Critical behavior of $O(n)$-symmetric systems with reversible mode-coupling terms: Stability against detailed-balance violation

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We investigate nonequilibrium critical properties of $O(n)$-symmetric models with reversible mode-coupling terms. Specifically, a variant of the model of Sasvári, Schwabl, and Szépfalusy (SSS) is studied, where violation of detailed balance is incorporated by allowing the order parameter and the dynamically coupled conserved quantities to be governed by heat baths of different temperatures T_S and T_M, respectively. Dynamic perturbation theory and the field-theoretic renormalization group are applied to one-loop order, and yield two new fixed points in addition to the equilibrium ones. The first fixed point corresponds to $\Theta = T_S/T_M = \infty$ and leads to model *A* critical behavior for the order parameter and to anomalous noise correlations for the generalized angular momenta; the second one is at $\Theta = 0$ and is characterized by mean-field behavior of the conserved quantities, by a dynamic exponent $z = d/2$ equal to that of the equilibrium SSS model, and by modified static critical exponents. However, both these new fixed points are unstable, and upon approaching the critical point detailed balance is restored, and the equilibrium static and dynamic critical properties are recovered. [S1063-651X(97)11504-5]

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I. INTRODUCTION

Nonequilibrium steady states (NESS) have been much investigated, the main goal being the discovery of their common and distinguishing features as compared to equilibrium states. A promising approach to this problem is the study of phase transitions: since equilibrium critical phenomena display a large degree of universality, it is natural to ask to what extent these universal features remain characteristic of nonequilibrium phase transitions.

The basic complication with NESS is that, in addition to the interactions that entirely define the equilibrium properties, the dynamics is also essential in determining the steady-state properties. Thus, for example, a classification of nonequilibrium phase transitions requires not only the understanding of the role of symmetries of the order parameter, the range of interactions, and the dimensionality of the system, but the clarification of both the relevance of conservation laws imposed by dynamical symmetries and the range of the dynamical processes. Possibly, new dimensionality and anisotropy effects in the dynamics may also be important.

The most frequently studied models with nonequilibrium phase transitions are generalizations of systems with model *A* type dynamics [1]. The transitions in these systems have been shown to be robust against local nonequilibrium perturbations, which do not conserve the order parameter [2,3], and, remarkably, this robustness was found to persist even if the *dynamical* perturbations broke the discrete symmetry of the system [4]. Both locality and the nonconserving character of the perturbations are essential for the phase transition to stay in the Ising universality class. Indeed, nonlocal nonequilibrium dynamics generates effective long-range forces and thus changes the universality class dramatically [5].

The nonequilibrium generalizations of model *B* type dynamics (with conserved order parameter) are more interesting. External fields or local, *anisotropic*, nonequilibrium perturbations may drive the system into a NESS with phase transitions that are not characterized by any known equilibrium universality class [6], or belong to universality classes with long-range interactions [7,8].

Nonequilibrium generalizations of the case when a nonconserved order parameter is coupled to a conserved quantity have been considered in Ref. [9] where it was found that linear coupling to a conserved quantity generates power-law correlations for the order parameter. This suggests that, in this situation, long-range effective interactions are generated in the system, which in turn govern the critical behavior at the phase transition.

There are several other nonequilibrium phase transitions that have been studied without considering any equilibrium context. Most notable among these are phase transitions associated with the presence of an absorbing state (directed percolation) [10], and the roughening transition in surface growth and equivalent models such as the Kardar-Parisi-Zhang equation [11].

In this paper we continue the investigation of nonequilibrium generalizations of models originally proposed to describe equilibrium critical dynamics. Our aim is to study an example where there is a reversible mode coupling between the order parameter and another (conserved) field, using the field-theoretic dynamic renormalization group (RG) [12,13]. A simple example of this type of system is the Heisenberg model for isotropic ferromagnets where precession terms introduce a coupling among the different spin components (model *J* according to the classification in Ref. [1]; for early

RG studies of this model see Ref. [14]; a comprehensive review of the critical dynamics of ferromagnets is given in Ref. [15]). However, similar to the purely relaxational dynamics of models *A* and *B*, the effect of a (spatially isotropic) violation of detailed balance can be removed by a simple rescaling (see Sec. II B). We shall thus mainly consider a more complicated model, which was originally introduced by Sasvári, Schwabl, and Szépfalusy (SSS) in the context of structural phase transitions [16]. This SSS model consists of a nonconserved n-component order parameter purely dynamically coupled to the $n(n-1)/2$ conserved generalized angular momenta related to the underlying $O(n)$ symmetry of the system. The $n=2$ realization describes the critical dynamics of planar ferromagnets and superfluid ^{4}He [17] (for reviews regarding dynamic critical phenomena in superfluid helium, see Ref. [18]), while the case $n=3$ corresponds to the dynamics of isotropic antiferromagnets [19]. The SSS model, with its dynamic exponent $z=d/2$ (below the upper critical dimension $d_c=4$) thus encompasses models *E* and *G* (according to Ref. [1]) as special cases.

We shall generalize the previous field-theoretic RG studies of the SSS model [20,21] to a nonequilibrium situation by assuming that the order-parameter components S^α and conserved angular momenta $M^{\alpha\beta}$ are attached to heat baths of *different* temperatures T_S and T_M, respectively. Thus the detailed-balance condition required for near-equilibrium dynamics is violated and the flow of energy between the two heat baths ensures that the steady state is out of equilibrium. The introduction of two temperatures leads to an additional variable in the problem, namely, the temperature ratio $\Theta=T_S/T_M$ (of which no analog can be constructed for model *J*). By studying the RG flow equations to one-loop order (first order in $\epsilon=4-d$), we find *two new fixed points* corresponding to the cases $\Theta=0$ and $\Theta=\infty$, respectively, in addition to the equilibrium fixed points of the SSS model. The latter are (i) the usual Gaussian fixed point (describing static and dynamic mean-field behavior, $z=2$), (ii) the model *A* fixed point, corresponding to a decoupling of the conserved fields from the order parameter, with the nontrivial static exponents of the $O(n)$-symmetric ϕ^4 model and dynamic exponent $z=2+O(\epsilon^2)$, and the three nontrivial SSS dynamic fixed points consisting of the two so-called *weak-scaling* fixed points with the order parameter and conserved quantities fluctuating on different time scales, characterized by the exponents (iii) $z_S=2-2(n-1)\epsilon/(2n-1)+O(\epsilon^2)$, $z_M=2-\epsilon/(2n-1)+O(\epsilon^2)$, and (iv) $z_S=2$ and $z_M=d-2$, and finally (v) the *strong-scaling* fixed point with $z_S=z_M=z=d/2$. The results for (iv) and (v) actually hold to all orders in ϵ, and follow from the exact sum rule $z_S+z_M=d$ [17,20,22] (see Sec. III A). Stability analysis shows that to one-loop order only the strong-scaling fixed point (v) is stable; however, at least for $n=2$ the actual fixed-point values are rather close to its stability boundary, which allows for the possibility that in fact for superfluid helium strong scaling may be violated at the lambda transition [20] (a two-loop study of model *F*, combined with Borel resummation techniques, actually suggests the stability of a weak-scaling fixed point [23]).

The stability of the above fixed points may change in a nonequilibrium situation where the order parameter and conserved variables are allowed to fluctuate at different temperatures, which explicitly introduces different characteristic time scales. Indeed, while one of the two new nonequilibrium fixed points, corresponding to (a) $\Theta=\infty$, is described by model *A* dynamics $z_S=z_M=2$ (with the usual ϕ^4 model statics), albeit accompanied by *anomalous noise correlations for the conserved fields*, the second new fixed point, characterized by (b) $\Theta=0$, yields, actually to all orders in ϵ, $z_S=d/2$ for the order parameter as in equilibrium, but $z_M=2$, i.e., ordinary diffusion for the generalized angular momenta (note that the above-mentioned equilibrium sum rule does not hold here); this unusual behavior is supplemented by *anomalous order-parameter noise correlations*, and even *modified static critical exponents*. However, stability analysis reveals that in fact both these fixed points (a) and (b) are unstable, and for any initial value of $0<\Theta<\infty$ the flow asymptotically leads to the *stable strong-scaling equilibrium fixed point* of the SSS model (see Sec. III B). Thus we conclude that while violation of detailed balance might be conceived as a relevant perturbation, in fact the underlying $O(n)$ symmetry in conjunction with spatial isotropy and the growing correlation length as the phase transition is approached effectively *restore detailed balance* (described by the fixed point with $\Theta=1$), and thus asymptotically yield the usual static and dynamic critical behavior of the equilibrium SSS model.

This paper is organized as follows. In Sec. II, we briefly review the derivation of Langevin equations describing the critial dynamics of $O(n)$-symmetric models including reversible mode-coupling terms, and consider the possible relevance of detailed-balance violation for the relaxational models *A* and *B*, as well as model J and the SSS model. Sec. III will then be devoted to the RG study of the nonequilibrium SSS model as outlined above, starting with stating some general exact relations and Ward identities, followed by a detailed study of the one-loop perturbation theory, the ensuing flow equations, and a discussion of the physical content and stability of the RG fixed points. Finally, in Sec. IV we shall summarize our results again, draw some conclusions, and provide an outlook on possible future research along the path followed in this paper. In the Appendix, we provide a list of the explicit results to one-loop order for the two-, three-, and four-point functions required for the renormalization of the nonequilibrium SSS model.

II. CRITICAL DYNAMICS OF MODELS WITH REVERSIBLE MODE-COUPLING TERMS

A. General considerations

The universal static critical behavior of a system that is invariant with respect to rotations of its n-component order parameter and displays a second-order phase transition is described by the following $O(n)$-symmetric ϕ^4 Landau-Ginzburg-Wilson Hamiltonian in d space dimensions

$$H[\{S_0^\alpha\}]=\int d^dx\left\{\frac{r_0}{2}\sum_{\alpha=1}^{n}S_0^\alpha(\mathbf{x})^2+\frac{1}{2}\sum_{\alpha=1}^{n}[\nabla S_0^\alpha(\mathbf{x})]^2+\frac{u_0}{4!}\left[\sum_{\alpha=1}^{n}S_0^\alpha(\mathbf{x})^2\right]^2\right\}; \tag{2.1}$$

here $r_0=(T-T_c^0)/T_c^0$ denotes the relative distance from the mean-field critical temperature T_c^0, and we denote unrenormalized quantities by a subscript ''0.'' This effective free energy determines the equilibrium probability distribution for the vector order parameter S_0^α,

$$P_{\rm eq}[\{S_0^\alpha\}]=\frac{e^{-H[\{S_0^\alpha\}]/k_{\rm B}T}}{\int \mathcal{D}[\{S_0^\alpha\}]e^{-H[\{S_0^\alpha\}]/k_{\rm B}T}}, \tag{2.2}$$

and furthermore provides the starting point for the construction of the field-theoretic static renormalization group, which by virtue of a perturbation (loop) expansion in the nonlinearity u_0 provides a systematic means to compute the two independent static critical exponents η and ν either in an ϵ expansion about the upper critical dimension $d_c=4$, or directly in fixed dimensionality d [24]. Here, η is the anomalous dimension which describes the power-law decay of the order-parameter correlations at the critical point, $\langle S^\alpha(\mathbf{x})S^\beta(\mathbf{x}')\rangle\propto 1/|\mathbf{x}-\mathbf{x}'|^{d-2+\eta}$, and the exponent ν characterizes the divergence of the correlation length as T_c is approached, $\xi\propto|T-T_c|^{-\nu}$.

The simplest dynamics that may be imposed on the order-parameter fluctuations $S_0^\alpha(\mathbf{x},t)$ in order to describe how the system relaxes to equilibrium (for which the mean-field stationarity condition $\delta H[\{S_0^\alpha\}]/\delta S_0^\alpha=0$ holds) is then given by the following Langevin-type equations of motion:

$$\frac{\partial S_0^\alpha(\mathbf{x},t)}{\partial t}=-\lambda_0(i\nabla)^a\frac{\delta H[\{S_0^\alpha\}]}{\delta S_0^\alpha(\mathbf{x},t)}+\zeta^\alpha(\mathbf{x},t), \tag{2.3}$$

where the temporal average of the stochastic forces is assumed to vanish, $\langle\zeta^\alpha(\mathbf{x},t)\rangle=0$. In equilibrium, furthermore, an Einstein relation connects the second moment of the uncorrelated (white) noise with the relaxation coefficient,

$$\langle\zeta^\alpha(\mathbf{x},t)\zeta^\beta(\mathbf{x}',t')\rangle=2\lambda_0 k_{\rm B}T(i\nabla)^a\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta^{\alpha\beta}; \tag{2.4}$$

this ensures that the probability distribution $P[\{S_0^\alpha\}]$ finally approaches the equilibrium distribution (2.2) in the limit $t\to\infty$, as can be readily checked with the aid of the associated Fokker-Planck equation. Equation (2.3) incorporates both the case of a nonconserved order parameter with purely relaxational dynamics ($a=0$) and the case of a conserved order parameter, which as a consequence of the ensuing continuity equation relaxes diffusively ($a=2$). In the classification scheme of Hohenberg and Halperin, these situations are referred to as models A and B, respectively, and the corresponding dynamic critical exponents describing the critical slowing down near the phase transition (characteristic time scales diverge as $t_c\propto\xi^z\propto|T-T_c|^{-z\nu}$) are given in terms of the static exponent η by $z=4-\eta$ (model B) and $z=2+c\,\eta$ (model A) [1]. In the latter case, however, c is a new universal number and therefore z is an independent exponent not determined by the static critical exponents.

One may already anticipate that an *isotropic* violation of the Einstein relation (2.4), which is a consequence of an underlying detailed balance condition, by choosing a coefficient $\tilde{\lambda}_0$ instead of $\lambda_0 k_{\rm B}T$ for the noise correlator, merely amounts to a change in the order-parameter temperature T. Therefore, as long as one remains sufficiently close to the critical point, the *universal* critical behavior (exponents, amplitude ratios, etc.) will not be affected, while only nonuniversal amplitudes become modified through a rescaled nonlinear coupling u_0 (see Sec. II B and Ref. [3]).

However, in an $O(n)$-symmetric system there are always additional slow diffusive modes present. In our case these modes are associated with the conserved generalized angular momenta $M_0^{\alpha\beta}$, which generate the rotations in order-parameter space. Generically, they couple to the order-parameter fluctuations, and therefore Eq. (2.3) does not correctly describe their dynamics. Two cases can now be distinguished: (i) the vector order parameter itself is identical to the generators of the group $O(n)$; this yields, for $n=3$, precisely the dynamics of isotropic Heisenberg ferromagnets [14], model J according to Ref. [1]. (ii) The order parameter is nonconserved, and the conserved angular momenta constitute new dynamical variables; this defines the $O(n)$-symmetric model introduced by Sasvári, Schwabl, and Szépfalusy [16], and encompasses both model E for the dynamics of the XY model, i.e., of planar ferromagnets and superfluid ^{4}He ($n=2$) [17], and model G for isotropic antiferromagnets ($n=3$) [19].

Upon collecting the order-parameter and angular-momentum components in a large vector $\psi^\alpha=(S^\alpha,M^{\alpha\beta})$, the general structure of the ensuing Langevin equations reads [14,16]

$$\frac{\partial\psi^\alpha(\mathbf{x},t)}{\partial t}=V^\alpha[\{\psi^\alpha\}](\mathbf{x},t)-L^\alpha\frac{\delta H[\{\psi^\alpha\}]}{\delta\psi^\alpha(\mathbf{x},t)}+\zeta^\alpha(\mathbf{x},t), \tag{2.5}$$

where $L^\alpha=\lambda$ or $L^\alpha=-D\nabla^2$ for all the nonconserved and conserved fields, respectively. The second term on the right-hand side of Eq. (2.5) describes irreversible relaxation processes as in models A and B [Eq. (2.3)]; the first term, on the other hand, consists of *reversible* ''mode couplings,'' which are given entirely by the Poisson brackets $Q^{\alpha\beta}[\{\psi^\alpha\}]\propto\{\psi^\alpha,\psi^\beta\}$. As can be shown with the Kawasaki-Mori-Zwanzig projector formalism, $V[\{\psi^\alpha\}]$ assumes the form of a ''streaming velocity'' in the space of the ψ^α, namely,

$$V^\alpha[\{\psi^\alpha\}]=g\sum_\beta\left(k_{\rm B}T\frac{\delta Q^{\alpha\beta}}{\delta\psi^\beta}-Q^{\alpha\beta}\frac{\delta H[\{\psi^\alpha\}]}{\delta\psi^\beta}\right). \tag{2.6}$$

Note that the mode-coupling constants g are *independent* of α, which guarantees that $V^\alpha[\{\psi^\alpha\}]e^{-H[\{\psi^\alpha\}]/k_{\rm B}T}$ is divergence-free,

$$\int d^dx\sum_\alpha\frac{\delta}{\delta\psi^\alpha}(V^\alpha[\{\psi^\alpha\}]e^{-H[\{\psi^\alpha\}]/k_{\rm B}T})=0, \tag{2.7}$$

and therefore the equilibrium distribution $P_{\rm eq}[\{\psi^\alpha\}]\propto e^{-H[\{\psi^\alpha\}]/k_{\rm B}T}$ is not affected by the mode-coupling terms, which are of purely dynamical origin.

We defer the explicit construction of the mode-coupling terms for model J and the SSS model to the following subsections, and close this general discussion with a brief outline of how one may construct an effective field theory from Langevin equations of the type

$$\frac{\partial\psi^\alpha(\mathbf{x},t)}{\partial t}=K^\alpha[\{\psi^\alpha\}](\mathbf{x},t)+\zeta^\alpha(\mathbf{x},t) \tag{2.8}$$

[see Eq. (2.5)], with $\langle\zeta^\alpha(\mathbf{x},t)\rangle=0$ and the general noise correlator

$$\langle\zeta^\alpha(\mathbf{x},t)\zeta^\beta(\mathbf{x}',t')\rangle=2L^\alpha\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')\,\delta^{\alpha\beta} \tag{2.9}$$

(see Refs. [12,13]). This form of the white noise may be inferred from a Gaussian distribution for the stochastic forces

$$W[\{\zeta^\alpha\}]\propto\exp\left[-\frac{1}{4}\int d^dx\int dt\sum_\alpha\ \zeta^\alpha(L^\alpha)^{-1}\zeta^\alpha\right]; \tag{2.10}$$

eliminating ζ^α via Eq. (2.8) then immediately yields the desired probability distribution for the fields ψ^α,

$$W[\{\zeta^\alpha\}]\mathcal{D}[\{\zeta^\alpha\}]=P[\{\psi^\alpha\}]\mathcal{D}[\{\psi^\alpha\}]\propto e^{G[\{\psi^\alpha\}]}\mathcal{D}[\{\psi^\alpha\}], \tag{2.11}$$

with the Onsager-Machlup functional

$$G[\{\psi^\alpha\}]=-\frac{1}{4}\int d^dx\int dt\sum_\alpha\ \left(\frac{\partial\psi^\alpha}{\partial t}-K^\alpha[\{\psi^\alpha\}]\right)\times(L^\alpha)^{-1}\left(\frac{\partial\psi^\alpha}{\partial t}-K^\alpha[\{\psi^\alpha\}]\right). \tag{2.12}$$

From this functional one could already construct a perturbation expansion for correlation functions of the fields ψ^α; however, as for conserved quantities the inverse of the Onsager coefficient L^α is singular, and furthermore high nonlinearities $\propto K^\alpha[\{\psi^\alpha\}]^2$ appear, it is convenient to introduce Martin-Siggia-Rose auxiliary fields via a Gaussian transformation to partially linearize the above functional. This leads to

$$P[\{\psi^\alpha\}]\propto\int\mathcal{D}[\{i\tilde{\psi}^\alpha\}]e^{J[\{\tilde{\psi}^\alpha\},\{\psi^\alpha\}]}, \tag{2.13}$$

with the Janssen-De Dominicis functional

$$J[\{\tilde{\psi}^\alpha\},\{\psi^\alpha\}]=\int d^dx\int dt\sum_\alpha\ \left[\tilde{\psi}^\alpha L^\alpha\tilde{\psi}^\alpha-\tilde{\psi}^\alpha\left(\frac{\partial\psi^\alpha}{\partial t}-K^\alpha[\{\psi^\alpha\}]\right)\right]. \tag{2.14}$$

Equation (2.14) will provide the starting point for our discussion of the nonequilibrium dynamics of the isotropic ferromagnet (model J) as well as that of the SSS model in the subsequent subsections. In Sec. III, we shall use the corresponding Janssen–De Dominicis functional for the construction of the dynamical field theory of the SSS model with broken detailed balance, and therefrom infer its RG flow equations. We finally remark that both in Eqs. (2.12) and (2.14) we have omitted contributions stemming from the functional determinant $\mathcal{D}[\{\zeta^\alpha\}]/\mathcal{D}[\{\psi^\alpha\}]$. As is shown in Refs. [13,20], these terms precisely cancel any *acausal* Feynman diagrams for the dynamic response function that could be constructed from the above functionals, and upon restricting the perturbation expansion to those contributions that are consistent with causality requirements, we may therefore safely neglect these additional terms.

B. Model J—isotropic ferromagnets

We now turn explicitly to the construction of the Langevin equation for the critical dynamics of isotropic ferromagnets [14]. In this case, $n=3$, and the order parameter consists of the three spin components, S^x, S^y, and S^z. The total magnetization is a conserved quantity (hence $a=2$), and in fact the S^α are identical with the generators of the rotation group $O(3)$: $M^{12}=S^z$, $M^{23}=S^x$, and $M^{13}=-S^y$. The Poisson brackets between the spin components read

$$\{S^\alpha,S^\beta\}=\sum_\gamma\ \epsilon^{\alpha\beta\gamma}S^\gamma, \tag{2.15}$$

which immediately yields the streaming velocity

$$V^\alpha[\{S^\alpha\}]=-g\sum_{\beta,\gamma}\ \epsilon^{\alpha\beta\gamma}S^\gamma\frac{\delta H[\{S^\alpha\}]}{\delta S^\beta}=-g\sum_{\beta,\gamma}\ \epsilon^{\alpha\beta\gamma}S^\beta\nabla^2S^\gamma, \tag{2.16}$$

because the contractions of the fully antisymmetric tensor $\epsilon^{\alpha\beta\gamma}$ with all the symmetric terms in Eq. (2.1) vanish, leaving only the contribution stemming from the gradient term in the Hamiltonian. The mode-coupling terms (2.16) represent the spin precession in the effective field generated by the other spins, and in the ordered phase lead to propagating spin waves (Goldstone modes) with quadratic dispersion $\omega(q)\propto q^2$.

The complete Langevin equation for the conserved order parameter of isotropic ferromagnets (model J according to Ref. [1]) finally reads

$$\frac{\partial S_0^\alpha}{\partial t}=-g_0\sum_{\beta,\gamma}\ \epsilon^{\alpha\beta\gamma}S_0^\beta\nabla^2S_0^\gamma+\lambda_0\nabla^2\frac{\delta H[\{S_0^\alpha\}]}{\delta S_0^\alpha}+\zeta^\alpha, \tag{2.17}$$

with $\langle\zeta^\alpha(\mathbf{x},t)\rangle=0$ and

$$\langle\zeta^\alpha(\mathbf{x},t)\zeta^\beta(\mathbf{x}',t')\rangle=-2\tilde{\lambda}_0\nabla^2\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')\,\delta^{\alpha\beta}. \tag{2.18}$$

Here we have already allowed for a violation of the detailed-balance condition by introducing a noise strength $\tilde{\lambda}_0$ that is not necessarily equal to λ_0k_BT, where λ_0 is the spin diffusion constant. However, the form of Eqs. (2.17) and (2.18) already suggests that similar to the case of the purely relaxational models A and B, the ratio $\tilde{\lambda}_0/\lambda_0$ may be absorbed into a rescaled temperature T, and modified nonlinear couplings u_0 and g_0.

This can be readily seen by employing the corresponding Janssen–De Dominicis functional (2.14); for our nonequilibrium model J this becomes a sum of the dynamic functionals for the relaxational models

$$J_{\rm rel}[\{\widetilde{S}_0^\alpha\},\{S_0^\alpha\}]=\int d^dx\int dt\sum_\alpha\left\{\widetilde{\lambda}_0\widetilde{S}_0^\alpha(i\nabla)^a\widetilde{S}_0^\alpha-\widetilde{S}_0^\alpha\left[\frac{\partial}{\partial t}+\lambda_0(i\nabla)^a(r_0-\nabla^2)\right]S_0^\alpha-\lambda_0\frac{u_0}{6}\sum_\beta\widetilde{S}_0^\alpha(i\nabla)^aS_0^\alpha S_0^\beta S_0^\beta,\right. \quad (2.19)$$

with $a=2$, and the additional contribution stemming from the reversible spin precession term,

$$J_{\rm mc}[\{\widetilde{S}_0^\alpha\},\{S_0^\alpha\}]=-g_0\int d^dx\int dt\sum_{\alpha,\beta,\gamma}\epsilon^{\alpha\beta\gamma}\widetilde{S}_0^\alpha S_0^\beta\nabla^2S_0^\gamma. \quad (2.20)$$

Rescaling the fields according to

$$\widetilde{S}_0^\alpha\to\left(\frac{\lambda_0}{\widetilde{\lambda}_0}\right)^{1/2}\widetilde{S}_0^\alpha,\quad S_0^\alpha\to\left(\frac{\widetilde{\lambda}_0}{\lambda_0}\right)^{1/2}S_0^\alpha \quad (2.21)$$

then renders the noise strength and the relaxation constant in the quadratic part (first and second line) of Eq. (2.19) equal, and if in addition the rescaled static and dynamic nonlinear couplings

$$\widetilde{u}_0=\frac{\widetilde{\lambda}_0}{\lambda_0}u_0,\quad \widetilde{g}_0=\left(\frac{\widetilde{\lambda}_0}{\lambda_0}\right)^{1/2}g_0 \quad (2.22)$$

are introduced, the ensuing Janssen–De Dominicis functionals for the above nonequilibrium generalizations of the relaxational models as well as model J appear in precisely the same form as in equilibrium where detailed balance holds. As both the renormalized counterparts of $\widetilde{u}_0$ and $\widetilde{g}_0^2/\lambda_0^2$ approach universal fixed-point values near the transition, the modifications in Eq. (2.22) merely enter *nonuniversal* amplitudes. It is therefore established that the *critical* properties of neither the relaxational models A and B nor isotropic ferromagnets (model J) are affected by violating the detailed-balance condition. It is, however, important to note that both the $O(n)$ symmetry *and* the spatial isotropy of the models have been left intact by the above nonequilibrium generalization. For the dynamics of Heisenberg ferromagnets, we finally remark that the dynamic critical exponent becomes $z=(d+2-\eta)/2$, as a consequence of a Ward identity stemming from the underlying $O(3)$ symmetry (see also Sec. III A) and, indeed, dynamic scaling with $z\approx 5/2$ is observed experimentally in a variety of substances. In some real ferromagnets, however, anisotropies and/or long-range dipolar forces are present, which do not conserve the total magnetization, and upon approaching the critical point lead to a dynamic crossover to model A scaling with $z\approx 2$. Further details regarding these issues and the dynamic critical behavior of ferromagnets may be found in Ref. [15].

C. The SSS model—planar ferromagnets, isotropic antiferromagnets

More interesting for the issue of violating detailed balance will clearly be a situation where there are *two* independent temperature scales conceivable, and therefore a simple temperature rescaling will not suffice to render the field theory identical to the equilibrium one. We therefore consider a nonequilibrium version of the $O(n)$-symmetric SSS model, where a nonconserved n-component order parameter couples to $n(n-1)/2$ conserved generalized angular momenta [16]; possible realizations of this are as follows: (i) for $n=2$: the critical dynamics of the XY model [17] (model E according to Ref. [1]), with the order-parameter components S^x and S^y, and the conserved quantity $M^{12}=S^z$, which generates rotations in the xy plane; (ii) for $n=3$: the dynamic critical behavior of isotropic antiferromagnets, with S^x, S^y, and S^z representing the components of the staggered magnetization, and $M^{12}=M^z$, $M^{23}=M^x$, and $M^{13}=-M^y$ denoting the components of the magnetization itself, which are conserved and can be identified with the generators of $O(3)$ (model G [19]).

The variables $M_0^{\alpha\beta}$ are noncritical quantities, and their coupling to the order-parameter fluctuations S_0^α is of purely dynamical character. Hence it suffices to simply add a quadratic term to the Hamiltonian (2.1),

$$H[\{S_0^\alpha\},\{M_0^{\alpha\beta}\}]=H[\{S_0^\alpha\}]+\int d^dx\frac{1}{2}\sum_{\alpha>\beta}M_0^{\alpha\beta}(\mathbf{x})^2, \quad (2.23)$$

and for the construction of the reversible mode-coupling terms, again all that is required are the following Poisson brackets:

$$\{S^\alpha,S^\beta\}=0,\quad \{M^{\alpha\beta},S^\gamma\}=\delta^{\alpha\gamma}S^\beta-\delta^{\beta\gamma}S^\alpha,$$
$$\{M^{\alpha\beta},M^{\gamma\delta}\}=\delta^{\alpha\gamma}M^{\beta\delta}+\delta^{\beta\delta}M^{\alpha\gamma}-\delta^{\alpha\delta}M^{\beta\gamma}-\delta^{\beta\gamma}M^{\alpha\delta}. \quad (2.24)$$

Upon inserting Eq. (2.24) into Eq. (2.6), one readily finds the following mode-coupling terms in the equations of motion of the order parameter,

$$V^\alpha[\{S^\alpha\},\{M^{\alpha\beta}\}]=g\sum_\beta S^\beta\frac{\delta H}{\delta M^{\alpha\beta}}=g\sum_\beta M^{\alpha\beta}S^\beta, \quad (2.25)$$

and in the equation of motion for the conserved angular momenta,

$$V^{\alpha\beta}[\{S^\alpha\},\{M^{\alpha\beta}\}]=g\left(S^\alpha\frac{\delta H}{\delta S^\beta}-S^\beta\frac{\delta H}{\delta S^\alpha}\right)+g\sum_\gamma\left(M^{\alpha\gamma}\frac{\delta H}{\delta M^{\beta\gamma}}-M^{\beta\gamma}\frac{\delta H}{\delta M^{\alpha\gamma}}\right)=-g(S^\alpha\nabla^2S^\beta-S^\beta\nabla^2S^\alpha), \quad (2.26)$$

respectively. Note that as for model J [Eq. (2.16)], here as a consequence of the antisymmetry of the Poisson brackets only the gradient terms in the Hamiltonian contribute. We remark that in the ordered phase the above reversible mode couplings produce propagating Goldstone modes with linear dispersion $\omega(q)\propto q$.

Thus we arrive at the following set of coupled nonlinear Langevin equations that define the SSS model:

$$\frac{\partial S_0^\alpha}{\partial t} = g_0 \sum_\beta M_0^{\alpha\beta} S_0^\beta - \lambda_0 \frac{\delta H[\{S_0^\alpha\}]}{\delta S_0^\alpha} + \zeta^\alpha, \quad (2.27)$$

$$\frac{\partial M_0^{\alpha\beta}}{\partial t} = -g_0 (S_0^\alpha \nabla^2 S_0^\beta - S_0^\beta \nabla^2 S_0^\alpha) + D_0 \nabla^2 M_0^{\alpha\beta} + \eta^{\alpha\beta}, \quad (2.28)$$

with $\langle \zeta^\alpha(\mathbf{x},t)\rangle = 0$, $\langle \eta^{\alpha\beta}(\mathbf{x},t)\rangle = 0$, and

$$\langle \zeta^\alpha(\mathbf{x},t)\zeta^\beta(\mathbf{x}',t')\rangle = 2\tilde{\lambda}_0 \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta^{\alpha\beta}, \quad (2.29)$$

$$\langle \eta^{\alpha\beta}(\mathbf{x},t)\eta^{\gamma\delta}(\mathbf{x}',t')\rangle = -2\tilde{D}_0 \nabla^2 \delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \times (\delta^{\alpha\beta}\delta^{\gamma\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma}). \quad (2.30)$$

Here we have allowed for violation of detailed balance via introducing noise coefficients $\tilde{\lambda}_0$ and $\tilde{D}_0$, which are in general *not* taken equal to $\lambda_0 k_B T_S$ and $D_0 k_B T_M$, respectively, where T_S and T_M are the temperatures of the heat baths of the order parameter and of the conserved variables. Yet we now also have the *additional* freedom to choose the ratio $\Theta_0 = T_S/T_M$ different from 1, which corresponds to a violation of the detailed balance in the dynamical *coupling* of the modes S_0^α and $M_0^{\alpha\beta}$. We should stress again that neither the underlying $O(n)$ symmetry *nor* the spatial isotropy are affected by this specific nonequilibrium perturbation.

This becomes clear upon considering the Janssen–De Dominicis functional (2.14), which corresponds to Eqs. (2.27)–(2.30). Its harmonic part now reads

$$\begin{aligned} J_{\rm har}[\{\tilde{S}_0^\alpha\},\{S_0^\alpha\},\{\tilde{M}_0^{\alpha\beta}\},\{M_0^{\alpha\beta}\}] &= \int d^dx \int dt \Bigg\{ \sum_\alpha \tilde{\lambda}_0 \tilde{S}_0^\alpha \tilde{S}_0^\alpha \\ &\quad - \sum_\alpha \tilde{S}_0^\alpha \left[\frac{\partial}{\partial t} + \lambda_0 (r_0 - \nabla^2) \right] S_0^\alpha \\ &\quad - \sum_{\alpha>\beta} \tilde{D}_0 \tilde{M}_0^{\alpha\beta} \nabla^2 \tilde{M}_0^{\alpha\beta} \\ &\quad - \sum_{\alpha>\beta} \tilde{M}_0^{\alpha\beta} \left(\frac{\partial}{\partial t} - D_0 \nabla^2 \right) M_0^{\alpha\beta} \Bigg\}, \end{aligned} \quad (2.31)$$

and can be readily rescaled to the equilibrium form via Eq. (2.21) combined with

$$\tilde{M}_0^{\alpha\beta} \to \left(\frac{D_0}{\tilde{D}_0} \right)^{1/2} \tilde{M}_0^{\alpha\beta}, \quad M_0^{\alpha\beta} \to \left(\frac{\tilde{D}_0}{D_0} \right)^{1/2} M_0^{\alpha\beta}. \quad (2.32)$$

Thereby, the relaxation vertex

$$J_{\rm rel}[\{\tilde{S}_0^\alpha\},\{S_0^\alpha\}] = -\lambda_0 \frac{u_0}{6} \int d^dx \int dt \sum_{\alpha,\beta} \tilde{S}_0^\alpha S_0^\alpha S_0^\beta S_0^\beta \quad (2.33)$$

attains the new effective coupling

$$\tilde{u}_0 = \frac{\tilde{\lambda}_0}{\lambda_0} u_0. \quad (2.34)$$

For the mode-coupling terms,

$$\begin{aligned} J_{\rm mc}[\{\tilde{S}_0^\alpha\},\{S_0^\alpha\},\{\tilde{M}_0^{\alpha\beta}\},\{M_0^{\alpha\beta}\}] &= \int d^dx \int dt \sum_{\alpha,\beta} \Bigg\{ g_0 \tilde{S}_0^\alpha M_0^{\alpha\beta} S_0^\beta \\ &\quad - \frac{g_0}{2} \tilde{M}_0^{\alpha\beta} (S_0^\alpha \nabla S_0^\beta - S_0^\beta \nabla S_0^\alpha) \Bigg\}, \end{aligned} \quad (2.35)$$

however, which originally have identical couplings g_0, the effect of this rescaling procedure is to generate two *different* dynamical coupling constants in the first and second terms of Eq. (2.35), respectively, namely,

$$\tilde{g}_0 = \left(\frac{\tilde{D}_0}{D_0} \right)^{1/2} g_0, \quad \tilde{g}_0' = \Theta_0 \tilde{g}_0, \quad (2.36)$$

where

$$\Theta_0 = \frac{\tilde{\lambda}_0}{\lambda_0} \frac{D_0}{\tilde{D}_0}. \quad (2.37)$$

Thus, even if both equations (2.27) and (2.28) obey detailed balance separately, two different dynamic couplings will be generated as long as $T_S \neq T_M$, and then the new variable $\Theta_0 = T_S/T_M$ describes the deviation from equilibrium. With the two independent couplings $\tilde{g}_0$ and $\tilde{g}_0'$, the renormalization-group equations will become different as compared to the equilibrium situation, and new critical behavior may be expected at least in the extreme cases where the temperature ratio is either $\Theta_0 = 0$ or $\Theta_0 = \infty$. In Sec. III, we shall proceed with a detailed investigation of the one-loop flow equations of the nonequilibrium SSS model, as given by the field theory (2.31), (2.33), and (2.35), as function of the couplings (2.34), (2.36), and (2.37).

III. RENORMALIZATION OF THE NONEQUILIBRIUM SSS MODEL

A. Response functions and Ward identities

By adding source terms to the Janssen–De Dominicis functional (2.14), one arrives at the generating functional

$$\begin{aligned} Z[\{\tilde{h}^\alpha\},\{h^\alpha\}] \propto \int \mathcal{D}[\{i\tilde{\psi}_0^\alpha\}] \mathcal{D}[\{\psi_0^\alpha\}] e^{J[\{\tilde{\psi}_0^\alpha\},\{\psi_0^\alpha\}]} \\ \times \exp \int d^dx \int dt \sum_\alpha (\tilde{h}^\alpha \tilde{\psi}_0^\alpha + h^\alpha \psi_0^\alpha), \end{aligned} \quad (3.1)$$

and the $(\tilde{N}N)$-point correlation functions (cumulants) $G^c_{0\tilde{\psi}^{\tilde{N}}\psi^N}$ can be obtained from $\ln Z$ via functional derivatives with respect to the sources $\tilde{h}^\alpha$ and h^α, and then taking all $\tilde{h}^\alpha = h^\alpha = 0$. Following the usual field-theoretic techniques [24,13], we furthermore define the generating functional for the one-particle irreducible vertex functions using $\tilde{\phi}_0^\alpha = \delta \ln Z/\delta \tilde{h}^\alpha$ and $\phi_o^\alpha = \delta \ln Z/\delta h^\alpha$ via the Legendre transform

$$\Gamma[\{\widetilde{\phi}_0^\alpha\},\{\phi_0^\alpha\}] = -\ln Z[\{\widetilde{h}^\alpha\},\{h^\alpha\}] + \int d^dx \int dt \sum_\alpha \left(\widetilde{h}^\alpha \widetilde{\phi}_0^\alpha + h^\alpha \phi_0^\alpha\right); \tag{3.2}$$

the $(\widetilde{N}N)$-point vertex functions $\Gamma_{0\,\widetilde{\psi}^{\widetilde{N}}\psi^N}$ then follow via functional derivatives of Eq. (3.2) with respect to $\widetilde{\phi}_0^\alpha$ and ϕ_0^α.

With $\langle \psi_0^\alpha \rangle = 0$ we can write $\langle \psi_0^\alpha(\mathbf{x},t)\widetilde{\psi}_0^\beta(\mathbf{x}',t')\rangle = G^c_{0\,\widetilde{\psi}\psi}(\mathbf{x}-\mathbf{x}',t-t')\,\delta^{\alpha\beta}$, etc., and upon introducing the Fourier transform according to $\psi^\alpha(\mathbf{q},\omega) = \int d^dx \int dt\, \psi^\alpha(\mathbf{x},t) e^{-i(\mathbf{q}\cdot\mathbf{x}-\omega t)}$, one finds the following connections between the two-point correlation and vertex functions:

$$G^c_{0\,\widetilde{\psi}\psi}(\mathbf{q},\omega) = \Gamma_{0\,\widetilde{\psi}\psi}(-\mathbf{q},-\omega)^{-1}, \tag{3.3}$$

$$G^c_{0\,\psi\psi}(\mathbf{q},\omega) = -\frac{\Gamma_{0\,\widetilde{\psi}\widetilde{\psi}}(\mathbf{q},\omega)}{|\Gamma_{0\,\widetilde{\psi}\psi}(\mathbf{q},\omega)|^2}. \tag{3.4}$$

For the $(\widetilde{N}N)$-point functions with $\widetilde{N}N > 2$, relations similar to Eq. (3.4) hold; see Eq. (3.9) below.

In order to assign a meaning to the auxiliary fields, we compute the response functions for the SSS model by first adding external fields to the Hamiltonian (2.23) [13,22],

$$H \to H - \int d^dx \left[\sum_\alpha \widetilde{h}^\alpha S_0^\alpha + \sum_{\alpha>\beta} \widetilde{H}^{\alpha\beta} M_0^{\alpha\beta}\right], \tag{3.5}$$

which produces the following additional terms in the dynamic functional:

$$J \to J + \int d^dx \int dt \left[\lambda_0 \sum_\alpha \widetilde{h}^\alpha \widetilde{S}_0^\alpha - D_0 \sum_{\alpha>\beta} \widetilde{H}^{\alpha\beta}\nabla^2 \widetilde{M}_0^{\alpha\beta} + g_0 \sum_{\alpha,\beta}\left(\widetilde{h}^\alpha \widetilde{M}_0^{\alpha\beta} S_0^\beta - \widetilde{H}^{\alpha\beta}\widetilde{S}_0^\alpha S_0^\beta - \sum_\gamma \widetilde{H}^{\alpha\beta}\widetilde{M}_0^{\alpha\gamma} M_0^{\beta\gamma}\right)\right]. \tag{3.6}$$

Therefore the dynamic order-parameter susceptibility becomes

$$\chi_0(\mathbf{x}-\mathbf{x}',t-t')\,\delta^{\alpha\beta} = \left.\frac{\delta\langle S_0^\alpha(\mathbf{x},t)\rangle}{\delta\widetilde{h}^\beta(\mathbf{x}',t')}\right|_{\widetilde{h}^\beta=0} = \lambda_0\langle S_0^\alpha(\mathbf{x},t)\widetilde{S}_0^\beta(\mathbf{x}',t')\rangle + g_0\sum_\gamma \langle S_0^\alpha(\mathbf{x},t)[\widetilde{M}_0^{\beta\gamma}S_0^\gamma](\mathbf{x}',t')\rangle, \tag{3.7}$$

and similarly the response function for the conserved quantities reads

$$X_0(\mathbf{x}-\mathbf{x}',t-t')(\delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma}) = \left.\frac{\delta\langle M_0^{\alpha\beta}(\mathbf{x},t)\rangle}{\delta\widetilde{H}^{\gamma\delta}(\mathbf{x}',t')}\right|_{\widetilde{H}^{\gamma\delta}=0} = -D_0\langle M_0^{\alpha\beta}(\mathbf{x},t)\nabla^2\widetilde{M}_0^{\gamma\delta}(\mathbf{x}',t')\rangle - 2g_0\langle M_0^{\alpha\beta}(\mathbf{x},t)[\widetilde{S}_0^\gamma S_0^\delta](\mathbf{x}',t')\rangle. \tag{3.8}$$

[Note that $\Sigma_\rho\langle M_0^{\alpha\beta}(\mathbf{x},t)[\widetilde{M}_0^{\gamma\rho}M_0^{\delta\rho}](\mathbf{x}',t')\rangle = 0$.] Hence we also need cumulants containing composite operators $Y_0^\alpha = \Sigma_\beta[\widetilde{M}_0^{\alpha\beta}S_0^\beta]$ and $Y_0^{\alpha\beta} = [\widetilde{S}_0^\alpha S_0^\beta]$, as well as the corresponding vertex functions, which are related to each other via

$$G^c_{0SY}(\mathbf{q},\omega) = -\frac{\Gamma_{0\,\widetilde{S}Y}(\mathbf{q},\omega)}{\Gamma_{0\,\widetilde{S}S}(-\mathbf{q},-\omega)}. \tag{3.9}$$

Using Eqs. (3.4) and (3.9), we can finally write

$$\chi_0(\mathbf{q},\omega) = \Gamma_{0\,\widetilde{S}S}(-\mathbf{q},-\omega)^{-1}[\lambda_0 - g_0\Gamma_{0\,\widetilde{S}[\widetilde{M}S]}(\mathbf{q},\omega)], \tag{3.10}$$

$$X_0(\mathbf{q},\omega) = \Gamma_{0\,\widetilde{M}M}(-\mathbf{q},-\omega)^{-1}[D_0q^2 + 2g_0\Gamma_{0\,\widetilde{M}[\widetilde{S}S]}(\mathbf{q},\omega)]. \tag{3.11}$$

We conclude this discussion of general properties of the SSS model with the derivation of Ward identities, which are a consequence of the $O(n)$ symmetry, and the fact that the $M_0^{\alpha\beta}$ are the generators of this symmetry group [17,20,22]. As a first version, consider that a spatially homogeneous, but time-dependent external field $\widetilde{H}^{\alpha\beta}(t)$ is switched on at $t=0$. According to Eq. (3.5) and the equation of motion (2.27), this produces the following additional contribution to the expectation value of the order-parameter component S_0^α,

$$\langle S_0^\alpha(\mathbf{x},t)\rangle_{\widetilde{H}} = -g_0\int_0^t dt'\,\widetilde{H}^{\alpha\beta}(t')\langle S_0^\beta(\mathbf{x},t')\rangle_{\widetilde{H}}. \tag{3.12}$$

Upon employing suitable variational derivatives, this leads to the following relation between the nonlinear susceptibility

$$R_{0S;SM}(\mathbf{x},t;\mathbf{x}',t';\mathbf{x}'',t'')(\delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma}) = \left.\frac{\delta^2\langle S_0^\alpha(\mathbf{x},t)\rangle}{\delta\widetilde{h}^\beta(\mathbf{x}',t')\,\delta\widetilde{H}^{\gamma\delta}(\mathbf{x}'',t'')}\right|_{\widetilde{h}^\beta=\widetilde{H}^{\gamma\delta}=0} \tag{3.13}$$

and the order-parameter response:

$$\int d^dx' R_{0S;SM}(\mathbf{x},t;\mathbf{0},0;\mathbf{x}',t') = -g_0\Theta(t-t')\chi_0(\mathbf{x},t). \tag{3.14}$$

An equivalent Ward identity for vertex functions can be obtained by noting that the ''mixed'' generating functional $W[\{\widetilde{\phi}_0^\alpha\},\{\phi_0^\alpha\},\{\widetilde{H}^{\alpha\beta}\},\{H^{\alpha\beta}\}]$ [compare Eqs. (3.1), (3.2)] is invariant with respect to the following nontrivial variations corresponding to Eq. (3.12): $\delta\widetilde{H}^{\alpha\beta} = \varepsilon\widetilde{H}^{\alpha\beta}$, $\delta\phi_0^\alpha = -\varepsilon g_0 \Sigma_\beta \widetilde{H}^{\alpha\beta}\phi_0^\beta t$ [22]. Hence

$$\delta W=\frac{\varepsilon}{2}\int d^dx\int dt\sum_{\alpha,\beta}\widetilde{H}^{\alpha\beta}\left[\frac{\delta W}{\delta\widetilde{H}^{\alpha\beta}}-2g_0\frac{\delta W}{\delta\phi_0^\alpha}\phi_0^\beta t\right]=0, \tag{3.15}$$

which with $\widetilde{\mu}_0^{\alpha\beta}=\delta\ln Z/\delta\widetilde{H}^{\alpha\beta}$ translates to a Ward identity for the generating functional (3.2) of the vertex functions,

$$\int d^dx\int dt\sum_{\alpha,\beta}\frac{\delta\Gamma}{\delta\widetilde{\mu}_0^{\alpha\beta}}\left[\widetilde{\mu}_0^{\alpha\beta}-2g_0\frac{\delta\Gamma}{\delta\phi_0^\alpha}\phi_0^\beta t\right]=0. \tag{3.16}$$

Specifically, this yields

$$\Gamma_{0\widetilde{M}\widetilde{S}S}(\mathbf{q}/2,\omega/2;\mathbf{q}/2,\omega/2;-\mathbf{q},-\omega)$$
$$=g_0\frac{\partial}{\partial(i\omega)}[\Gamma_{0\widetilde{M}\widetilde{M}}(\mathbf{q}/2,\omega/2)\Gamma_{0\widetilde{S}S}(\mathbf{q}/2,\omega/2)]. \tag{3.17}$$

Note that Eqs. (3.14) and (3.17) hold quite independently of any detailed-balance condition.

We can use these Ward identities now to demonstrate that the mode-coupling constant g, as a consequence of the underlying $O(n)$ symmetry, does not renormalize [20]. First, we note that the *static* response function for the conserved angular momenta is *exactly*

$$X_0(\mathbf{q},\omega=0)\equiv 1, \tag{3.18}$$

as follows from the Hamiltonian (2.23) and the fact that in the limit $\omega\to 0$ there is no coupling between the critical fluctuations S_0^α and $M_0^{\alpha\beta}$, which is true even for our nonequilibrium model. Therefore there cannot be any field renormalization for the angular momenta: $M^{\alpha\beta}=Z_M^{1/2}M_0^{\alpha\beta}$ (and similarly for $\widetilde{M}^{\alpha\beta}$) with $Z_M\equiv 1$. Second, as a result of the q dependence of the mode-coupling vertices (2.35), to *all orders* in perturbation theory

$$\frac{\partial}{\partial(i\omega)}\Gamma_{0\widetilde{M}M}(\mathbf{q}=\mathbf{0},\omega)\equiv 1, \tag{3.19}$$

and hence $Z_{\widetilde{M}}Z_M\equiv 1$. We remark that an analogous equation for model B leads to the identity $z=4-\eta$ for the dynamic exponent [13]; a similar result for the Kardar-Parisi-Zhang (KPZ) equation implies the absence of field renormalizations there as well [25]. At last, we utilize the above Ward identities (3.14), (3.17), both of which imply that the renormalization factor for the mode-coupling constant is identical to $Z_g\equiv Z_M\equiv 1$. Physically, this means that the reversible mode couplings are not affected by critical fluctuations. This fact will lead to certain general identities for the dynamic exponent in the scaling regimes; see Sec. III C. Again, similar Ward identities may be derived and corresponding conclusions can be drawn for the mode-coupling constant in model J, as mentioned above, leading to the exact result $z=(d+2-\eta)/2$ [14], and also for the nonlinearity in the KPZ problem, there originating in the Galilean invariance of the equivalent Burgers equation, and implying the nontrivial scaling relation $z+\chi=2$ between the dynamic and roughness exponents [25].

B. Renormalization to one-loop order

Bearing the results of the previous subsection in mind, we introduce multiplicatively renormalized fields and parameters according to

$$\widetilde{S}^\alpha=Z_{\widetilde{S}}^{1/2}\widetilde{S}_0^\alpha,\quad S^\alpha=Z_S^{1/2}S_0^\alpha, \tag{3.20}$$

$$\widetilde{\lambda}=Z_{\widetilde{S}}^{-1}Z_{\widetilde{\lambda}}\widetilde{\lambda}_0,\quad \widetilde{D}=Z_{\widetilde{D}}\widetilde{D}_0, \tag{3.21}$$

$$\lambda=(Z_{\widetilde{S}}Z_S)^{-1/2}Z_\lambda\lambda_0,\quad D=Z_DD_0, \tag{3.22}$$

$$\tau=Z_S^{-1}Z_\tau\tau_0\mu^{-2},\quad \tau_0=r_0-r_{0c}, \tag{3.23}$$

$$u=Z_S^{-2}Z_uu_0A_d\mu^{d-4}. \tag{3.24}$$

Here, $A_d=\Gamma(3-d/2)/2^{d-1}\pi^{d/2}$ denotes a d-dependent geometric factor, and μ is a momentum scale. Thus all the renormalized couplings are dimensionless, as is $g=g_0A_d^{1/2}\mu^{(d-4)/2}$. Note that both the static and the mode-coupling constants u_0 and g_0 become dimensionless at the upper critical dimension $d_c=4$. We determine the renormalization constants (Z factors) by demanding that they absorb all the (ultraviolet) divergences in the corresponding vertex functions to one-loop order (see the Appendix). We furthermore employ the dimensional regularization scheme with minimal subtraction in $d=4-\epsilon$ dimensions, i.e., only include the ultraviolet poles $\propto 1/\epsilon$ in the Z factors, along with their residues in four dimensions (further details on these procedures can be found in Ref. [24]). In order to avoid the infrared singularities near the critical point, we take $\tau=1$ ($\tau_0=\mu^2$ to one-loop order) and $\mathbf{q}=\mathbf{0}$, $\omega=0$ as the normalization point. This, of course, follows closely the renormalization procedure for the equilibrium SSS model [20] (see also Ref. [21]).

Using $\Gamma_{\widetilde{S}\widetilde{S}}(\mathbf{q},\omega)=Z_{\widetilde{S}}^{-1}\Gamma_{0\,\widetilde{S}\widetilde{S}}(\mathbf{q},\omega)$, the renormalization of the noise strengths $\widetilde{\lambda}_0$ and $\widetilde{D}_0$, as well as of the diffusion constant D_0 is readily inferred from Eqs. (A3), (A4), and (A8), respectively, with the results

$$Z_{\widetilde{\lambda}}=1+\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{n-1}{1+w_0}w_0\overline{f_0}, \tag{3.25}$$

$$Z_{\widetilde{D}}=1+\frac{A_d\mu^{-\epsilon}}{2\epsilon}w_0\overline{f_0}\Theta_0^2, \tag{3.26}$$

$$Z_D=1+\frac{A_d\mu^{-\epsilon}}{2\epsilon}w_0\overline{f_0}\Theta_0, \tag{3.27}$$

where we have used the definitions (2.37) and

$$w_0=\frac{\lambda_0}{D_0},\quad \overline{f_0}=g_0^2\frac{\widetilde{D}_0}{\lambda_0^2D_0} \tag{3.28}$$

for the ratio of relaxation constants w_0 and effective dynamical coupling $\overline{f_0}$.

Next, we consider $\Gamma_{\widetilde{S}S}(\mathbf{q},\omega)=(Z_{\widetilde{S}}Z_S)^{-1/2}\Gamma_{0\widetilde{S}S}(\mathbf{q},\omega)$; see Eq. (A5). First, we determine the fluctuation-induced T_c shift r_{0c} from the condition of criticality $\chi_0(\mathbf{q}=\mathbf{0},\omega=0)^{-1}=0$,

which because of Eq. (3.10) is equivalent to demanding that $\Gamma_{0\tilde{S}S}(\mathbf{0},0)=0$ for $r_0=r_{0c}$. Equation (A7) then yields with Eq. (2.34)

$$r_{0c}=-\frac{n+2}{6}\tilde{u}_0\int_k\frac{1}{r_{0c}+k^2}-(n-1)w_0\overline{f_0}(1-\Theta_0)\int_k\frac{1}{w_0 r_{0c}+(1+w_0)k^2}; \tag{3.29}$$

note that for $d\leqslant 2$ the integrals on the right-hand side of Eq. (3.29) are infrared divergent. Evaluating the momentum integrals for $2<d<4$ gives explicitly

$$|r_{0c}|=\left(\frac{2A_d}{(d-2)(4-d)}\left[\frac{n+2}{6}\tilde{u}_0+(n-1)\times\left(\frac{w_0}{1+w_0}\right)^{d/2}\overline{f_0}(1-\Theta_0)\right]\right)^{2/(4-d)} \tag{3.30}$$

(notice the pole at $d_{lc}=2$ and the essential singularity at $d_c=4$). The first term here corresponds to the downwards shift of the critical temperature of the ϕ^4 model; the second contribution, which is of purely dynamical origin, may either reduce T_c further, namely, for $T_S<T_M$, or enhance it with respect to the equilibrium situation, if $T_S>T_M$.

Upon defining $\tau_0=r_0-r_{0c}$, the true distance from the critical point, and inserting Eq. (3.29) into (A7), setting $r_{0c}=0+O(u_0,g_0^2)$ in the integrals, one finds

$$Z_\tau\frac{Z_\lambda}{Z_S}=1-\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{n+2}{6}\tilde{u}_0+\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{n-1}{1+w_0}w_0\overline{f_0}\Theta_0-\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{(n-1)w_0^2}{(1+w_0)^2}\overline{f_0}(1-\Theta_0). \tag{3.31}$$

Then, rendering $\partial\Gamma_{\tilde{S}S}(\mathbf{0},\omega)/\partial(i\omega)$ and $\partial\Gamma_{\tilde{S}S}(\mathbf{q},0)/\partial q^2$ finite gives

$$(Z_{\tilde{S}}Z_S)^{1/2}=1-\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{(n-1)w_0^2}{(1+w_0)^2}\overline{f_0}(1-\Theta_0), \tag{3.32}$$

$$Z_\lambda=1+\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{n-1}{1+w_0}w_0\overline{f_0}\Theta_0-\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{(n-1)w_0^2}{(1+w_0)^3}\overline{f_0}(1-\Theta_0). \tag{3.33}$$

Equation (3.32) also absorbs the divergences in the three-point function (A11), which confirms that indeed $Z_g=1$. Equation (A12) can then be used to determine the still unknown field renormalization itself, with the result

$$Z_S=1-\frac{A_d\mu^{-\epsilon}}{2\epsilon}\frac{n-1}{(1+w_0)^2}w_0\overline{f_0}(1-\Theta_0); \tag{3.34}$$

notice that $Z_S\neq 1$ and $Z_{\tilde{S}}\neq 1$ already to one-loop order if $\Theta_0\neq 1$. At last, the multiplicative renormalization of the nonequilibrium SSS model vertex functions is concluded by rendering the four-point function (A13) finite with

$$Z_u\frac{Z_\lambda}{Z_S}=1-\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{n+8}{6}\tilde{u}_0+\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{n-1}{1+w_0}w_0\overline{f_0}-\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{(n-1)w_0^2}{(1+w_0)^2}\overline{f_0}(1-\Theta_0)-\frac{6A_d\mu^{-\epsilon}}{\epsilon}\frac{n-1}{1+w_0}\frac{(w_0\overline{f_0})^2}{\tilde{u}_0}\Theta_0(1-\Theta_0). \tag{3.35}$$

When detailed balance holds, $\Theta_0=1$, these one-loop Z factors reduce to the well-known equilibrium results [20].

Whereas the vertex functions and hence also the two-point correlation functions (3.4) are rendered finite with the above Z factors, the dynamic response functions may require additional additive renormalizations, as a consequence of the involved composite operators. The response function for the conserved angular momenta (3.11), using Eq. (3.18) for its static limit, can generally be written in the following form:

$$X_0(\mathbf{q},\omega)=\frac{\Delta_0(\mathbf{q},\omega)q^2}{-i\omega+\Delta_0(\mathbf{q},\omega)q^2}. \tag{3.36}$$

Using Eqs. (A6) and (A10), one finds the one-loop result

$$\Delta_0(\mathbf{q},\omega)=D_0\left[1+\frac{2}{d}w_0\overline{f_0}\Theta_0\int_k\frac{k^2}{\tau_0+(\mathbf{q}/2+\mathbf{k})^2}\frac{1}{\tau_0+(\mathbf{q}/2-\mathbf{k})^2}\frac{1}{-i\omega/2\lambda_0+\tau_0+q^2/4+k^2}\right]; \tag{3.37}$$

hence, as the ultraviolet singularity in Eq. (3.37) is absorbed by the Z factor (3.27), no additive renormalization is needed. This comes as no surprise, as the contribution from Eq. (A10) is nonsingular.

However, the integral Eq. (A9) is divergent, and therefore a corresponding additive renormalization has to be introduced. The structure of the order-parameter susceptibility (3.10) is

$$\chi_0(\mathbf{q},\omega)=\frac{\Lambda_0(\mathbf{q},\omega)}{-i\omega+\Lambda_0(\mathbf{q},\omega)/\chi_0(\mathbf{q},0)}. \tag{3.38}$$

Here, using Eqs. (A5) and (A9), the static susceptibility reads to one-loop order

$$\chi_0(\mathbf{q},0)^{-1}=\tau_0\Bigg[1-\frac{n+2}{6}\tilde{u}_0\int_k\frac{1}{k^2(\tau_0+k^2)}-\frac{n-1}{1+w_0}w_0^2\overline{f_0}(1-\Theta_0)\int_k\frac{1}{k^2[w_0\tau_0-(1-w_0)(\mathbf{q}\cdot\mathbf{k})+(1+w_0)(q^2/4+k^2)]}+q^2$$
$$+\frac{n-1}{1+w_0}w_0\overline{f_0}(1-\Theta_0)\int_k\frac{(1-w_0)(\mathbf{q}\cdot\mathbf{k})-(1+w_0)q^2/4}{k^2[w_0\tau_0-(1-w_0)(\mathbf{q}\cdot\mathbf{k})+(1+w_0)(q^2/4+k^2)]}\Bigg], \tag{3.39}$$

where Eq. (3.29) has been inserted, and the renormalized Onsager coefficient is

$$\Lambda_0(\mathbf{q},\omega)=\lambda_0\Bigg[1+(n-1)w_0\overline{f_0}\Theta_0\int_k\frac{1}{\tau_0+(\mathbf{q}/2+\mathbf{k})^2}\,\frac{1}{-i\omega/D_0+w_0[\tau_0+(\mathbf{q}/2+\mathbf{k})^2]+(\mathbf{q}/2-\mathbf{k})^2}\Bigg]; \tag{3.40}$$

as expected, the above multiplicative renormalizations with Eqs. (3.31)–(3.34) do *not* suffice to remove the divergences in Eqs. (3.39) and (3.40). We determine the necessary additive renormalization by requiring that

$$\frac{\partial}{\partial q^2}\chi_0(\mathbf{q},0)^{-1}_{\rm sing}=Z_S+A_S, \tag{3.41}$$

and Eqs. (3.39) and (3.34) then yield

$$A_S=-\frac{A_d\mu^{-\epsilon}}{\epsilon}\frac{n-1}{(1+w_0)^2}\left(\frac{1}{2}+\frac{w_0}{1+w_0}\right)w_0\overline{f_0}(1-\Theta_0). \tag{3.42}$$

Indeed, $\chi_0(\mathbf{0},0)^{-1}$ and $\Lambda_0(\mathbf{0},0)$ are then rendered finite with the combinations of Z factors $(Z_S+A_S)Z_\tau/Z_S$ and $Z_\lambda/(Z_S+A_S)$, respectively.

C. RG flow equations and fixed points

The renormalization-group equations serve to connect the asymptotic theory, where the infrared divergences become manifest, with a region in parameter space (in our case consisting of $\{a\}=\tilde{\lambda},\tilde{D},\lambda,D,g,u,\tau$) where the couplings are finite and ordinary ''naive'' perturbation expansion is applicable. They are derived by observing that the ''bare'' vertex functions do not depend on the renormalization scale μ,

$$\mu\frac{d}{d\mu}\Big|_0\,\Gamma_{0\tilde{S}^r\tilde{M}^k S^s M^l}(\{a_0\})=0. \tag{3.43}$$

Introducing Wilson's flow functions

$$\zeta_{\tilde{S}}=\mu\frac{\partial}{\partial\mu}\Big|_0\ln Z_{\tilde{S}},\zeta_S=\mu\frac{\partial}{\partial\mu}\Big|_0\ln Z_S, \tag{3.44}$$

$$\zeta_a=\mu\frac{\partial}{\partial\mu}\Big|_0\ln\frac{a}{a_0}, \tag{3.45}$$

Eq. (3.43) may be written as a partial differential equation for the renormalized vertex functions:

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_{\{a\}}\zeta_a a\frac{\partial}{\partial a}+\frac{r}{2}\zeta_{\tilde{S}}+\frac{s}{2}\zeta_S\right]\Gamma_{\tilde{S}^r\tilde{M}^k S^s M^l}(\mu,\{a\})=0. \tag{3.46}$$

Note that $\zeta_{\tilde{M}}=\zeta_M\equiv 0$ and $\zeta_g\equiv-\epsilon/2$ as a consequence of the exact results in Sec. III A. Equation (3.46) can be solved with the method of characteristics $\mu\to\mu\ell$; this defines running couplings as the solutions to the first-order differential RG flow equations

$$\ell\frac{da(\ell)}{d\ell}=\zeta_a(\ell)a(\ell),\quad a(1)=a. \tag{3.47}$$

The solution of the Callan-Symanzik equation (3.46) then reads

$$\Gamma_{\tilde{S}^r\tilde{M}^k S^s M^l}(\mu,\{a\},q,\omega)$$
$$=\exp\left\{\frac{1}{2}\int_1^\ell[r\zeta_{\tilde{S}}(\ell')+s\zeta_S(\ell')]\frac{d\ell'}{\ell'}\right\}$$
$$\times\Gamma_{\tilde{S}^r\tilde{M}^k S^s M^l}(\mu\ell,\{a(\ell)\},q/\mu\ell,\omega/\mu^2\ell^2). \tag{3.48}$$

Upon introducing the renormalized ratios

$$w=\frac{\lambda}{D},\quad \Theta=\frac{\tilde{\lambda}}{\lambda}\frac{D}{\tilde{D}} \tag{3.49}$$

and renormalized effective couplings

$$\overline{f}=g^2\frac{\tilde{D}}{\lambda^2 D},\quad \tilde{u}=\frac{\tilde{\lambda}}{\lambda}u, \tag{3.50}$$

and collecting the definitions Eqs. (3.20)–(3.23) and one-loop results (3.25)–(3.27) and (3.31)–(3.34), one finds

$$\zeta_S=\frac{n-1}{2}\frac{1}{(1+w)^2}w\overline{f}(1-\Theta), \tag{3.51}$$

$$\zeta_{\tilde{S}}=-\frac{n-1}{2}\frac{1-4w}{(1+w)^2}w\overline{f}(1-\Theta), \tag{3.52}$$

$$\zeta_\tau=-2+\frac{n+2}{6}\tilde{u}+\frac{(n-1)w^2}{(1+w)^3}w\overline{f}(1-\Theta), \tag{3.53}$$

$$\zeta_{\tilde{\lambda}}=-\frac{n-1}{1+w}w\overline{f}+\frac{n-1}{2}\frac{1-4w}{(1+w)^2}w\overline{f}(1-\Theta), \tag{3.54}$$

$$\zeta_{\tilde{D}}=-\frac{1}{2}w\overline{f}\Theta^2, \tag{3.55}$$

$$\zeta_\lambda = -\frac{n-1}{1+w} w\overline{f\Theta} - \frac{(n-1)w^2}{(1+w)^3} w\overline{f}(1-\Theta), \quad (3.56)$$

$$\zeta_D = -\frac{1}{2} w\overline{f\Theta}. \quad (3.57)$$

Notice that nonzero values of $\zeta_{\tilde{\lambda}}$ and $\zeta_{\tilde{D}}$ induce anomalous noise correlations, while ζ_λ and ζ_D determine the dynamic critical exponents; see Eq. (3.66) below.

We furthermore need the flows for the running couplings $v(\ell)$, with $\{v\} = w, \Theta, \overline{f}, \tilde{u}$,

$$\ell \frac{dv(\ell)}{d\ell} = \beta_v(\ell), \quad v(1) = v, \quad (3.58)$$

as given by the β functions

$$\beta_v = \mu \frac{\partial}{\partial \mu}\bigg|_0 v; \quad (3.59)$$

with Eqs. (3.24), (3.35), and (3.54)–(3.57) these become to one-loop order

$$\beta_w = w(\zeta_\lambda - \zeta_D) = w^2 \overline{f}\left[\left(\frac{1}{2} - \frac{n-1}{1+w}\right)\Theta - \frac{(n-1)w^2}{(1+w)^3}(1-\Theta)\right], \quad (3.60)$$

$$\beta_\Theta = \Theta(\zeta_{\tilde{\lambda}} - \zeta_{\tilde{D}} - \zeta_\lambda + \zeta_D) = -\frac{1}{2} w\overline{f\Theta}(1-\Theta)\left[\Theta + (n-1)\frac{1+7w+4w^2}{(1+w)^3}\right], \quad (3.61)$$

$$\beta_{\overline{f}} = \overline{f}(-\epsilon + \zeta_{\tilde{D}} - 2\zeta_\lambda - \zeta_D) = \overline{f}\left[-\epsilon - \frac{1}{2} w\overline{f\Theta}^2 + \left(\frac{1}{2} + \frac{2(n-1)}{1+w}\right) w\overline{f\Theta} + \frac{2(n-1)w^2}{(1+w)^3} w\overline{f}(1-\Theta)\right], \quad (3.62)$$

$$\beta_{\tilde{u}} = \tilde{u}\left[-\epsilon + \frac{n+8}{6}\tilde{u} - \frac{2(n-1)w}{(1+w)^3} w\overline{f}(1-\Theta) - \frac{2(n-1)}{1+w}\left(1 - \frac{3w\overline{f\Theta}}{\tilde{u}}\right) w\overline{f}(1-\Theta)\right]. \quad (3.63)$$

We are now ready to explore the fixed points of the RG flow equations, as given by the zeros of the β functions (3.60)–(3.63). First, we can check that indeed for $\Theta^* = 1$ the equilibrium fixed points (see Ref. [20]) emerge. The above flow equations then simplify considerably, and the effective dynamical coupling in Eqs. (3.54)–(3.57) becomes

$$f = w\overline{f} = \frac{g^2}{\lambda D}, \quad (3.64)$$

because as now $\zeta_{\tilde{\lambda}} \equiv \zeta_\lambda$ and $\zeta_{\tilde{D}} \equiv \zeta_D$, we can identify the noise strengths $\tilde{\lambda}$ and $\tilde{D}$ with the Onsager coefficients λ and D, respectively. The corresponding β function for f reads

$$\beta_f = f(-\epsilon - \zeta_\lambda - \zeta_D) = f\left[-\epsilon + \left(\frac{1}{2} + \frac{n-1}{1+w}\right) f\right]. \quad (3.65)$$

The first equation in Eq. (3.65) implies that for *any* nontrivial fixed point $0 < f^* < \infty$ the *exact* relation $\zeta_\lambda^* + \zeta_D^* = -\epsilon = d - 4$ holds. Furthermore the analysis of the RG equation in the vicinity of f^* reveals that the dynamic exponents for the fluctuations of the order-parameter and conserved quantities are given by

$$z_S = 2 + \zeta_\lambda^*, \quad z_M = 2 + \zeta_D^*, \quad (3.66)$$

which then leads to the following identity [17,20,22]:

$$z_s + z_M = d. \quad (3.67)$$

Therefore, in a *strong-scaling* situation where the characteristic time scales for the order parameter and angular momenta are the same, $0 < w^* < \infty$, and hence $z_S = z_M = z$, one finds the well-known *exact* result

$$z = d/2. \quad (3.68)$$

Indeed, the above one-loop flow equations (3.60) and (3.65) provide the strong-scaling fixed point

$$w_{\rm eq}^* = 2n - 3, \quad f_{\rm eq}^* = \epsilon. \quad (3.69)$$

However, in addition there are two nontrivial *weak-scaling* fixed points with $\zeta_\lambda^* \neq \zeta_D^*$, namely,

$$w_w^* = 0, \quad f_w^* = \frac{2\epsilon}{2n-1}, \quad (3.70)$$

with

$$z_S = 2 - \frac{2(n-1)\epsilon}{2n-1} + O(\epsilon^2),$$

$$z_M = 2 - \frac{\epsilon}{2n-1} + O(\epsilon^2), \quad (3.71)$$

and

$$w_{w'}^* = \infty, \quad f_{w'}^* = 2\epsilon, \quad (3.72)$$

implying that

$$z_S = 2, \quad z_M = d - 2. \quad (3.73)$$

Note that at both fixed points the relation $z_S + z_M = d$ holds, of course. For the fixed point (3.72) this sum rule actually even implies that Eq. (3.73) is probably exact, for ζ_λ should vanish if $w^* = \infty$. Finally, there is also the model A fixed point with $f_0^* = 0$ (and w^* unspecified because now the order-parameter and angular momenta are decoupled), with $z_S = 2 + O(\epsilon^2)$; yet, according to Eq. (3.65), it is clearly unstable against f for $d < 4$: $\ell df/d\ell = \beta_f(\ell) = -\epsilon f$ at

$f_0^*=0$, and hence f will increase in the asymptotic limit $\ell\to 0$. Similarly, both weak-scaling fixed points are unstable (to one-loop order at least) for $d<4$: Near $w_{w'}^*=\infty$ one has $\beta_w=\epsilon w$, and hence w will decrease as $\ell\to 0$, while in the vicinity of $w_w^*=0$ one finds $\beta_w=-\epsilon w(2n-3)/(2n-1)$, and upon decreasing ℓ, w will go down as well. Stability analysis therefore demonstrates that to one-loop order the *strong-scaling* fixed point with $z=d/2$ is *stable* [17,19,20]; however, as $w_w^*=0$ is actually close to its stability boundary for $n=2$, it may well be that to higher-loop orders the strong-scaling fixed point actually becomes unstable for the planar model, and in fact Eq. (3.68) does not hold [20,23].

We shall not pursue this issue further here, but rather turn to the newly emerging, genuinely *nonequilibrium* fixed points, at which even the static critical behavior might be changed, as opposed to the equilibrium situation with $\Theta^*=1$ for which statics and dynamics decouple; see Eqs. (3.51), (3.53), and (3.63). For all the above fixed points, we therefore have the nontrivial static Heisenberg fixed point (to one-loop order)

$$u_{\mathrm{H}}^*=\frac{6\epsilon}{n+8}, \tag{3.74}$$

which is stable for $d<4$, and leads to the critical exponents of the $O(n)$-symmetric $\boldsymbol{\phi}^4$ model

$$\eta=-\zeta_S^*=0+lO(\epsilon^2),$$

$$1/\nu=-\zeta_\tau^*=2-\frac{(n+2)\epsilon}{n+8}+O(\epsilon^2). \tag{3.75}$$

At the critical point there also appear anomalous long-range noise correlations, both for the fluctuations of the order-parameter and of the conserved fields. For the order-parameter noise, given by $\Gamma_{\tilde S\tilde S}(\mathbf{q},\omega)$, one finds

$$\langle\zeta^\alpha(\mathbf{q},\omega)\zeta^\beta(\mathbf{q}',\omega')\rangle = \Lambda(\mathbf{q},\omega)\,\delta(\mathbf{q}+\mathbf{q}')\,\delta(\omega+\omega')\,\delta^{\alpha\beta}, \tag{3.76}$$

with

$$\Lambda(\mathbf{q},0)\propto q^{z_S-2},\quad \Lambda(\mathbf{0},\omega)\propto\omega^{(z_S-2)/2}. \tag{3.77}$$

Similarly, from $\Gamma_{\tilde M\tilde M}(\mathbf{q},\omega)$ we can infer the noise for the generalized angular momenta at the critical point, taking into account its diffusive character,

$$\langle\eta^{\alpha\beta}(\mathbf{q},\omega)\,\eta^{\gamma\delta}(\mathbf{q}',\omega')\rangle=\Delta(\mathbf{q},\omega)\,\delta(\mathbf{q}+\mathbf{q}')\,\delta(\omega+\omega') \times(\delta^{\alpha\gamma}\delta^{\beta\delta}-\delta^{\alpha\delta}\delta^{\beta\gamma}); \tag{3.78}$$

taking into account its diffusive character, we find the limiting behavior

$$\Delta(\mathbf{q},0)\propto q^{z_M},\quad \Delta(\mathbf{q}\to\mathbf{0},\omega/q^2)\propto(\omega/q^2)^{z_M/2}. \tag{3.79}$$

This concludes our discussion of the equilibrium fixed points, and we now turn to the two new universality classes appearing as a consequence of our genuinely nonequilibrium perturbation.

The β function for the temperature ratio $\Theta_0=T_S/T_M$ (3.61) reveals that there can only be nonequilibrium fixed points with either (i) $\Theta^*=0$ or (ii) $\Theta^*=\infty$, which as $T_S\approx T_c$ effectively either correspond to a ''renormalized'' temperature $T_M=\infty$ or $T_M=0$. In the first case, $\Theta^*=0$, one finds $\beta_w=-(n-1)wfw^3/(1+w)^3$, with the stable fixed point $w^*=\infty$. Inserting this into Eqs. (3.62) and (3.63) yields

$$\Theta^*=0:\quad w^*=\infty,\quad \overline{f^*}=\frac{\epsilon}{2(n-1)},$$

$$\tilde u^*=2u_H^*=\frac{12\epsilon}{n+8}. \tag{3.80}$$

The divergence of w^* already shows that this fixed point describes a kind of weak-scaling behavior somewhat similar to the equilibrium fixed point (3.72) with (3.73). Indeed, inserting Eq. (3.80) into Eqs. (3.56) and (3.57) yields

$$z_S=d/2,\quad z_M=2, \tag{3.81}$$

i.e., the dynamic exponent for the order parameter is identical to its equilibrium value, while the angular momenta are described by mean-field theory (simple diffusion). Actually, any nontrivial fixed point $0<f^*<\infty$ via Eqs. (3.62) and (3.66) implies the identity $2z_S+z_M=d+2+\zeta_D^*$. As for $\Theta^*=0$ both the anomalous dimensions (3.57) and (3.55) should vanish according to the general structure of the couplings, the result (3.81) is probably exact. There is, however, a nonzero anomalous dimension for the order-parameter noise, $\zeta_{\tilde\lambda}^*=-\epsilon$; therefore the constant $\tilde\lambda_0$ in Eq. (2.29) is to be replaced by a wave-vector- and frequency-dependent function,

$$\langle\zeta^\alpha(\mathbf{q},\omega)\zeta^\beta(\mathbf{q}',\omega')\rangle = \tilde\Lambda(\mathbf{q},\omega)\,\delta(\mathbf{q}+\mathbf{q}')\,\delta(\omega+\omega')\,\delta^{\alpha\beta}, \tag{3.82}$$

with the singular large-wavelength and low-frequency behavior ($d<4$)

$$\tilde\Lambda(\mathbf{q},0)\propto q^{d-4},\quad \tilde\Lambda(\mathbf{0},\omega)\propto\omega^{(d-4)/2}, \tag{3.83}$$

which follows from the matching conditions $\mu\ell=q$ and $(\mu\ell)^2=\omega$, respectively. Finally, the new nonequilibrium static fixed point $\tilde u^*=2u_H^*$, along with the fact that now the dynamics affects the static anomalous dimension (3.53), yields the *new* static critical exponents

$$\eta=-\zeta_S^*=0+O(\epsilon^2),$$

$$1/\nu=-\zeta_\tau^*=2-\left(\frac{1}{2}+\frac{2(n+2)}{n+8}\right)\epsilon+O(\epsilon^2). \tag{3.84}$$

According to Eq. (2.36), the fixed point (3.80) corresponds to the situation where there exists a coupling $\propto g_0$ of the order parameter to the angular momenta, leading to the dynamic exponent $z_S=d/2$, the generation of long-range noise correlations, and even to anomalous static exponents, yet the dynamics of the conserved quantities themselves remains unaffected by the critical fluctuations and hence displays mean-field behavior.

The above analysis has tacitly assumed that $\widetilde{w}=w\Theta=\widetilde{\lambda}/\widetilde{D}$ remains finite for $\Theta^*=0$ and $w^*=\infty$. However, $\beta_{\widetilde{w}}=\widetilde{w}(\zeta_{\widetilde{\lambda}}-\zeta_{\widetilde{D}})=-\epsilon\widetilde{w}$, and hence the fixed point (3.80) is *unstable* for $d<4$ against an increasing coupling $\widetilde{w}$, or equivalently, against the generation of the new effective coupling

$$\widetilde{f}=w\overline{f}\Theta^2=g^2\frac{\widetilde{\lambda}^2}{\lambda^3\widetilde{D}}, \tag{3.85}$$

which characterizes the second nonequilibrium fixed point where $\Theta^*=\infty$, and which describes a coupling of the critical order-parameter fluctuations into the diffusive dynamics of the angular momenta, but no effect of the latter on the equation of motion for the order parameter itself. Indeed, the anomalous dimensions (3.51)–(3.54), (3.56), (3.57) all vanish when $\Theta\to\infty$ with $\widetilde{f}$ held finite, and with Eq. (3.63) one finds the standard $\boldsymbol{\phi}^4$ Heisenberg static exponents (3.75) along with model A and purely diffusive dynamics for the order-parameter and conserved quantities, respectively,

$$z_S=2+O(\epsilon^2), \quad z_M=2; \tag{3.86}$$

note that w^* cannot be specified because of this decoupling of the modes. The β function for the effective mode coupling (3.85) becomes $\beta_{\widetilde{f}}=\widetilde{f}(-\epsilon+2\zeta_{\widetilde{\lambda}}-3\zeta_\lambda-\zeta_{\widetilde{D}})=\widetilde{f}(-\epsilon+\widetilde{f}/2)$, and hence we arrive at the following fixed-point values:

$$\Theta^*=\infty: \quad \widetilde{f}^*=2\epsilon, \quad \widetilde{u}^*=u_H^*. \tag{3.87}$$

At this fixed point anomalous noise correlations for the angular momenta emerge, which in analogy with Eq. (3.82) can be written in the form

$$\langle\eta^{\alpha\beta}(\mathbf{q},\omega)\,\eta^{\gamma\delta}(\mathbf{q}',\omega')\rangle=\widetilde{\Delta}(\mathbf{q},\omega)\,\delta(\mathbf{q}+\mathbf{q}')\,\delta(\omega+\omega')\times(\delta^{\alpha\gamma}\delta^{\beta\delta}-\delta^{\alpha\delta}\delta^{\beta\gamma}), \tag{3.88}$$

replacing Eq. (2.30). Their singular behavior follows from Eq. (3.55) with (3.87),

$$\widetilde{\Delta}(\mathbf{q},0)\propto q^\rho, \quad \rho=d-2, \tag{3.89}$$

which is probably an exact result again, because the existence of a nontrivial fixed point $\widetilde{f}^*$ with $\rho=2+\zeta_{\widetilde{D}}^*$ implies the relation $3z_S+\rho=d+4+2\zeta_{\widetilde{\lambda}}^*$, and both the structure of the perturbation theory and the above physical interpretation require that $\zeta_\lambda^*=\zeta_{\widetilde{\lambda}}^*=0$. Similarly, the frequency dependence of $\widetilde{\Delta}(\mathbf{q},\omega)$ displays anomalous behavior, but because of the underlying diffusive dynamics we now have to take the limit $\mathbf{q}\to\mathbf{0}$ more carefully, namely, with ω/q^2 held fixed. One then gets

$$\widetilde{\Delta}(\mathbf{q}\to\mathbf{0},\omega/q^2)\propto(\omega/q^2)^{\rho/2}. \tag{3.90}$$

It is a remarkable fact that these anomalous noise correlations always appear for those degrees of freedom *towards* which the energy flows, i.e., those quantitites that are in contact with the heat bath at *lower* temperature. It should be noticed that the power laws in Eqs. (3.83), (3.89), and (3.90) are *not* determined by the corresponding dynamic exponents, as opposed to the equilibrium situation described by Eqs. (3.77) and (3.79), which are consequences of detailed balance. Yet again, inspection of Eq. (3.62) in the vicinity of Eq. (3.87), $\beta_{\overline{f}}=-2\epsilon f$, shows that this second new fixed point is *unstable* with respect to the flow of f.

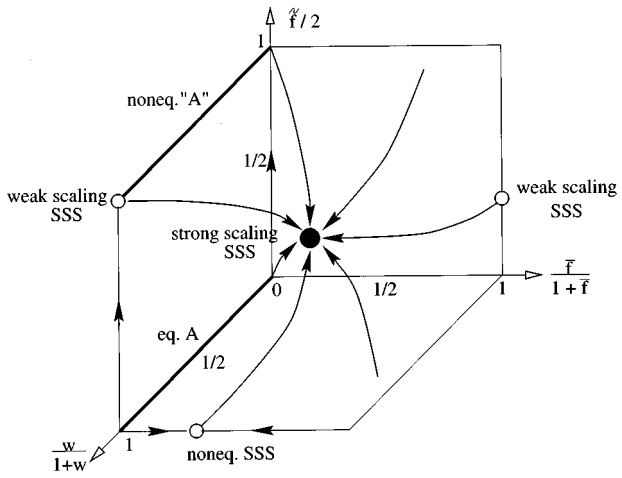

FIG. 1. One-loop flow diagram for the nonequilibrium SSS model in the space of dynamical couplings $w/(1+w)$, $f/(1+f)$, and $\widetilde{f}/2$ (displayed for the case $n=2$, $\epsilon=1$). Asymptotically, the equilibrium strong-scaling fixed point $w_{\rm eq}^*=2n-3$, $\widetilde{f}_{\rm eq}^*=\epsilon$, $f_{\rm eq}^*=\epsilon/(2n-3)$ is stable; in the figure, it is located at the point (1/2,1/2,1/2) (full circle).

Therefore, as both new nonequilibrium fixed points are actually unstable for $d<4$, the *asymptotic critical behavior* must be governed by the *equilibrium* (strong-scaling) fixed point (3.69), (3.74) with dynamic exponent (3.68) and the usual static critical exponents (3.75). The stability against perturbations away from $\Theta^*=1$ is readily demonstrated by observing that $\beta_\Theta=-C(1-\Theta)$, with $C>0$ according to Eq. (3.61). Hence any deviation from the equilibrium fixed point will be counteracted by the flow of the coupling Θ; i.e., whenever the initial value of Θ is neither zero or infinite, the flow will asymptotically approach the stable equilibrium fixed point, and thereby detailed balance is *dynamically restored*. This is illustrated in Fig. 1, which displays the location of the fixed points and their flow in the space of the dynamical couplings w, f, and $\widetilde{f}$; notice that both the equilibrium and ''nonequilibrium'' model A fixed points are represented by lines here because of the undetermined value of w^*. Concluding this section, we remark that at each of the fixed points discussed above, including both new nonequilibrium fixed points, the anomalous dimension stemming from the additive renormalization (3.42) vanishes, as does ζ_S from the field renormalization. We therefore did not have to take its effects into account explicitly.

IV. SUMMARY AND CONCLUSIONS

We have studied the critical dynamics of $O(n)$-symmetric systems, including reversible mode-coupling terms, in the framework of effective Langevin equations and dynamic field theory, and generalized the equations of mo-

tions to nonequilibrium situations where detailed balance is broken. In Sec. II B we have argued that for the dynamics of isotropic Heisenberg ferromagnets the effect of violating the Einstein relation between the spin diffusion constant and the Langevin noise strength can be absorbed via rescaling the static nonlinearity and the mode-coupling constant. Hence universal properties cannot be affected by this specific form of detailed-balance violation, and the critical point is described by the usual Heisenberg model exponents with the equilibrium dynamic critical exponent $z=(d+2-\eta)/2$ [14]. This fact generalizes previous results regarding the stability of the relaxational models *A* and *B* against nonequilibrium perturbations [2,3,4] to a situation where reversible mode-coupling terms are present as well.

On the other hand, for the critical dynamics of planar ferromagnets (model *E*) [17] or isotropic antiferromagnets (model *G*) [19], both incorporated in the $O(n)$-symmetric SSS model [16], such a simple rescaling does not remove the effects of detailed-balance violation completely, as discussed in Sec. II C. This is because there appears a new degree of freedom, namely, the temperature ratio Θ_0 of the heat baths to which the order parameter and the conserved angular momenta are attached. This new variable induces different renormalizations for the noise strengths as compared to the Onsager coefficients (Sec. III B), and therefore genuinely new dynamic and static critical behavior may emerge. Indeed, there appear two new fixed points of the resulting RG flow equations, describing continuous phase transitions of entirely nonequilibrium character, namely, corresponding to either $\Theta=0$ or $\Theta=\infty$, where in the former case even the static critical exponents become modified. The ensuing dynamic exponents may be interpreted physically by noting that in both cases either the coupling of the order parameter into the diffusion equation for the conserved fields vanishes while the reverse coupling remains effective or vice versa. To one-loop order, however, the stability analysis in Sec. III C shows that both of these fixed points are actually unstable, and provided $0<\Theta_0<\infty$ the asymptotic critical behavior is governed by the nontrivial strong-scaling equilibrium fixed point of the SSS model characterized again by the static exponents of the $O(n)$-symmetric Heisenberg model and by the dynamic exponent $z=d/2$ [16,20].

This suggests that the role of detailed balance is actually a weaker one as compared to internal symmetries; for while breaking a discrete or continuous symmetry typically results in a change of the universality class, detailed balance becomes restored here at the critical point, obviously as a consequence of the underlying $O(n)$ rotation symmetry and the spatial isotropy of the model. We remark that there are some notable exceptions, though, for the relevance of symmetry-breaking terms. For example, the $O(n)$ symmetry of the Heisenberg model (2.1) is restored as a consequence of the large critical fluctuations, even when cubic anisotropies are added, if $n<n_c$ (with $n_c=4$ to one-loop order) [26]. We should also cautiously state that our above results relied on the one-loop approximation only, and specifically the stability boundaries might change in more accurate calculations, as seems indeed to be the case for the equilibrium planar model [23]. However, in the β functions $\beta_{\tilde{w}}$ and $\beta_{\bar{f}}$ describing the instabilities of the fixed points (3.80) and (3.87), respectively, no dangerous n dependences appeared, and it is therefore probably safe to say that these nonequilibrium fixed points will remain unstable even to higher orders in perturbation theory, albeit for $n=2$ the asymptotic region may ultimately be governed by the weak-scaling equilibrium fixed point (3.70); see also Fig. 1.

The remarkable result that violation of detailed balance appears to be an irrelevant perturbation in the RG sense, at least in rotation-invariant and isotropic systems, is of course strengthening the notion of universality even in nonequilibrium situations. Probably in many experiments probing the critical dynamics with electromagnetic radiation or neutron scattering, the system is not perfectly thermalized and variations in the effective temperatures for the different degrees of freedom cannot be avoided completely due to the inevitable critical slowing down that prevents fast relaxation processes. Now if such a perturbation were relevant, its effects would become enhanced drastically in the vicinity of the phase transition, however small the initial deviations from equilibrium might have been. Such a behavior is apparently not observed; however, it might be interesting to prepare a nonthermalized system on purpose, say by arranging for the in-plane spins S^x and S^y in a planar ferromagnet to be on a different temperature than S^z, conceivably attainable with polarized electromagnetic radiation. Another possibility would be to introduce a long-wavelength magnetic field with random time variation. In an antiferromagnetic material, its effect would cancel for the staggered magnetization but the coupling to the magnetization would mimic a situation with $T_M=\infty$. For both experimental realizations, or at least in equivalent computer simulations studies, the effects of the above two new fixed points may then be explored, for these should manifest themselves in nontrivial crossover behavior, and perhaps even the associated anomalous critical exponents and noise correlations might be detected.

We finally remark again that the specific violation of detailed balance investigated here was *isotropic* in character, thereby disturbing neither of the underlying symmetries of the SSS model, namely, $O(n)$ symmetry and its spatial isotropy. Obviously, it would be interesting to see if the above stability against nonequilibrium perturbations persists even when the detailed-balance violation is applied in an anisotropic manner, e.g., by coupling the order parameter to conserved angular momenta as above, but arranging the noise strength of the conserved fields to be related to different temperatures in different space directions. The study of this, or similar direction- or scale-dependent nonequilibrium perturbations provides a promising venue for further research.

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APPENDIX A: EXPLICIT ONE-LOOP RESULTS FOR THE VERTEX FUNCTIONS

In this Appendix, we present a list of the results to one-loop order in perturbation theory for those vertex function that are required for the renormalization of the nonequilibrium SSS model defined by Eqs. (2.27)–(2.30), or, equivalently, by the field theory (2.31), (2.33), and (2.35). In all the subsequent expressions, the internal frequency integrations have already been carried out via the residue theorem. We use the abbreviation $\int_k \cdots \equiv (2\pi)^{-d} \int d^d k \cdots$. We do not explicitly provide the Feynman diagrams themselves, as they are identical with those of the equilibrium model [20].

We begin with the two-point vertex functions renormalizing the noise strengths $\tilde{\lambda}_0$ and $\tilde{D}_0$,

$$\Gamma_{0\tilde{S}\tilde{S}}(\mathbf{q},\omega) = -2\tilde{\lambda}_0 \Bigg[1 + (n-1) g_0^2 \frac{\tilde{D}_0}{\lambda_0 D_0} \int_k \frac{1}{r_0 + (\mathbf{q}/2+\mathbf{k})^2} \Bigg(\frac{D_0 (\mathbf{q}/2-\mathbf{k})^2}{(\omega - i\lambda_0 [r_0 + (\mathbf{q}/2+\mathbf{k})^2])^2 + D_0^2 (\mathbf{q}/2-\mathbf{k})^4} + \frac{\lambda_0 [r_0 + (\mathbf{q}/2+\mathbf{k})^2]}{[\omega + i D_0 (\mathbf{q}/2-\mathbf{k})^2]^2 + \lambda_0^2 [r_0 + (\mathbf{q}/2+\mathbf{k})^2]^2} \Bigg) \Bigg], \tag{A1}$$

$$\Gamma_{0\tilde{M}\tilde{M}}(\mathbf{q},\omega) = -2\tilde{D}_0 q^2 \Bigg[1 + 4 g_0^2 \frac{\tilde{\lambda}_0^2}{\lambda_0 \tilde{D}_0} \int_k (\mathbf{q}\cdot\mathbf{k})^2 \Bigg(\frac{1}{r_0 + (\mathbf{q}/2+\mathbf{k})^2} \frac{1}{(\omega - i\lambda_0 [r_0 + (\mathbf{q}/2+\mathbf{k})^2])^2 + \lambda_0^2 [r_0 + (\mathbf{q}/2-\mathbf{k})^2]^2} + \frac{1}{r_0 + (\mathbf{q}/2-\mathbf{k})^2} \frac{1}{(\omega + i\lambda_0 [r_0 + (\mathbf{q}/2-\mathbf{k})^2])^2 + \lambda_0^2 [r_0 + (\mathbf{q}/2+\mathbf{k})^2]^2} \Bigg) \Bigg]. \tag{A2}$$

Consequently, we have

$$\Gamma_{0\tilde{S}\tilde{S}}(\mathbf{0},0) = -2\tilde{\lambda}_0 \Bigg[1 + (n-1) g_0^2 \frac{\tilde{D}_0}{\lambda_0 D_0} \int_k \frac{1}{r_0 + k^2} \frac{1}{\lambda_0 (r_0 + k^2) + D_0 k^2} \Bigg], \tag{A3}$$

$$\frac{\partial}{\partial q^2} \Gamma_{0\tilde{M}\tilde{M}}(\mathbf{q},0) \Big|_{\mathbf{q}=\mathbf{0}} = -2\tilde{D}_0 \Bigg[1 + \frac{2}{d} g_0^2 \frac{\tilde{\lambda}_0^2}{\lambda_0^3 \tilde{D}_0} \int_k \frac{k^2}{(r_0 + k^2)^3} \Bigg]. \tag{A4}$$

For the computation of the response functions, one needs the two-point functions

$$\begin{aligned} \Gamma_{0\tilde{S}S}(\mathbf{q},\omega) = {} & i\omega + \lambda_0 (r_0 + q^2) + \frac{n+2}{6} \tilde{\lambda}_0 u_0 \int_k \frac{1}{r_0 + k^2} \\ & + (n-1)(r_0 + q^2) g_0^2 \frac{\tilde{\lambda}_0}{\lambda_0} \int_k \frac{1}{r_0 + (\mathbf{q}/2+\mathbf{k})^2} \frac{1}{i\omega + \lambda_0 [r_0 + (\mathbf{q}/2+\mathbf{k})^2] + D_0 (\mathbf{q}/2-\mathbf{k})^2} \\ & + (n-1) g_0^2 \frac{\tilde{D}_0}{D_0} \Bigg(1 - \frac{\tilde{\lambda}_0}{\lambda_0} \frac{D_0}{\tilde{D}_0} \Bigg) \int_k \frac{1}{i\omega + \lambda_0 [r_0 + (\mathbf{q}/2+\mathbf{k})^2] + D_0 (\mathbf{q}/2-\mathbf{k})^2}, \end{aligned} \tag{A5}$$

$$\Gamma_{0\tilde{M}M}(\mathbf{q},\omega) = i\omega + D_0 q^2 - 4 g_0^2 \frac{\tilde{\lambda}_0}{\lambda_0} \int_k \frac{(\mathbf{q}\cdot\mathbf{k})}{r_0 + (\mathbf{q}/2+\mathbf{k})^2} \frac{1}{i\omega + 2\lambda_0 (r_0 + q^2/4 + k^2)}; \tag{A6}$$

specifically,

$$\begin{aligned} \Gamma_{0\tilde{S}S}(\mathbf{0},0) = {} & \lambda_0 \Bigg[r_0 \Bigg(1 + (n-1) g_0^2 \frac{\tilde{\lambda}_0}{\lambda_0^2} \int_k \frac{1}{r_0 + k^2} \frac{1}{\lambda_0 (r_0 + k^2) + D_0 k^2} \Bigg) + \frac{n+2}{6} \frac{\tilde{\lambda}_0}{\lambda_0} u_0 \int_k \frac{1}{r_0 + k^2} \\ & + (n-1) g_0^2 \frac{\tilde{D}_0}{\lambda_0 D_0} \Bigg(1 - \frac{\tilde{\lambda}_0}{\lambda_0} \frac{D_0}{\tilde{D}_0} \Bigg) \int_k \frac{1}{\lambda_0 (r_0 + k^2) + D_0 k^2} \Bigg], \end{aligned} \tag{A7}$$

$$\frac{\partial}{\partial q^2} \Gamma_{0\tilde{M}M}(\mathbf{q},0) \Big|_{\mathbf{q}=\mathbf{0}} = D_0 \Bigg[1 + \frac{2}{d} g_0^2 \frac{\tilde{\lambda}_0}{\lambda_0^2 D_0} \int_k \frac{k^2}{(r_0 + k^2)^3} \Bigg]. \tag{A8}$$

Furthermore, the following vertex functions containing composite operators are required [note Eq. (3.9)]:

$$\Gamma_{0\tilde{S}[\tilde{M}S]}(\mathbf{q},\omega)=-(n-1)g_0\frac{\tilde{\lambda}_0}{\lambda_0}\int_k\frac{1}{r_0+(\mathbf{q}/2+\mathbf{k})^2}\,\frac{1}{-i\omega+\lambda_0[r_0+(\mathbf{q}/2+\mathbf{k})^2]+D_0(\mathbf{q}/2-\mathbf{k})^2}, \tag{A9}$$

$$\Gamma_{0\tilde{M}[\tilde{S}S]}(\mathbf{q},\omega)=2g_0\frac{\tilde{\lambda}_0}{\lambda_0}\int_k\frac{(\mathbf{q}\cdot\mathbf{k})}{r_0+(\mathbf{q}/2+\mathbf{k})^2}\,\frac{1}{-i\omega+2\lambda_0(r_0+q^2/4+k^2)}. \tag{A10}$$

Calculating the three- and four-point functions is already a rather tedious task even to one-loop order, and we merely quote the final results needed for renormalization purposes:

$$\Gamma_{0\tilde{S}SM}(\mathbf{0},0;\mathbf{0},0;\mathbf{0},0)=-g_0\left[1-(n-1)g_0^2\frac{\tilde{D}_0}{D_0}\int_k\frac{1}{[\lambda_0(r_0+k^2)+D_0k^2]^2}+(n-1)g_0^2\frac{\tilde{\lambda}_0}{\lambda_0}\int_k\frac{k^2}{r_0+k^2}\,\frac{1}{[\lambda_0(r_0+k^2)+D_0k^2]^2}\right], \tag{A11}$$

$$\begin{aligned}\frac{\partial}{\partial(\mathbf{q}\cdot\mathbf{p})}\Gamma_{0\tilde{M}SS}(-\mathbf{q},0;\mathbf{q}/2-\mathbf{p},0;\mathbf{q}/2+\mathbf{p},0)\big|_{\mathbf{q}=\mathbf{p}=\mathbf{0}}=2g_0\Bigg[&1-\frac{2}{d}(n-1)g_0^2\frac{\tilde{D}_0}{\lambda_0}\int_k\frac{k^2}{r_0+k^2}\,\frac{1}{[\lambda_0(r_0+k^2)+D_0k^2]^2}\\&+\frac{2}{d}(n-1)g_0^2\frac{\tilde{\lambda}_0D_0}{\lambda_0^2}\int_k\frac{k^4}{(r_0+k^2)^2}\,\frac{1}{[\lambda_0(r_0+k^2)+D_0k^2]^2}\Bigg];\end{aligned} \tag{A12}$$

and finally

$$\begin{aligned}\Gamma_{0\tilde{S}SSS}(\mathbf{0},0;\mathbf{0},0;\mathbf{0},0;\mathbf{0},0)=\lambda_0u_0\Bigg[&1-\frac{n+8}{6}\frac{\tilde{\lambda}_0}{\lambda_0}u_0\int_k\frac{1}{(r_0+k^2)^2}+(n-1)g_0^2\left(1-\frac{3g_0^2}{\lambda_0D_0u_0}\right)\frac{\tilde{D}_0}{\lambda_0D_0}\int_k\frac{1}{r_0+k^2}\,\frac{1}{\lambda_0(r_0+k^2)+D_0k^2}\\&-(n-1)g_0^2\left(1+\frac{3g_0^2}{\lambda_0D_0u_0}\right)\frac{\tilde{D}_0}{D_0}\int_k\frac{1}{[\lambda_0(r_0+k^2)+D_0k^2]^2}+(n-1)g_0^2\left(1+\frac{3g_0^2}{\lambda_0D_0u_0}\right)\frac{\tilde{\lambda}_0}{\lambda_0}\\&\times\int_k\frac{k^2}{r_0+k^2}\,\frac{1}{[\lambda_0(r_0+k^2)+D_0k^2]^2}+(n-1)\frac{3g_0^4}{\lambda_0D_0u_0}\frac{\tilde{\lambda}_0}{\lambda_0^2}\int_k\frac{k^2}{(r_0+k^2)^2}\,\frac{1}{\lambda_0(r_0+k^2)+D_0k^2}\\&-(n-1)\frac{3g_0^4}{\lambda_0D_0u_0}\frac{\tilde{D}_0}{\lambda_0}\int_k\frac{k^2}{r_0+k^2}\,\frac{1}{[\lambda_0(r_0+k^2)+D_0k^2]^2}\\&+(n-1)\frac{3g_0^4}{\lambda_0D_0u_0}\frac{\tilde{\lambda}_0D_0}{\lambda_0^2}\int_k\frac{k^4}{(r_0+k^2)^2}\,\frac{1}{[\lambda_0(r_0+k^2)+D_0k^2]^2}\Bigg].\end{aligned} \tag{A13}$$

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