(1-e^{-t_2})^{l_2}AB.\tag{92}$$

Then $[AB]_{12}$ can be split into two terms as follows:

$$[AB]_{12}=[2n\Lambda^3(1+\cosh\alpha)]^{-2}[AB]_{12}^{(c)}+\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}(l_1+1)(l_2+1)\times[n^2\Lambda^6(1+e^{-\alpha})^{l_1+l_2+4}]^{-1}[AB]_{12}^{(l_1l_2)}.\tag{93}$$

The second term corresponds to the correction to the "classical" contribution for which the distribution functions are taken to be Maxwellian. The classical collision bracket integrals $[AB]_{12}^{(c)}$ are available in the literature [15]. We will be content with the lowest-order approximation for the collision bracket integrals. Thus, we take

$$[AB]_{12}\approx[2n\Lambda^3(1+\cosh\alpha)]^{-2}[AB]_{12}^{(c)}.\tag{94}$$

For the value of α the result in Eq. (69) must be used with the last term neglected for consistency with the approximation made for Eq. (94). We use this result and Eq. (84) to calculate the viscosity of granular matter. For the viscosity,

$$A=B=m[\mathbf{CC}]^{(2)}=m(\mathbf{CC}-\tfrac{1}{3}C^2\boldsymbol{\delta}),\tag{95}$$

with $\boldsymbol{\delta}$ denoting the unit second-rank tensor. Calculating the front factor involving α in Eq. (94) with the aforementioned approximation neglecting the ξ-dependent term in Eq. (69) and using the literature result for the collision bracket integral $[AB]_{12}^{(c)}$ for hard spheres [15], we obtain from Eq. (84) the viscosity in the form

$$\eta=\frac{5}{16\sigma^2}\left(\frac{mk_BT}{\pi}\right)^{1/2}\left[1+\frac{\Lambda^3n(0.177-0.070\Lambda^3n)}{(1-\frac{1}{8}\Lambda^3n)^3}\right]^2,\tag{96}$$

for which we have used the equation of state, Eq. (79). If the equation of state given in Eq. (78) is used for viscosity, then it is given by the formula

$$\eta=\frac{5}{16\sigma^2}\left(\frac{mk_BT}{\pi}\right)^{1/2}\times\left[1+\frac{\Lambda^3n(0.177-0.070\Lambda^3n)}{(1-\frac{1}{8}\Lambda^3n)^3}+\frac{2\pi\sigma^3n}{3}\chi(\sigma)\right]^2.\tag{97}$$

It must be noted that in contrast to the Newtonian viscosity of molecular fluids the viscosity formulas presented here depend on the shear rate γ on account of the γ dependence of T; see Eqs. (59) and (63). Since the quantity in the square bracket in Eq. (96) is proportional to the compressibility factor that reaches unity as γ increases according to the formula (75)—the equation of state before resumming by a Padé approximant—η approximately increases with γ like $\sqrt{\gamma}$. That is, the granular fluid is seen to be dilatant and this is a qualitatively correct behavior as is well known [1] of granular matter.

The viscosity formula (96) is plotted in Fig. 2. It shows a rather flat viscosity that slowly reaches a minimum at $\xi=\Lambda^3n\approx5.7$ before it rapidly increases. This behavior is reminiscent of the experimental data by Hanes and Inman [31]. This minimum appears because the third virial coeffi-

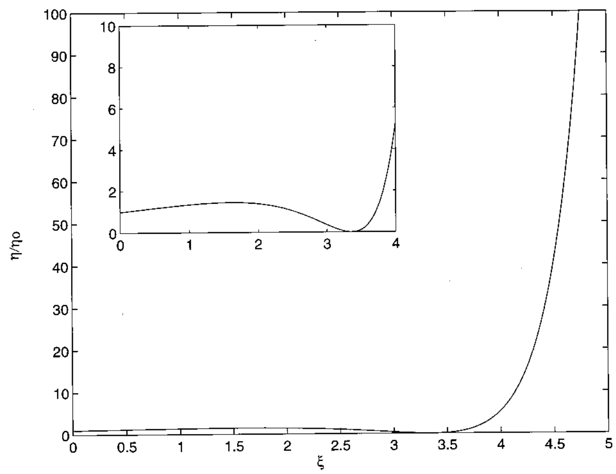

FIG. 2. Viscosity vs reduced density $\xi = n\Lambda^3$ (a scaled packing fraction). The inset is the blowup of the minimum region. The Chapman-Cowling formula for hard spheres is used for n_0: $n_0 = (5/16\sigma^2)(mk_BT/\pi)^{1/2}$.

cient B_2 is negative owing to the momentum correlations. We note that if Eq. (78) is used instead of Eq. (75) for the equation of state, the minimum will be practically washed out, since the term containing $\chi(\sigma)$ does not exhibit a minimum and is dominant.

IX. CONCLUDING REMARKS

In this paper, we have formulated a kinetic theory of fluidized granular matter by taking into account the excluded-volume effects and significant momentum correlations present in congested elastic granular fluids. These two kinds of effects combine to give rise to an equilibrium distribution function which has a mathematical form similar to the equilibrium distribution function of a Fermi gas, which also has the excluded-volume effects and momentum correlations. We believe that *the momentum correlations in a congested assembly of particles with an excluded volume is the key feature that controls the kinetic evolution of statistical distribution in a fluidized granular system*. Another significant point in the present theory is the assertion that in analogy to the statistical mechanics of ordinary molecular fluids the distribution function obeying the kinetic equation postulated is the thermodynamic branch of solution, which yields a thermodynamic theory of processes in fluidized granular matter, and this assertion enables us to construct a thermodynamic theory from the statistical theory. Note that this procedure is just the opposite of the one used in the conventional ensemble theory of statistical mechanics, where the thermodynamic branch of the distribution function is constructed in correspondence with the phenomenological thermodynamics established from the thermodynamic laws by means of the Clausius inequality. This thermodynamic formalism for fluidized granular fluid, with the help of the attendant statistical theory, elucidates the meaning of granular temperature when the system is subject to an external perturbation such as shearing. This theory also provides the equation of state in terms of the granular temperature so determined. The pressure increases with the density, which is a qualitatively correct behavior.

The density distribution function obtained from the kinetic equation is in agreement with experiment and the computer simulation result on a vibrated granular matter and the viscosity computed in the theory also exhibits qualitatively correct behavior when compared with the viscosity of granular matter in that it remains almost constant over a density interval and then steeply increases with the density. The present theory also predicts that granular matter is dilatant as experimentally known. Such behaviors would not have arisen if the equilibrium distribution function was simply given by the Boltzmann distribution function that arises as the equilibrium solution of the kinetic equations appearing in Refs. [10–14]. Therefore, they may be taken as evidence in support of the kinetic equation used in the present work and, in particular, the collision integral therein. Nevertheless, the theory presented here is only a first step toward the goal of understanding kinetic processes in fluidized granular matter and we hope that the present work stimulates further studies in kinetic theory of granular matter.

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