

Spectral form factor in a random matrix theory

E. Brézin¹ and S. Hikami²

¹*Laboratoire de Physique Théorique, Ecole Normale Supérieure, 24 rue Lhomond, 75231, Paris Cedex 05, France**

²*Department of Pure and Applied Sciences, University of Tokyo, Meguro-ku, Komaba, Tokyo 153, Japan*

(Received 26 August 1996)

In the theory of disordered systems the spectral form factor $S(\tau)$, the Fourier transform of the two-level correlation function with respect to the difference of energies, is linear for $\tau < \tau_c$ and constant for $\tau > \tau_c$. Near zero and near τ_c it exhibits oscillations which have been discussed in several recent papers. In problems of mesoscopic fluctuations and quantum chaos a comparison is often made with a random matrix theory. It turns out that, even in the simplest Gaussian unitary ensemble, these oscillations have not yet been studied there. For random matrices, the two-level correlation function $\rho(\lambda_1, \lambda_2)$ exhibits several well-known universal properties in the large- N limit. Its Fourier transform is linear as a consequence of the short-distance universality of $\rho(\lambda_1, \lambda_2)$. However the crossover near zero and τ_c requires one to study these correlations for finite N . For this purpose we use an exact contour-integral representation of the two-level correlation function which allows us to characterize these crossover oscillatory properties. This representation is then extended to the case in which the Hamiltonian is the sum of a deterministic part H_0 and of a Gaussian random potential V . Finally, we consider the extension to the time-dependent case. [S1063-651X(97)08304-9]

PACS number(s): 05.45.+b, 05.40.+j

I. INTRODUCTION

The properties of universality at short distance of correlation functions between eigenvalues for invariant random matrix ensembles [1] were conjectured by Dyson [2] many years ago. Modern applications to areas such as fluctuations in mesoscopic systems, randomly triangulated surfaces, or quantum chaos [3] led many people to look again into these properties. Dyson's short-distance universality was finally understood [4,5] and also extended in several directions, for instance to smoothed correlation functions at arbitrary distances [4,6], and also to noninvariant ensembles [7] such as ensembles in which matrix elements are independent random variables [8]. In this work we consider Gaussian ensembles of random Hermitian matrices in the presence of a nonvanishing external source which breaks unitary symmetry. In particular we study a Fourier transform of the two-level correlation function, the spectral form factor $S(\tau)$. For the Gaussian unitary ensemble, in the large- N limit, this form factor $S(\tau)$ has a simple linear behavior with τ up to a critical value $\tau_c = 2N$, beyond which it becomes 1. This remarkable behavior is due to the short-distance universality of the two-level correlation function $\rho(\lambda_1, \lambda_2)$ [2]. In the problem of quantum chaos, it is known that the level statistics of chaotic systems in a certain energy range, agrees with the result of a random matrix theory, and the linear behavior of $S(\tau)$ has been derived by the method of perturbation of periodic orbits [9].

In this paper, we evaluate this spectral form factor within the random matrix theory, in order to characterize the crossover to the linear behavior in the large- N limit. We will investigate the subdominant term, to the large- N limit, which

has an oscillatory behavior. The motivation comes from the work of Kravtsov and Mirlin [10], who showed that deviations to the Wigner-Dyson large- N correlation function were present in the level statistics of a weakly disordered metal. These oscillations, due to subdominant terms, are more visible if we consider the derivative of $S(\tau)$ with respect to τ , since they become of the same order as the leading linear term. We believe that these oscillatory terms near $\tau=0$ and $\tau=2N$, although small, are relevant for discussions of current interest on oscillations in disordered metals or in quantum chaos in nonuniversal regions [10–13].

For a discussion of the crossover to the universal linear behavior, we derive an exact expression for finite N of $S(\tau)$. Our analysis is based upon the recent calculation of the two-level correlation function [14], in which the Kazakov contour-integral representation [15] has been used. This representation has also been used recently for the Laguerre ensemble; it made it easy to characterize the crossover behavior near the edge and near-zero energy for the density of state [16]. Here we consider a similar crossover behavior for the two-level correlation function or the spectral form factor.

We also find, after averaging this form factor over the energy, that the corresponding form factor $\langle S(\tau) \rangle$ is remarkably related, through a simple integration, to the density of state $\rho(\tau)$ in the Laguerre ensemble; this density is known to possess a universal oscillatory behavior near the origin [16–19]. We extend the form factor calculations to the time-dependent case, which is shown to be equivalent to the two-matrix model. In this case, the singular behavior at the Heisenberg time is smeared out.

We further discuss matrix models with an external source for the correlation functions. We find interesting characteristic properties of the kernel $K_N(\lambda, \mu)$, and its universal behavior. The kernel $K_N(\lambda, \mu)$ has lines of zeros in the real (λ, μ) plane. We briefly study the zeros of the kernels in the two-matrix model and in the model with external source [14].

*Unité propre du Centre National de la Recherche Scientifique, associée à l'Ecole Normale Supérieure et à l'Université de Paris-Sud.

II. UNIVERSAL BEHAVIOR OF THE FORM FACTOR

The two-level correlation function $\rho^{(2)}(\lambda, \mu)$ for the random matrix model is defined by

$$\rho^{(2)}(\lambda, \mu) = \left\langle \frac{1}{N} \text{Tr} \delta(\lambda - M) \frac{1}{N} \text{Tr} \delta(\mu - M) \right\rangle, \quad (2.1)$$

where M is an $N \times N$ random Hermitian matrix, and the bracket means an averaging with respect to the Gaussian distribution,

$$P(M) = \frac{1}{Z} \exp\left(-\frac{N}{2} \text{Tr} M^2\right). \quad (2.2)$$

The connected correlation function $\rho_c^{(2)}(\lambda, \mu)$ is obtained by a subtraction of the disconnected part, which is a product of the density of states $\rho(\lambda)$ and $\rho(\mu)$. This function has a complicated expression with strong oscillations, which simplifies only in the short-distance limit, in which there are a finite number of levels between λ and μ , i.e., for $N(\lambda - \mu)$ finite in the large- N limit. Introducing the scaling variable

$$x = \pi N(\lambda - \mu) \rho\left(\frac{1}{2}\lambda + \frac{1}{2}\mu\right), \quad (2.3)$$

and taking the large- N limit with a finite x , one finds [1]

$$\rho_c^{(2)}(\lambda, \mu) \simeq \frac{1}{N} \delta(\lambda - \mu) \rho(\lambda) - \rho(\lambda) \rho(\mu) \frac{\sin^2 x}{x^2}. \quad (2.4)$$

The spectral form factor $S(\tau)$ is defined by

$$S(\tau) = \int_{-\infty}^{+\infty} d\omega e^{i\omega\tau} \rho_c^{(2)}(E, E + \omega). \quad (2.5)$$

Using the large- N , small- ω limit, we have, leaving aside the δ -function term in Eq. (2.4),

$$\rho_c^{(2)}\left(E - \frac{\omega}{2}, E + \frac{\omega}{2}\right) \simeq -\frac{1}{\pi^2 N^2} \frac{\sin^2[\pi N \omega \rho(E)]}{\omega^2}. \quad (2.6)$$

Then the Fourier integral is evaluated easily, since

$$\int_{-\infty}^{+\infty} e^{i\omega t} \frac{\sin^2(a\omega)}{\omega^2} d\omega = \begin{cases} \frac{\pi}{2} (2a - |t|), & |t| < 2a \\ 0, & |t| > 2a. \end{cases} \quad (2.7)$$

This leads to

$$S(\tau) = \begin{cases} \frac{|\tau|}{2\pi N^2} - \frac{\rho(E)}{N}, & |\tau| < 2\pi N \rho(E) \\ 0, & |\tau| > 2\pi N \rho(E). \end{cases} \quad (2.8)$$

Adding the δ -function term of Eq. (2.4), we find that $S(\tau)$ vanishes for $\tau=0$. From this result, we find that if τ is of the order of N , then the integration over ω is dominated by a range of order $1/N$, and therefore, the approximation of $\rho^{(2)}(\lambda, \mu)$ by its short-distance behavior (2.6) is justified. However, if τ is of the order of 1, then we have to deal with an integration over a range in which ω is not small, and we can no longer use the short-distance universal behavior for

$\rho^{(2)}(\lambda, \mu)$. Therefore, one expects a universal linear behavior in the range in which τ is of order N .

In a previous paper [14] we derived the oscillating short-distance behavior of Eq. (2.6) by using a method introduced by Kazakov. This method gives exact expressions of the correlation functions for finite N . It is very convenient for characterizing the crossovers in comparison with the standard approach based on orthogonal polynomials. It consists of adding a matrix source to the probability distribution, and this external source is set to zero at the end of the calculation. (In some cases one is interested in keeping a finite external source, as studied recently in [14]). We thus modify the probability distribution of the matrix by a source A , an $N \times N$ Hermitian matrix with eigenvalues (a_1, \dots, a_N) :

$$P_A(M) = \frac{1}{Z_A} \exp\left(-\frac{N}{2} \text{Tr} M^2 - N \text{Tr} A M\right). \quad (2.9)$$

We consider the average evolution operator with this modified distribution

$$U_A(t) = \left\langle \frac{1}{N} \text{Tr} e^{itM} \right\rangle. \quad (2.10)$$

The density of state $\rho(\lambda)$ is its Fourier transform

$$\rho(\lambda) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{-it\lambda} U_A(t). \quad (2.11)$$

We first integrate over the unitary matrix V which diagonalizes M (we may assume, without loss of generality, that A is a diagonal matrix). This is done by the well-known Itzykson-Zuber integral [22]

$$\int dV \exp(\text{Tr} A V B V^\dagger) = \frac{\det(\exp(a_i b_j))}{\Delta(A) \Delta(B)}, \quad (2.12)$$

where $\Delta(A)$ is the Van der Monde determinant constructed with the eigenvalues of A :

$$\Delta(A) = \prod_{i < j}^N (a_i - a_j). \quad (2.13)$$

We are then led to

$$U_A(t) = \frac{1}{Z_A \Delta(A)} \frac{1}{N} \sum_{\alpha=1}^N \int dr_1 \cdots dr_N e^{itr_\alpha \Delta(r_1, \dots, r_N)} \times \exp\left(-\frac{N}{2} \sum r_i^2 - N \sum a_i r_i\right). \quad (2.14)$$

After integrating over the r_i , we obtain

$$U_A(t) = \frac{1}{N} \sum_{\alpha=1}^N \prod_{\gamma \neq \alpha} \left(\frac{a_\alpha - a_\gamma - (it/N)}{a_\alpha - a_\gamma} \right) e^{-(t^2/2N) - ita_\alpha}. \quad (2.15)$$

Instead of summing over N terms, one can write a contour integral in the complex plane,

$$U_A(t) = -\frac{1}{it} \oint \frac{du}{2\pi i} \prod_{\gamma=1}^N \left(\frac{u - a_\gamma - (it/N)}{u - a_\gamma} \right) e^{-itu - (t^2/2N)}. \quad (2.16)$$

We may now, and only at this stage, let the a_γ go to zero; we obtain

$$U_0(t) = -\frac{1}{it} e^{-t^2/2N} \oint \frac{du}{2\pi i} e^{-itu} \left(1 - \frac{it}{Nu} \right)^N. \quad (2.17)$$

Similarly the two-level correlation function, $\rho^{(2)}(\lambda, \mu)$ is obtained from the Fourier transform $U_A(t_1, t_2)$, after letting A go to zero [14],

$$\rho^{(2)}(\lambda, \mu) = \int \int \frac{dt_1 dt_2}{(2\pi)^2} e^{-it_1\lambda - it_2\mu} U_0(t_1, t_2), \quad (2.18)$$

where $U_A(t_1, t_2)$ is

$$U_A(t_1, t_2) = \left\langle \frac{1}{N} \text{Tr} e^{it_1 M} \frac{1}{N} \text{Tr} e^{it_2 M} \right\rangle. \quad (2.19)$$

The same procedure leads to

$$U_A^{(2)}(t_1, t_2) = \frac{1}{N^2} \sum_{\alpha_1, \alpha_2} \int \prod dr_i \frac{\Delta(r)}{\Delta(A)} \times e^{-N\sum[(1/2)r_i^2 + r_i a_i] + i(t_1 r_{\alpha_1} + t_2 r_{\alpha_2})}. \quad (2.20)$$

By integration over r_i , we obtain, after subtraction of the disconnected part, a representation in terms of an integral over two complex variables

$$U_0(t_1, t_2) = -\frac{1}{N^2} \oint \frac{du dv}{(2\pi i)^2} e^{-(t_1^2/2N) - (t_2^2/2N) - it_1 u - it_2 v} \times \left(1 - \frac{it_1}{Nu} \right)^N \left(1 - \frac{it_2}{Nv} \right)^N \times \frac{1}{(u - v - it_1/N)[u - v + (it_2/N)]} \quad (2.21)$$

where the contours are taken around $u=0$ and $v=0$. If we let the contour include the pole, $v = u - it_1/N$, it gives precisely the disconnected term $U_0(t_1 + t_2)$, whose Fourier transform is the δ -function part of Eq. (2.4).

We now write the two-level correlation function as the Fourier transform of $U_0(\lambda_1, \lambda_2)$. In order to show that it takes a factorized form, we shift the variables t_1 and t_2 to $t_1 \rightarrow t_1 - iuN$ and $t_2 \rightarrow t_2 - ivN$. Then one finds

$$\rho_c(\lambda_1, \lambda_2) = \frac{1}{N^2} \int \frac{dt_1}{2\pi} \oint \frac{dv}{2\pi i} \left(\frac{it_1}{Nv} \right)^N \times \frac{1}{v + \frac{it_1}{N}} e^{-(N/2)v^2 - (t_1^2/2N) - it_1\lambda_1 - Nv\lambda_2} \times \int \frac{dt_2}{2\pi} \oint \frac{du}{2\pi i} \left(\frac{it_2}{Nu} \right)^N \quad (2.22)$$

$$\times \frac{1}{u + \frac{it_2}{N}} e^{-(N/2)u^2 - (t_2^2/2N) - it_2\lambda_2 - Nu\lambda_1}$$

$$= -\frac{1}{N^2} K_N(\lambda_1, \lambda_2) K_N(\lambda_2, \lambda_1).$$

We have obtained the integral representation for the kernel $K_N(\lambda, \mu)$,

$$K_N(\lambda, \mu) = -\int_{-\infty}^{\infty} \frac{dt}{2\pi} \oint \frac{du}{2\pi i} \left(-\frac{it}{Nu} \right)^N \times \frac{1}{u + \frac{it}{N}} e^{-(N/2)u^2 - (1/2N)t^2 - it\lambda - Nu\mu}. \quad (2.23)$$

It may be interesting to note that the same integral expression is obtained through the orthogonal polynomial method. In Appendix A, we give this derivation. A similar formula has been obtained by Guhr [21], who used the method of integration over Grassmannian variables. At this stage our formalism does not provide more information, but we shall see later that it may be extended to the much more difficult problem of a nonzero external source.

The expression of $K_N(\lambda_1, \lambda_2)$ may be simplified further by the shift $t_1 \rightarrow t + ivN$,

$$K_N(\lambda_1, \lambda_2) = \int \frac{dt}{2\pi} \oint \frac{dv}{2\pi i} \times \frac{1}{it} \left(1 - \frac{it}{Nv} \right)^N e^{-(t^2/2N) - ivt - it\lambda_1 + Nv(\lambda_1 - \lambda_2)}. \quad (2.24)$$

We may now find the short-distance behavior of $\rho^{(2)}(\lambda_1, \lambda_2)$ in the large- N limit with a finite value of the variable $y = N(\lambda_1 - \lambda_2)$. There are several procedures to obtain the oscillating universal form. One possibility has been discussed in [14]. Here we follow another procedure for the purpose of later use. If we substitute to v , $v \rightarrow itv$, we may then perform the t integration

$$K_N(\lambda_1, \lambda_2) = \frac{1}{2\pi} \oint \frac{dv}{2\pi i} \left(\frac{\pi}{\frac{1}{2N} - v} \right)^{1/2} e^{-[(v\lambda_1 - \lambda_1)^2]/2[(1/N) - 2v] - (1/v)} \times \left(1 - \frac{1}{Nv} \right)^N. \quad (2.25)$$

In the large- N limit, we may neglect $1/N$ terms, and exponentiate the term which is a power of N . We obtain

$$K_N(\lambda_1, \lambda_2) \simeq -\frac{i}{2\pi} \oint \frac{dv}{2\pi i} \left(\frac{\pi}{v}\right)^{1/2} e^{(y^2/4)v - [1 - (\lambda_1^2/4)]/v + (\lambda_1 y/2)}. \quad (2.26)$$

We change the contour in the complex plane, and we use the following result:

$$K_{1/2}(z) = \int_0^\infty e^{-z \cosh x} \cosh \frac{x}{2} dx = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}, \quad (2.27)$$

where $K_{1/2}(z)$ is a modified Bessel function. Then we obtain

$$K_N(\lambda_1, \lambda_2) \simeq \frac{1}{\pi y} e^{(\lambda_1/2)y} \sin \left[y \left(1 - \frac{\lambda_1^2}{4}\right)^{1/2} \right]. \quad (2.28)$$

The other term $K_N(\lambda_2, \lambda_1)$ is obtained in a similar way. Thus in the large- N , finite- y , limit, we obtain

$$\begin{aligned} \rho(\lambda_1, \lambda_2) &= -\frac{e^{-[(\lambda_1 - \lambda_2)/2]y}}{\pi^2 y^2} \sin\left(\frac{\sqrt{4 - \lambda_1^2}}{2} y\right) \sin\left(\frac{\sqrt{4 - \lambda_2^2}}{2} y\right) \\ &\simeq -\frac{1}{\pi^2 y^2} \sin^2\left(\frac{\sqrt{4 - \lambda_1^2}}{2} y\right). \end{aligned} \quad (2.29)$$

We also derive a more precise expression for the kernel $K_N(\lambda_1, \lambda_2)$ from Eq. (2.22). We have

$$\begin{aligned} K_N(\lambda_1, \lambda_2) &= \int \frac{dt}{2\pi} \oint \frac{dv}{2\pi i} \left(\frac{it}{Nv}\right)^N \\ &\quad \times \frac{1}{v + \frac{it}{N}} e^{-(N/2)v^2 - (t^2/2N) - it\lambda_1 - Nv\lambda_2} \\ &= N \int \frac{dt}{2\pi} \oint \frac{dv}{2\pi i} \frac{i^N}{v + it} e^{-Nf}, \end{aligned} \quad (2.30)$$

where f is

$$f = \frac{v^2}{2} + \frac{t^2}{2} + it\lambda_1 + v\lambda_2 - \ln t + \ln v. \quad (2.31)$$

The saddle-point equations for f become

$$\begin{aligned} \frac{\partial f}{\partial v} &= v + \lambda_2 + \frac{1}{v} = 0, \\ \frac{\partial f}{\partial t} &= t + i\lambda_1 - \frac{1}{t} = 0, \end{aligned} \quad (2.32)$$

Four solutions are obtained: $v = ie^{i\varphi}$, $v = -ie^{-i\varphi}$, $t = e^{-i\theta}$, and $t = -e^{i\theta}$. We define $\lambda_1 = 2 \sin \theta$ and $\lambda_2 = 2 \sin \varphi$. The Gaussian fluctuation around the saddle point is evaluated by

$$\frac{1}{\left(\frac{\partial^2 f}{\partial v^2}\right)^{1/2}} = \frac{e^{\pm(i/2)\theta}}{\sqrt{2 \cos \varphi}} \quad (2.33)$$

$$\frac{1}{\left(\frac{\partial^2 f}{\partial t^2}\right)^{1/2}} = \frac{e^{\pm(i/2)\theta}}{\sqrt{2 \cos \theta}}.$$

Adding these four saddle-point contributions, we have

$$\begin{aligned} K_N(\lambda_1, \lambda_2) &= \frac{e^{(N/2)(\cos 2\theta - \cos 2\varphi)}}{8\pi N \sqrt{\cos \theta \cos \varphi}} \\ &\quad \times \left[-\frac{\cos \left[N \left(\theta + \varphi + \frac{\sin 2\theta}{2} + \frac{\sin 2\varphi}{2} \right) \right] \cos \frac{1}{2}(\theta + \varphi)}{1 + \cos(\theta + \varphi)} \right. \\ &\quad \left. + \frac{\sin \left[N \left(\theta - \varphi + \frac{1}{2} \sin 2\theta - \frac{1}{2} \sin 2\varphi \right) \right] \sin \frac{1}{2}(\theta - \varphi)}{1 - \cos(\theta - \varphi)} \right]. \end{aligned} \quad (2.34)$$

When $\lambda_1 - \lambda_2$ is order of $1/N$, we make approximations in Eq. (2.34),

$$\begin{aligned} \theta - \varphi + \frac{\sin 2\theta}{2} - \frac{\sin 2\varphi}{2} &\simeq \sin(\theta - \varphi) + (\sin \theta - \sin \varphi) \cos \theta \\ &\simeq \pi(\lambda_1 - \lambda_2) \rho(\lambda_1), \end{aligned} \quad (2.35)$$

where $\rho(\lambda_1) = \sqrt{4 - \lambda_1^2}/2\pi$, and the denominator of Eq. (2.34) is approximated as $1 - \cos(\theta - \varphi) \simeq 1/2 \sin^2(\theta - \varphi)$. Then, in the large- N limit for fixed $N(\lambda_1 - \lambda_2)$ we obtain the short-range universal form of Eq. (2.29). Later we will discuss the generalization of Eq. (2.34) to the time dependent case.

We now consider the form factor $S(\tau)$, which is defined by Eq. (2.5). From the expressions of $K_N(0, \omega)$ and $K_N(\omega, 0)$ in Eq. (2.25), we have

$$\begin{aligned} S(\tau) &= \frac{1}{2\pi} \int d\omega \oint \frac{du dv}{(2\pi i)^2} \frac{e^{i\omega\tau}}{\sqrt{\frac{1}{N} - 2u} \sqrt{\frac{1}{N} - 2v}} \\ &\quad \times e^{-(N^2 v^2 \omega^2)/2[(1/N) - 2v] - [\omega^2(Nu - 1)^2]/2[(1/N) - 2u]} \\ &\quad \times \left(1 - \frac{1}{Nu}\right)^N \left(1 - \frac{1}{Nv}\right)^N. \end{aligned} \quad (2.36)$$

In the large- N limit, if τ is of the order of N , we may use the previous expressions for rederiving the universal short-distance behavior in Eq. (2.29), and obtain the linear behavior up to $\tau = 2N$. However, for finite N , this function is complicated, and we need the study of the oscillating part based on Eq. (2.36).

III. OSCILLATORY BEHAVIOR OF THE FORM FACTOR

We first integrate out ω in Eq. (2.36), and by shifting $u \rightarrow (1/N)u$ and $v \rightarrow (1/N)v$, we obtain

$$S(\tau) = \frac{1}{\sqrt{2\pi N^2}} \oint \frac{du dv}{(2\pi i)^2} e^{-\tau^2/2ND} \times \frac{1}{\sqrt{v^2(1-2u) + (u-1)^2(1-2v)}} \times \left(1 - \frac{1}{u}\right)^N \left(1 - \frac{1}{v}\right)^N, \quad (3.1)$$

where D is given by

$$D = \frac{v^2}{1-2v} + \frac{(u-1)^2}{1-2u}. \quad (3.2)$$

A quasilinear behavior with small oscillations follows from this expression. It is interesting first to compute this contour integral (3.1) for finite N . We have evaluated, for example, the $N=7$ case. The correction to the linear behavior is small, but the derivative of $S(\tau)$ with respect to τ shows an oscillating behavior. Returning to the analytic calculation we can obtain exact expressions for this oscillatory behavior by a saddle-point analysis of Eq. (3.1). For this purpose, we scale τ by $\tau = N\tilde{\tau}$. Then we have

$$S(\tau) = \frac{1}{\sqrt{2\pi N^2}} \oint \frac{du dv}{(2\pi i)^2} \times \frac{1}{\sqrt{v^2(1-2u) + (u-1)^2(1-2v)}} e^{-Nf}, \quad (3.3)$$

where the exponent f is

$$f = \frac{\tilde{\tau}^2}{2D} - \ln\left(1 - \frac{1}{u}\right) - \ln\left(1 - \frac{1}{v}\right). \quad (3.4)$$

In the large N limit, we look for the saddle points of u and v in the complex plane. They are obtained by

$$\frac{\partial f}{\partial u} = 0, \quad \frac{\partial f}{\partial v} = 0. \quad (3.5)$$

We thus obtain the two equations

$$\frac{(1-2u)^2}{u^2(1-u)^2} = \frac{\tilde{\tau}^2}{D^2}, \quad \frac{(1-2v)^2}{v^2(1-v)^2} = \frac{\tilde{\tau}^2}{D^2}. \quad (3.6)$$

As solutions of these equations, we have

$$\frac{1-2u}{u(1-u)} = \pm \frac{1-2v}{v(1-v)}. \quad (3.7)$$

There are four solutions to these equations, but two of them only are saddle points: (a) for the (+) case, $u=v$; and (b) for the (-) case, $u=v/(2v-1)$. Although $v=(1-u)/(1-2u)$ for the (+) case and $v=1-u$ for the (-) case are solutions, they are not saddle points, since D vanishes. Case (a), $u=v$, still leaves us with four different solutions. The first one is

$$u=v = \frac{1}{2} + \frac{i}{2} \left(\frac{2-\tilde{\tau}}{2+\tilde{\tau}} \right)^{1/2}. \quad (3.8)$$

The three other solutions are obtained by the replacements $i \rightarrow -i$ and $\tilde{\tau} \rightarrow -\tilde{\tau}$. Therefore, it is sufficient to consider the first case explicitly, and make the necessary replacements at the end for the other solutions. The quantity D becomes

$$D = \frac{i\tilde{\tau}}{\sqrt{4-\tilde{\tau}^2}}. \quad (3.9)$$

For this saddle point, f becomes

$$f = i[2\theta - \sin(2\theta)] - 2\pi i, \quad (3.10)$$

where we have put $\tilde{\tau} = 2\cos\theta$. The fluctuation around this saddle point is obtained by the consideration of the second derivatives with respect to u and v . They are

$$\frac{\partial^2 f}{\partial u^2} = \frac{\partial^2 f}{\partial v^2} = \frac{2i(2+\tilde{\tau})^{3/2}}{\sqrt{2-\tilde{\tau}}} \left(\frac{2}{\tilde{\tau}} - \tilde{\tau} \right), \quad (3.11)$$

$$\frac{\partial^2 f}{\partial u \partial v} = \frac{4i}{\tilde{\tau}} \frac{(2+\tilde{\tau})^{3/2}}{(2-\tilde{\tau})^{1/2}}.$$

The Gaussian fluctuations around the saddle point produce the inverse of the square root of a determinant, which is

$$\det f'' = \left(\frac{\partial^2 f}{\partial u^2} \right)^2 - \left(\frac{\partial^2 f}{\partial u \partial v} \right)^2 = 4(2+\tilde{\tau})^4. \quad (3.12)$$

Thus from this result we obtain

$$S(\tau) \sim \frac{e^{-iN(2\theta - \sin 2\theta)}}{\sqrt{1-2u} \sqrt{1-2v} \sqrt{D} \sqrt{\det f'' N^3}} \sim \frac{i}{2} \frac{1}{\sqrt{2i \sin 2\theta} (2+2\cos\theta) N^3} e^{-iN(2\theta - \sin 2\theta)}. \quad (3.13)$$

We now add the other solutions by making the replacements $i \rightarrow -i$ and $\tilde{\tau} \rightarrow -\tilde{\tau}$, which corresponds to $\theta \rightarrow \theta + \pi$. Adding these terms, we have

$$S_a(\tau) = \frac{\cos\left(\frac{\pi}{4} - N(2\theta - \sin 2\theta)\right)}{N^3 \sqrt{2 \sin 2\theta}} \times \left[\frac{1}{(2+2\cos\theta)} + \frac{1}{(2-2\cos\theta)} \right], \quad (3.14)$$

where $\tilde{\tau} = 2\cos\theta$; thus $\theta=0$ corresponds to $\tilde{\tau}=2$, while $\theta=\pi/2$ corresponds to $\tilde{\tau}=0$.

Case (b) is quite similar. We have

$$u = \frac{1}{2} + \frac{i}{2} \left(\frac{2-\tilde{\tau}}{2+\tilde{\tau}} \right)^{1/2}, \quad v = \frac{1}{2} - \frac{i}{2} \left(\frac{2+\tilde{\tau}}{2-\tilde{\tau}} \right)^{1/2}. \quad (3.15)$$

Using the notation $\tilde{\tau} = 2 \cos \theta$, we obtain

$$f = i(2\theta - \sin 2\theta) - \pi i \tag{3.16}$$

and

$$D = \frac{i\tilde{\tau}}{\sqrt{4 - \tilde{\tau}^2}} \tag{3.17}$$

$\det f''$ becomes

$$\det f'' = \left(\frac{\partial^2 f}{\partial u^2}\right)^2 - \left(\frac{\partial^2 f}{\partial u \partial v}\right)^2 = 4(4 - \tilde{\tau}^2)^2. \tag{3.18}$$

Adding the four terms obtained by $\tilde{\tau} \rightarrow -\tilde{\tau}$ and $i \rightarrow -i$, we have

$$S_b(\tau) = \frac{\cos\left(N(2\theta - \sin 2\theta) - N\pi - \frac{\pi}{4}\right)}{2\sqrt{\tilde{\tau}N^3(4 - \tilde{\tau}^2)^{3/4}}}. \tag{3.19}$$

From these analyses, we obtain the oscillating part of $S(\tau)$ in the large- N limit. It is a sum of $S_a(\tau)$ and $S_b(\tau)$. Noting that the linear part of $S(\tau)$ in Eq. (2.8) is of the order $\tau/N^2 = \tilde{\tau}/N$, we find that the oscillating part is a nothing but a correction of order $1/N$. However, if we take a derivative, it becomes of the same order as the linear term. We also find that when $\tilde{\tau}$ is close to 2, the coefficient of the oscillating part of $S(\tau)$ becomes large, as shown in Eqs. (3.14) and (3.19), and even diverges at $\tilde{\tau} = 2$. Therefore, there should be again a crossover near the critical $\tau_c = 2N$. Up to now we have considered a fixed energy $E = 0$. In Sec. IV, we will instead take an average over E , and see that the expression for the form factor simplifies.

IV. AVERAGE OF THE FORM FACTOR

We will now consider the average of $S(\tau)$ over E , by simply integrating over E ,

$$\langle S(\tau) \rangle = \int_{-\infty}^{+\infty} dE S(\tau). \tag{4.1}$$

Remarkably, we find that this $\langle S(\tau) \rangle$ is given analytically in terms of known functions.

From Eqs. (2.5) and (2.18), $S(\tau)$ is written as

$$S(\tau) = \int d\omega e^{i\omega\tau} \rho(E, E + \omega) = \int dt_1 e^{-it_1 E - i\tau E} U_0(t_1, \tau). \tag{4.2}$$

Thus the integration over E gives simply

$$\langle S(\tau) \rangle = \int dE \int dt_1 e^{-i(t_1 + \tau)E} U_0(t_1, \tau) = U_0(-\tau, \tau). \tag{4.3}$$

Then we write the following contour-integral representation for $\langle S(\tau) \rangle$ from Eq. (3.19):

$$\begin{aligned} \langle S(\tau) \rangle &= \frac{1}{N^2} e^{-\tau^2/N} \oint \frac{du dv}{(2\pi i)^2} e^{-i\tau(u-v)} \left(1 - \frac{i\tau}{Nu}\right)^N \\ &\times \left(1 + \frac{i\tau}{Nv}\right)^N \frac{1}{\left(u-v - \frac{i\tau}{N}\right)^2}. \end{aligned} \tag{4.4}$$

Replacing u by τu and v by τv , and putting $\tau^2 = x$, we have

$$\begin{aligned} \langle S(x) \rangle &= \frac{1}{N^2} \oint \frac{du dv}{(2\pi i)^2} \frac{e^{-ix[u-v-(i/N)]}}{\left(u-v - \frac{i}{N}\right)^2} \left(1 - \frac{i}{Nu}\right)^N \\ &\times \left(1 + \frac{i}{Nv}\right)^N. \end{aligned} \tag{4.5}$$

Taking two derivatives with respect to x , we obtain a simple factorized expression

$$\begin{aligned} \frac{d^2 \langle S(x) \rangle}{dx^2} &= -\frac{e^{-x/N}}{N^2} \oint \frac{du}{2\pi i} e^{-ixu} \left(1 - \frac{i}{Nu}\right)^N \oint \frac{dv}{2\pi i} e^{ixv} \\ &\times \left(1 + \frac{i}{Nv}\right)^N \\ &= -\frac{e^{-x/N}}{N^2} \left[\frac{d}{dx} L_N\left(\frac{x}{N}\right) \right]^2, \end{aligned} \tag{4.6}$$

where $L_N(x)$ is a Laguerre polynomial. Remarkably enough, an identical expression has been found earlier, but for a completely different ensemble and a different quantity. Indeed expression (4.6) has been found in previous work on the Laguerre ensemble of random matrices [16], in which it was the derivative of the density of state. The Laguerre ensemble, also called the chiral Gaussian unitary ensemble (CHGUE), since eigenvalues appear by pairs of opposite signs, thus has a curious relation to the Gaussian unitary ensemble (GUE): the form factor $\langle S(\tau) \rangle$ in the GUE is related, for any finite N , to the density of state $\rho(\tau)$ of the CHGUE. We have not been able to find a direct proof of this exact relation valid for any finite N , without calculating both expressions and verifying that they are identical.

The oscillating behavior of Eq. (4.6) is similar to that of $S(\tau)$. This oscillation in $\langle S(\tau) \rangle$ is slightly different from the proposal by Kravtsov and Mirlin [10], which is a simple $\sin(\tau)$ oscillation. Our result (4.6) is not a sine oscillation. The oscillating behavior of the density of state for the Laguerre ensemble near the origin can be seen in Fig. 2 of [17]. In the large- N limit, we know that the oscillations of the density of state near zero energy become universal, and are given in terms of Bessel functions. In view of the previous correspondence we now have to consider the variable τ as an energy (although it is a time in the GUE problem). Near zero energy, the density of state of the Laguerre ensemble is given by

$$\rho(\tau) = \tau [J_0^2(\tau) + J_1^2(\tau)], \tag{4.7}$$

and it shows an oscillating behavior around 1. Consequently, one understands that the integral of this density of state is proportional to τ . This is why we have obtained a linear

behavior for $\langle S(\tau) \rangle$. However, since the density of state for large N is a semicircle, and not a constant, its integral is no longer proportional to τ . Indeed, in the large- N limit, by integrating the semicircle line, we have

$$\begin{aligned} \langle S(\tau) \rangle &= \int_0^\tau dx \left[1 - \left(\frac{x}{2N} \right)^2 \right]^{1/2} \\ &= \frac{\tau \sqrt{1 - \left(\frac{\tau}{2N} \right)^2}}{4N} + \frac{\arcsin\left(\frac{\tau}{2N} \right)}{2}. \end{aligned} \quad (4.8)$$

Beyond the critical value $\tau=2N$, it remains equal to 1, and it approaches this limit smoothly. Therefore, taking into account the fact that the density of state is not constant, the singularity is smoothed out. Near zero energy, the oscillatory behavior of $\langle S(\tau) \rangle$, following from the Bessel functions of Eq. (4.7), becomes universal. In [16], by the same contour-integral representation, we showed that there is a crossover from the bulk to the zero-energy region, which is described universally by function (4.7). We have also found, in a model consisting of a lattice of coupled matrices, that this oscillating behavior is model independent. Near $\tau=2N$, the crossover behavior has been also studied in [16]. It is given by the square of an Airy function [see Eq. (3.37) of [16]]; this crossover is also known to be universal.

V. TIME-DEPENDENT CASE

We now proceed to investigate the time-dependent correlation function and its Fourier transform, the dynamical form factor. In the large- N limit, the universal form of this time-dependent correlation function has been discussed [23–25]. We will consider this problem by the contour integral representation, which is valid for finite N , and evaluate the form factor $S(\tau)$ for a fixed time t . For a finite t , we will find that $S(\tau)$ shows different behavior compared to the previous linear behavior about τ .

We consider the $N \times N$ Hermitian matrix M , which depends upon a time t . The time-dependent correlation function is defined by

$$\rho(\lambda, \mu; t) = \left\langle \frac{1}{N} \text{Tr} \delta[\lambda - M(t_1)] \frac{1}{N} \text{Tr} \delta[\mu - M(t_2)] \right\rangle, \quad (5.1)$$

where $t=t_1-t_2$, and t_1 and t_2 are different times. This is written as a Fourier transform of the following quantity $U(\alpha, \beta)$:

$$U(\alpha, \beta) = \frac{1}{N^2} \langle \text{Tr} e^{i\alpha M(t_1)} \text{Tr} e^{i\beta M(t_2)} \rangle. \quad (5.2)$$

We use a set of variables α and β for the Fourier transform variables, instead of t_1 and t_2 . To avoid the confusion, we use t_1 and t_2 as time.

We show exactly that the correlation function (5.1) reduces to the correlation function of the two-matrix model in the Gaussian ensemble; the $c=1$ problem is described by the two-matrix model. This correspondence may be derived by other arguments [26]. Here we follow the path-integral

method, which can show explicitly that this equivalence to two-matrix model holds for any finite N .

By considering the Hamiltonian H ,

$$H = \frac{1}{2} \text{Tr}(p^2 + M^2), \quad (5.3)$$

where $p = \dot{M}$ and M is an $N \times N$ Hermitian matrix, we write $U(\alpha, \beta)$ in Eq. (5.2) as

$$U(\alpha, \beta) = \frac{1}{N^2} \langle 0 | e^{Ht_1} (\text{Tr} e^{i\alpha M}) e^{H(t_2-t_1)} (\text{Tr} e^{i\beta M}) e^{-Ht_2} | 0 \rangle. \quad (5.4)$$

We use the path-integral formulation, and define

$$\langle A | e^{-\tilde{\beta} H} | B \rangle = \int_{M(\tilde{\beta})=A, M(0)=B} DM e^{-(1/2) \text{Tr} \int_0^{\tilde{\beta}} (\dot{M}^2 + M^2) dt}. \quad (5.5)$$

Then $U(\alpha, \beta)$ is expressed by

$$\begin{aligned} U(\alpha, \beta) &= \frac{1}{N^2} \int dA dB \langle 0 | e^{Ht_1} | A \rangle \langle A | (\text{Tr} e^{i\alpha M}) e^{H(t_2-t_1)} \\ &\quad \times (\text{Tr} e^{i\beta M}) | B \rangle \langle B | e^{-Ht_2} | 0 \rangle. \end{aligned} \quad (5.6)$$

Noting that the ground-state energy of the free independent N^2 fermions is $N^2/2$, we have

$$\langle 0 | e^{Ht_1} | A \rangle = e^{(N^2/2)t_1} e^{-(1/2) \text{Tr} A^2}. \quad (5.7)$$

The solution of $\ddot{M} = M$, becomes

$$M(t) = B \text{cht} + \frac{\text{sh}t}{\text{sh}\tilde{\beta}} (A - B \text{ch}\tilde{\beta}). \quad (5.8)$$

Then we are able to write the action in Eq. (5.5) by the matrices A and B ,

$$\begin{aligned} \frac{1}{2} \text{Tr} \int_0^{\tilde{\beta}} (\dot{M}^2 + M^2) dt &= \frac{1}{2} \text{Tr} (\dot{M} M) \Big|_0^{\tilde{\beta}} \\ &= \frac{1}{2 \text{sh}\tilde{\beta}} [(A^2 + B^2) \text{ch}\tilde{\beta} - 2AB]. \end{aligned} \quad (5.9)$$

Denoting $\tilde{\beta}$ by a time t , and taking the fluctuation part, we obtain

$$\begin{aligned} U(\alpha, \beta) &= \frac{1}{N^2} \left(\frac{e^t}{\text{sh}t} \right)^{N^2/2} \int dA dB (\text{Tr} e^{i\alpha A}) \\ &\quad \times (\text{Tr} e^{i\beta B}) e^{-(1/2 \text{sh}t) \text{Tr} [(A^2 + B^2) e^t - 2AB]}. \end{aligned} \quad (5.10)$$

Thus the problem reduces exactly to a calculation of the correlation function for the two-matrix model, in which matrices A and B are linearly coupled.

The correlation function for two-matrix model has been studied by D'anna, Brezin, and Zee [27] by the orthogonal polynomial method for finite N . Although we can use their result, it is more convenient to use the contour integral representation for the correlation function.

By making the change of variables of A, B , and α, β by a factor $\sqrt{e^{-t} \text{sh } t}$, we obtain a simple expression for Eq. (5.10),

$$U(\alpha, \beta) = \frac{1}{Z} \int dA dB \text{Tre}^{i\alpha A} \text{Tre}^{i\beta B} e^{-(1/2)\text{Tr}(A^2+B^2-2cAB)}, \tag{5.11}$$

where $c = e^{-t}$. Note that we scaled α and β by a factor $\sqrt{e^{-t} \text{sh } t}$, the variables λ and μ of the two-point correlation function should be modified by this factor for the mapping of the time-dependent case to the two-matrix model. We now go back to the notation in which two matrices are given by M_1 and M_2 . We denote the matrices A and B in Eq. (5.11) by M_1 and M_2 . We introduce the external matrix A , which is coupled to matrix M_1 . The Gaussian distribution is given by

$$P_A(M_1, M_2) = \frac{1}{Z_A} e^{-H_{1,2}}, \tag{5.12}$$

$$H_{1,2} = \frac{1}{2} \text{Tr} M_1^2 + \frac{1}{2} \text{Tr} M_2^2 - c \text{Tr} M_1 M_2 + \text{Tr} A M_1.$$

The density of state $\rho(\lambda)$ is given by the Fourier transform of

$$U_A(z) = \left\langle \frac{1}{N} \text{Tre}^{izM_1} \right\rangle. \tag{5.13}$$

The calculation of this $\rho(\lambda)$ is similar to the one-matrix case. The integration over M_2 , which has eigenvalues ξ_i , is performed by the help of the Itzykson-Zuber formula. We denote the eigenvalues of M_1 by r_i . The integration over ξ becomes

$$\begin{aligned} & \int d\xi \prod_{i < j} (\xi_i - \xi_j) e^{-(N/2)\sum \xi_i^2 - cN\sum \xi_i r_i} \\ &= \prod_{i < j} (r_i - r_j) e^{(Nc^2/2)\sum r_i^2}. \end{aligned} \tag{5.14}$$

Then we are left with the integration about r_i ,

$$\begin{aligned} U_A(z) &= \frac{1}{\Delta(A)} \sum_{\alpha=1}^N \int dr \prod_{i < j} (r_i - r_j) \\ &\quad \times e^{-(N/2)(1-c^2)\sum_i r_i^2 - N\sum_i a_i r_i + izr_\alpha}. \end{aligned} \tag{5.15}$$

Therefore, by the contour integration, we have, by letting a_i go to zero,

$$\begin{aligned} U_0(z) &= -\frac{\sqrt{1-c^2}}{it} \oint \frac{du}{2\pi i} \\ &\quad \times \left(1 - \frac{iz}{Nu\sqrt{1-c^2}} \right)^N e^{-(izu/\sqrt{1-c^2}) - [z^2/2N(1-c^2)]}. \end{aligned} \tag{5.16}$$

We have the same density of state as the one-matrix case except for the scaling factor $(1-c^2)$,

$$\rho(\lambda) = \sqrt{1-c^2} \rho_0(\sqrt{1-c^2}\lambda), \tag{5.17}$$

where $\rho_0(\lambda)$ is the density of state for the one-matrix model. In the large- N limit, this density of state becomes

$$\rho(\lambda) = \frac{\sqrt{1-c^2}}{2\pi} \sqrt{4 - (1-c^2)\lambda^2}, \tag{5.18}$$

which is normalized to be 1 by the integration.

The two-level correlation function is given by

$$\rho^{(2)}(\lambda, \mu) = \int \int \frac{dz_1 dz_2}{(2\pi)^2} e^{-iz_1\lambda - iz_2\mu} U_0(z_1, z_2), \tag{5.19}$$

where $U_0(z_1, z_2)$ is

$$U_0(z_1, z_2) = \left\langle \frac{1}{N} \text{Tre}^{iz_1 M_1} \frac{1}{N} \text{Tre}^{iz_2 M_2} \right\rangle. \tag{5.20}$$

By integration over the eigenvalues r_i of M_1 , and ξ_i of M_2 , and keeping the eigenvalues a_i of the external matrix A , we have the following expression:

$$\begin{aligned} U_A(z_1, z_2) &= \frac{1}{N^2} \sum_{\alpha_1, \alpha_2} \frac{\prod_{i < j} \left(a_i - a_j - \frac{iz_1}{N} (\delta_{i, \alpha_1} - \delta_{j, \alpha_1}) - \frac{iz_2}{cN} (\delta_{i, \alpha_2} - \delta_{j, \alpha_2}) \right)}{\prod_{i < j} (a_i - a_j)} \\ &\quad \times e^{-[iz_1/(1-c^2)]a_{\alpha_1} - [iz_2c/(1-c^2)]a_{\alpha_2} - [z_1^2/2N(1-c^2)] - [z_2^2/2N(1-c^2)] - [cz_1z_2/N(1-c^2)]\delta_{\alpha_1, \alpha_2}}. \end{aligned} \tag{5.21}$$

The double sum for α_1 and α_2 is divided into two parts. The part $\alpha_1 = \alpha_2$ is written by the contour-integral representation

$$\begin{aligned} U'_A(z_1, z_2) &= -\frac{1}{iN \left(z_1 + \frac{z_2}{c} \right)} \oint \frac{du}{2\pi i} \left[1 - \frac{i}{Nu} \left(z_1 + \frac{z_2}{c} \right) \right]^N \\ &\quad \times e^{-[iz_1/(1-c^2)] - [iz_2c/(1-c^2)]u - [z_1^2/2N(1-c^2)] - [z_2^2/2N(1-c^2)] - [cz_1z_2/N(1-c^2)]}. \end{aligned} \tag{5.22}$$

The Fourier transform of this quantity becomes by the change of variable u to $u[z_1 + (z_2/c)]$,

$$\rho^I(\lambda, \mu) = \frac{i}{N} \int \frac{dz_1 dz_2}{(2\pi)^2} \oint \frac{du}{2\pi i} \left(1 - \frac{i}{Nu}\right)^N e^{-i(z_1+z_2c)[z_1+(z_2/c)u/(1-c^2)]} e^{-[(z_1^2+z_2^2+2cz_1z_2)/2N(1-c^2)] - iz_1\lambda - iz_2\mu}. \quad (5.23)$$

This part is further simplified by the change of variables z_1 to $(1/\sqrt{1-c^2})(z_1 - cz_2)$ and z_2 to $(1/\sqrt{1-c^2})(z_2 - cz_1)$. By integration over z_1 , we have

$$\rho^I(\lambda, \mu) = \frac{i}{N} \frac{1}{(1-c^2)^2} e^{-(N/2)(\lambda - c\mu)^2} \int \frac{dz}{2\pi} \oint \frac{du}{2\pi i} \left(1 - \frac{i}{Nu}\right)^N e^{-i\mu z + Nu z[\mu - (\lambda/c)]} e^{-[i/(1-c^2)]uz^2 - (N/2c^2)u^2z^2 - [1/2N(1-c^2)]z^2}. \quad (5.24)$$

The remaining $\alpha_1 \neq \alpha_2$ part is given after letting a_γ go to zero,

$$U_0(z_1, z_2) = -\frac{c}{z_1 z_2} \oint \frac{du dv}{(2\pi i)^2} \left(1 - \frac{iz_1}{Nu}\right)^N \left(1 - \frac{iz_2}{cNv}\right)^N \left[1 - \frac{z_1 z_2}{cN^2 \left(u - v - \frac{iz_1}{N}\right) \left(u - v + \frac{iz_2}{cN}\right)}\right] \\ \times e^{-[iz_1 u/(1-c^2)] - [iz_2 cv/(1-c^2)] - [z_1^2/2N(1-c^2)] - [z_2^2/2N(1-c^2)]}. \quad (5.25)$$

This expression includes both a disconnected part and a connected part. The disconnected part has a factorized form and it corresponds to the first term in the bracket. This term, indeed by shifting $v \rightarrow v/c$, becomes the product of the density of states $\rho(\lambda)$ and $\rho(\mu)$. Therefore, after subtracting this disconnected part, we obtain the connected part

$$U_0(z_1, z_2) = -\frac{1}{N^2} \oint \frac{dudv}{(2\pi i)^2} \left(1 - \frac{iz_1}{Nu}\right)^N \left(1 - \frac{iz_2}{cNv}\right)^N \frac{1}{\left(u - v - \frac{iz_1}{N}\right) \left(u - v + \frac{iz_2}{cN}\right)} \\ \times e^{-[iz_1 u/(1-c^2)] - [iz_2 v c/(1-c^2)] - [z_1^2/2N(1-c^2)] - [z_2^2/2N(1-c^2)]}, \quad (5.26)$$

where the contour integrals are taken around $u=0$ and $v=0$. If we include the contour integration around the pole $v = u - iz_1/N$, we obtain precisely the same term as Eq. (5.22). Therefore we use this representation for the whole expression, including the term of Eq. (5.21), by taking the contour around both $v=0$ and $v = u - iz_1/N$.

The expression for the two-matrix connected correlation function $\rho_c^{(2)}(\lambda, \mu)$, which is obtained by the Fourier transform of $U_0(z_1, z_2)$, has a factorized form when we consider the contribution from $u=0$ and $v=0$. If z_1 and z_2 are replaced by $z_1 = z_1 - iuN$ and $z_2 = z_2 - ivcN$, we have a multiplicative form

$$\rho_c^{(2)}(\lambda, \mu) = -\oint \frac{du}{2\pi i} \int \frac{dz_2}{2\pi} \left(\frac{z_2}{cuN}\right)^N \frac{1}{u + \frac{iz_2}{cN}} \\ \times e^{-[Nu^2/2(1-c^2)] - [z_2^2/2N(1-c^2)] - iz_2\mu - uN\lambda} \\ \times \oint \frac{dv}{2\pi i} \int \frac{dz_1}{2\pi} \left(\frac{z_1}{vN}\right)^N \frac{1}{v + \frac{iz_1}{N}} \\ \times e^{-[v^2c^2N/2(1-c^2)] - [z_1^2/2N(1-c^2)] - iz_1\lambda - v c\mu N}. \quad (5.27)$$

We write this expression as

$$\rho^{II}(\lambda, \mu) = -K_N(\lambda, \mu) \bar{K}_N(\lambda, \mu). \quad (5.28)$$

We find the previous first part $\rho^I(\lambda, \mu)$ of Eq. (5.24) is also expressed by

$$\rho^I(\lambda, \mu) = \frac{1}{N} K_N(\lambda, \mu) e^{-(N/2)(\lambda - c\mu)^2}, \quad (5.29)$$

where $K_N(\lambda, \mu)$ is given by

$$K_N(\lambda, \mu) = i^N (-iN) \oint \frac{du}{2\pi i} \int \frac{dz}{2\pi} \left(1 - \frac{i}{uN}\right)^N \\ \times e^{-(Nu^2z^2/2c^2) - [iu^2z/(1-c^2)] + zu\mu N - iz\mu} \\ \times e^{-(N\lambda u/c) - [z^2/2N(1-c^2)]}. \quad (5.30)$$

This kernel $K_N(\lambda, \mu)$ is written as a sum of Hermite polynomials $H_n(x)$,

$$K_N(\lambda, \mu) = e^{-(N/2)(1-c^2)\mu^2} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{c^n} \frac{H_n(\beta\lambda) H_n(\beta\mu)}{n!}, \quad (5.31)$$

where $\beta = \sqrt{(N/2)(1-c^2)}$. The other kernel $\bar{K}_N(\lambda, \mu)$ is given

$$\bar{K}_N(\lambda, \mu) = \frac{1}{N} e^{-(N/2)(1-c^2)\lambda^2} \sum_{n=0}^{N-1} c^n \frac{H_n(\beta\lambda) H_n(\beta\mu)}{n!}. \quad (5.32)$$

It follows from these expressions that $\rho^I(\lambda, \mu)$ and $\rho^{II}(\lambda, \mu)$ are invariant under exchange of λ and μ . The ex-

pressions of Eqs. (5.28) and (5.29) for the correlation function by the kernel $K_N(\lambda, \mu)$ agree with the result obtained by the method of orthogonal polynomials [27]. The only difference is the exponential Gaussian factor in $K_N(\lambda, \mu)$ and $\bar{K}_N(\lambda, \mu)$, and this difference disappears for the product of these two kernels.

In the large- N limit, we expect to recover the usual universality. We return to the expression of Eq. (5.26), and neglect the terms iz_1/N and iz_2/cN in the denominator. We then exponentiate the powers of N . The integrals over z_1 and z_2 are Gaussian, and this leads to

$$\begin{aligned} \rho^{II}(\lambda, \mu) &= \frac{(1-c^2)}{2\pi N} \oint \frac{du dv}{(2\pi i)^2} \frac{1}{(u-v)^2} \\ &\quad \times e^{-[N/2(1-c^2)](u+[(1-c^2)/u]+(1-c^2)\lambda)^2} \\ &\quad \times e^{-\frac{N}{2(1-c^2)}(vc+[(1-c^2)/cv]+(1-c^2)\mu)^2} \\ &= \frac{(1-c^2)}{2\pi N} \oint \frac{du dv}{(2\pi i)^2} \frac{1}{(u-v)^2} e^{-N[f_1(u)+f_2(v)]}. \end{aligned} \quad (5.33)$$

We use the saddle points of $u(\lambda)$ and $u(\mu)$, which are the solutions of $\partial f_1 \partial u = 0$ and $\partial f_2 \partial v = 0$, i.e.,

$$u^2 + \lambda(1-c^2)u + (1-c^2) = 0, \quad (5.34)$$

$$v^2 + \frac{1-c^2}{c}\mu v + \frac{1-c^2}{c^2} = 0.$$

Taking into account the fluctuations around the saddle point, we have

$$\rho^{II}(\lambda, \mu) = -\frac{1-c^2}{2\pi^2 N^2} \sum \frac{1}{[u(\lambda) - v(\mu)]^2} \frac{1}{\left(\frac{\partial^2 f_1}{\partial u^2} \frac{\partial^2 f_2}{\partial v^2}\right)^{1/2}}, \quad (5.35)$$

where the sum is taken over four different saddle points; note that f_1 and f_2 vanish at these saddle points. We write the expressions for $u(\lambda)$ and $v(\mu)$ explicitly by solving Eq. (5.34) as

$$\begin{aligned} u(\lambda) &= \frac{1-c^2}{2} \left[-\lambda \pm \left(\lambda^2 - \frac{4}{1-c^2} \right)^{1/2} \right], \\ v(\mu) &= \frac{1-c^2}{2c} \left[-\mu \pm \left(\mu^2 - \frac{4}{1-c^2} \right)^{1/2} \right], \end{aligned} \quad (5.36)$$

where we put $\lambda = \sqrt{4/(1-c^2)} \sin \theta$ and $\mu = \sqrt{4/(1-c^2)} \sin \varphi$. The saddle points become $u = i\sqrt{1-c^2} e^{i\theta}$, $-i\sqrt{1-c^2} e^{-i\theta}$ and $v = (i/c)\sqrt{1-c^2} e^{i\varphi}$, $-(i/c)\sqrt{1-c^2} e^{-i\varphi}$. Then, adding these solutions of u and v in terms of θ and φ , from (5.35) we obtain

$$\begin{aligned} \rho^{II}(\lambda, \mu) &= \frac{1-c^2}{8N^2 \pi^2 c} \frac{1}{\cos \theta \cos \varphi} \left(\frac{\frac{2}{c} - \left(1 + \frac{1}{c^2}\right) \cos(\theta - \varphi)}{\left[\left(1 + \frac{1}{c^2}\right) - \frac{2}{c} \cos(\theta - \varphi)\right]^2} \right. \\ &\quad \left. + \frac{\frac{2}{c} + \left(1 + \frac{1}{c^2}\right) \cos(\theta + \varphi)}{\left[\left(1 + \frac{1}{c^2}\right) + \frac{2}{c} \cos(\theta + \varphi)\right]^2} \right). \end{aligned} \quad (5.37)$$

This expression in the large- N limit coincides with the previous result [23]. The denominator of Eq. (5.37) does not vanish for $\lambda \rightarrow \mu$. Note that c is related to the time t as $c = e^{-t}$. When t is small, we have $c \sim 1 - t$. Then the denominator is approximated as $[1 - (1/c)]^2 + (\theta - \varphi)^2/c \sim t^2 + (t/2)(\lambda - \mu)^2$, when λ and μ are small. Note that we have to rescale λ and μ for the time-dependent case by a factor $\sqrt{e^{-t} \sinh t} = \sqrt{(1-c^2)/2}$, as explained in Eq. (5.11). Then we have $t^2 + 1/2(\lambda - \mu)^2$ as a denominator, and the result agrees with [23]. In the time-dependent case in Eq. (5.1), λ and μ are interpreted as one-dimensional space coordinates.

In order to discuss the oscillatory behavior, we return to expression (5.27). We then change z_2 into Nzc , and obtain

$$K_N(\lambda, \mu) = cN \oint \frac{du}{2\pi i} \int \frac{dz}{2\pi} \frac{1}{u + iz} e^{-Nf(z, u)}, \quad (5.38)$$

where $f(z, u)$ is given by

$$f(z, u) = \frac{c^2 z^2}{2(1-c^2)} + ic\mu z - \ln z + \frac{u^2}{2(1-c^2)} + \lambda u + \ln u. \quad (5.39)$$

Note that the variables z and u are decoupled, and the saddle-point equations are simplified. Then, using the previous notations $\lambda = \sqrt{4/(1-c^2)} \sin \theta$ and $\mu = \sqrt{4/(1-c^2)} \sin \varphi$, we find the relevant saddle points for z and u from the solutions of $\partial f / \partial z = 0$ and $\partial f / \partial u = 0$,

$$z = \frac{\sqrt{1-c^2}}{c} e^{-i\varphi}, \quad -\frac{\sqrt{1-c^2}}{c} e^{i\varphi}, \quad (5.40)$$

$$u = i\sqrt{1-c^2} e^{i\theta}, \quad -i\sqrt{1-c^2} e^{-i\theta}.$$

For the saddle point values, $z = (\sqrt{1-c^2}/c) e^{-i\varphi}$ and $u = i\sqrt{1-c^2} e^{i\theta}$, f and the fluctuation determinant $(\partial^2 f / \partial u^2)(\partial^2 f / \partial z^2)$ become

$$\begin{aligned} f(z, u) &= i(\theta + \varphi) + \frac{1}{2}(e^{2i\theta} - e^{-2i\varphi}), \\ \frac{1}{\left(\frac{\partial^2 f}{\partial u^2} \frac{\partial^2 f}{\partial z^2}\right)^{1/2}} &= \left(\frac{1-c^2}{2c}\right) \frac{e^{i(2)(\theta-\varphi)}}{\sqrt{\cos \theta \cos \varphi}}. \end{aligned} \quad (5.41)$$

Adding other saddle-points values, we obtain

$$K_N(\lambda, \mu) = -\frac{\sqrt{1-c^2} e^{(N/4)(1-c^2)(\lambda^2-\mu^2)}}{4\pi N \sqrt{\cos\theta \cos\varphi}} \left(\frac{\cos[N(h(\theta)+h(\varphi))-(\theta+\varphi)] + c\cos[N(h(\theta)+h(\varphi))]}{\frac{1+c^2}{2c} + \cos(\theta+\varphi)} - \frac{\cos[N(h(\theta)-h(\varphi))-(\theta-\varphi)] - c\cos[N(h(\theta)-h(\varphi))]}{\frac{1+c^2}{2c} - \cos(\theta-\varphi)} \right), \quad (5.42)$$

where $h(\theta) = \theta + \frac{1}{2}\sin 2\theta + (1/2N)\theta$. Thus we easily find the oscillating behavior of $K_N(\lambda, \mu)$ from the integral representation in the large- N limit. In [27], this result had been obtained through the integral representation of Hermite polynomials. Our derivation is more direct.

The oscillating behavior of another kernel $\bar{K}_N(\lambda, \mu)$ is obtained similarly from Eq. (5.27). If we make the smooth average for the product of $K_N(\lambda, \mu)$ and $\bar{K}_N(\lambda, \mu)$, we obtain the previous result, Eq. (5.37). Also, the expression of Eq. (5.42) is the generalization of the one-matrix result of Eq. (2.34), which may be obtained in the limit $c \rightarrow 1$.

For the Fourier transform $K(\tau)$, the form factor is obtained from Eq. (5.42). We consider the simple case $E=0$,

$$S(\tau) = \int d\omega e^{i\omega\tau} \rho(E, E+\omega)|_{E=0}. \quad (5.43)$$

Using Eq. (5.42), for $\theta=0, (2/\sqrt{1-c^2})\sin\varphi = \omega$, we have

$$K_N(0, \omega) \bar{K}_N(0, \omega) = -\frac{1-c^2}{16\pi^2 N^2} \frac{1}{\cos\varphi} \frac{1}{\left[\frac{(1-c)^2}{4c^2} + \sin^2\varphi \right]^{2f}}, \quad (5.44)$$

where f is

$$f = [-2 \cos\varphi \cos[Nh(\varphi) - \varphi] + (1+c^2)\cos Nh(\varphi)]^2. \quad (5.45)$$

The Fourier transform of Eq. (5.43) is an integral over ω which may be performed by taking the residue at the pole $\omega = i(\sqrt{1-c}/c)$. Then we obtain

$$\int d\omega e^{i\omega\tau} K_N(0, \omega) \bar{K}_N(0, \omega) \simeq \tau e^{-(\sqrt{1-c}/c)\tau}. \quad (5.46)$$

Thus we find that the form factor has a linear term in τ , but modified by $e^{-(\sqrt{1-c}/c)\tau}$. Note that the integrand of Eq. (5.43) has poles on both sides of the real axis, and the integral does not vanish for τ larger than $2a$, unlike Eq. (2.7). However, for large τ , the form factor becomes exponentially small. The Fourier integral of the first term ρ^I also becomes exponentially small for large τ . Therefore, the singularity of the form factor $S(\tau)$ at the Heisenberg time $\tau=2N$ in the one-matrix model is smeared out for the time-dependent case.

VI. CORRELATION FUNCTIONS WITH AN EXTERNAL FIELD

The contour-integral method can be applied to the case in which the Hamiltonian is a sum of a deterministic term and of a random one [14],

$$H = H_0 + V. \quad (6.1)$$

H_0 is a given deterministic term, and V is a random matrix with a Gaussian distribution P given by

$$P(H) = \frac{1}{Z} e^{-(N/2)\text{Tr}V^2} = \frac{1}{Z} e^{-(N/2)\text{Tr}(H^2 - 2H_0H + H_0^2)}. \quad (6.2)$$

Then we deal with a Gaussian unitary ensemble modified by a matrix source $A = -H_0$. Therefore, keeping the finite eigenvalues a_i of A and putting $H = M$, we readily obtain the correlation function in the presence of the deterministic term A .

By the contour-integral method we will show that the connected part of the n -point correlation function is also given by the product of the two-point kernel $K_N(\lambda, \mu)$ in the case of a nonvanishing external source A .

Let us recall the two-point correlation function, which, becomes from Eq. (2.20) [14], becomes

$$U_A(t_1, t_2) = -\frac{1}{N^2} \oint \frac{du dv}{(2\pi i)^2} e^{-(t_1^2/2N) - (t_2^2/2N) - it_1 u - it_2 v} \times \frac{1}{\left(u-v - \frac{it_1}{N}\right) \left(u-v + \frac{it_2}{N}\right)} \times \prod_{\gamma=1}^N \left(1 - \frac{it_1}{N(u-a_\gamma)}\right) \left(1 - \frac{it_2}{N(v-a_\gamma)}\right). \quad (6.3)$$

By the shift $t_1 \rightarrow t_1 - iuN$, we obtain the kernel of the two-point correlation function

$$K_N(\lambda, \mu) = \frac{1}{N} \int \frac{dt}{2\pi} \oint \frac{dv}{2\pi i} \prod_{\gamma=1}^N \left(\frac{a_\gamma + \frac{it}{N}}{v - a_\gamma} \right) \times \frac{1}{v + \frac{it}{N}} e^{-(N/2)v^2 - (t^2/2N) - it\lambda - Nv\mu}. \quad (6.4)$$

The connected part of the two-level correlation function $\rho_c^{(2)}(\lambda, \mu)$ is given by $-(1/N^2)K_N(\lambda, \mu)K_N(\mu, \lambda)$. The Fourier transform of the three-point correlation function is given by

$$U(t_1, t_2, t_3) = \frac{1}{N^3} \langle \text{Tr} e^{it_1 M} \text{Tr} e^{it_2 M} \text{Tr} e^{it_3 M} \rangle. \quad (6.5)$$

We evaluate this quantity by the same procedure for the two-point correlation function. We integrate out the eigenvalues

of M by the use of Itzykson- Zuber formula. Then it becomes

$$U(t_1, t_2, t_3) = \frac{1}{N^3} \sum_{\alpha_i=1}^N \frac{\prod_{i<j} (b_i - b_j)}{\prod_{i<j} (a_i - a_j)} e^{(N/2)\sum b_i^2}, \quad (6.6)$$

where b_i is given

$$b_i = a_i - \frac{it_1}{N} \delta_{i,\alpha_1} - \frac{it_2}{N} \delta_{i,\alpha_2} - \frac{it_3}{N} \delta_{i,\alpha_3}. \quad (6.7)$$

We have to consider the cases $\alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_1 = \alpha_2 \neq \alpha_3$. These cases give the δ -function part as Eq. (2.4).

When all α_j 's are different, the contour-integral representation is straightforward,

$$U(t_1, t_2, t_3) = \oint \frac{du dv dw}{(2\pi i)^3} \frac{\prod \left(u - a_\gamma - \frac{it_1}{N} \right) \left(v - a_\gamma - \frac{it_2}{N} \right) \left(w - a_\gamma - \frac{it_3}{N} \right)}{\prod (u - a_\gamma)(v - a_\gamma)(w - a_\gamma)} \times \frac{\left(u - v - \frac{it_1}{N} + \frac{it_2}{N} \right) \left(u - w - \frac{it_1}{N} + \frac{it_3}{N} \right) \left(v - w - \frac{it_2}{N} + \frac{it_3}{N} \right)}{\left(u - v - \frac{it_1}{N} \right) \left(u - v + \frac{it_2}{N} \right) \left(u - w - \frac{it_1}{N} \right) \left(u - w + \frac{it_3}{N} \right) \left(v - w - \frac{it_2}{N} \right)} \times \frac{(u-v)(v-w)(w-u)}{\left(v - w + \frac{it_3}{N} \right)} \frac{1}{it_1 t_2 t_3} e^{-(t_1^2/2N) - (t_2^2/2N) - (t_3^2/2N) - it_1 u - it_2 v - it_3 w}. \quad (6.8)$$

Writing the part of the numerators of Eq. (6.8) as

$$\left[\left(u - v - \frac{it_1}{N} \right) \left(u - v + \frac{it_2}{N} \right) - \frac{t_1 t_2}{N^2} \right] \left[\left(v - w - \frac{it_2}{N} \right) \left(v - w + \frac{it_3}{N} \right) - \frac{t_2 t_3}{N^2} \right] \left[\left(u - w - \frac{it_1}{N} \right) \left(u - w + \frac{it_3}{N} \right) - \frac{t_1 t_3}{N^2} \right], \quad (6.9)$$

we have the following disconnected parts:

$$U(t_1)U(t_2)U(t_3) - U(t_1)U(t_2, t_3) - U(t_2)U(t_1, t_3) - U(t_3)U(t_1, t_2) + \left(u + \frac{it_2}{N} \right) \left(v + \frac{it_3}{N} \right) \left(w + \frac{it_1}{N} \right) + \left(u + \frac{it_3}{N} \right) \left(v + \frac{it_1}{N} \right) \left(w + \frac{it_2}{N} \right) \times \left(w + \frac{it_2}{N} \right). \quad (6.11)$$

The remaining term is a connected term. We consider the Fourier transform of this connected part $U_c(t_1, t_2, t_3)$, and make the changes of variables $t_1 \rightarrow t_1 - iuN$, $t_2 \rightarrow t_2 - ivN$, and $t_3 \rightarrow t_3 - iwN$. Then the remaining term of Eq. (6.9) becomes simply

After cancellation of these terms with the corresponding terms in the denominator, we obtain the connected part of the three-point correlation function $\rho^{(3)}(\lambda, \mu, \nu)$,

$$\rho_c^{(3)}(\lambda, \mu, \nu) = K_N(\lambda, \mu)K_N(\mu, \nu)K_N(\nu, \lambda) + K_N(\lambda, \nu)K_N(\nu, \mu)K_N(\mu, \lambda), \quad (6.12)$$

where $K_N(\lambda, \mu)$ is given by Eq. (6.4). Thus we find that the three-point correlation function has the same form as for the unitary Gaussian ensemble in terms of the kernel $K_N(\lambda, \mu)$, but the kernel should be modified as Eq. (6.4) for the external field case.

In the presence of the external source A , the kernel $K_N(\lambda, \mu)$ in Eq. (6.4) is not expressed as a sum of products of orthogonal polynomials. However, it satisfies remarkably the following equation:

$$\int_{-\infty}^{\infty} d\mu K_N(\lambda, \mu) K_N(\mu, \nu) = K_N(\lambda, \nu). \quad (6.13)$$

The proof of this equation is shown in Appendix B.

Although the kernel $K_N(\lambda, \mu)$ is no longer expressed in terms of Hermite polynomials when the external source is present, it still possesses N simple eigenfunctions since, for $n \leq N-1$,

$$\int_{-\infty}^{\infty} K_N(\lambda, \mu) H_n(\sqrt{N}\mu) e^{-(N/2)\mu^2} d\mu = H_n(\sqrt{N}\lambda) e^{-(N/2)\lambda^2}, \quad (6.14)$$

where $H_n(x)$ are the usual Hermite polynomials. However for $n \geq N$, Eq. (6.14) does not hold. Indeed in the case of zero external source, the right-hand side of Eq. (6.14) vanishes. When the external source is nonzero, the result is nonzero and a_i dependent. The proof of Eq. (6.14) is given in Appendix B.

The n -point correlation $R_n(\lambda_1, \dots, \lambda_n)$ is defined [1] by

$$R_n(\lambda, \dots, \lambda_n) = \frac{N!}{(N-n)!} \int \dots \int P_N(\lambda_1, \dots, \lambda_N) d\lambda_{n+1} \dots d\lambda_N \quad (6.15)$$

$$P_N(\lambda_1, \dots, \lambda_N) = \left\langle \prod_{i=1}^N \text{Tr} \delta(\lambda_i - M) \right\rangle. \quad (6.16)$$

Without external source, this n -point correlation function is expressed in terms of the kernel $K_N(\lambda_i, \lambda_j)$ as

$$R_n = \det(K_N(\lambda_i, \lambda_j)), \quad (6.17)$$

where $i, j = 1, \dots, n$. This result was derived Eq. [1], by the use of (6.3) since one has

$$P_N(\lambda_1, \dots, \lambda_n) = \frac{1}{N!} \det(K_N(\lambda_i, \lambda_j)), \quad (6.18)$$

where $i, j = 1, \dots, N$. For a nonvanishing external source, from representation (6.3), whose derivation is given in Appendix B, it follows that Eq. (6.17) still holds. This result leads to the universality of the level spacing probability $P(s)$. As shown in a previous work [14], the kernel $K_N(\lambda, \mu)$ has a universal short-distance behavior

$$K_N(\lambda, \mu) \simeq \frac{\sin \left[N\pi(\lambda - \mu) \rho \left(\frac{\lambda + \mu}{2} \right) \right]}{(\lambda - \mu)\pi}. \quad (6.19)$$

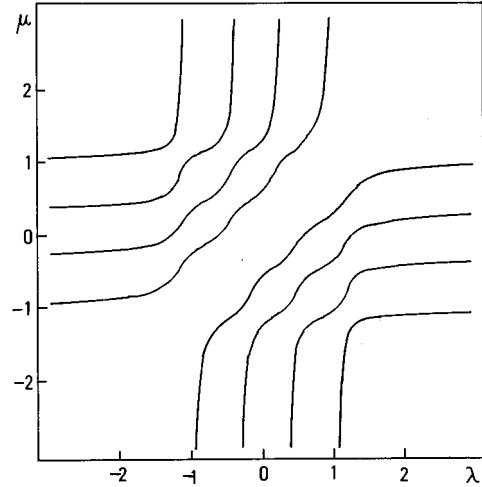


FIG. 1. The zeros of $K_N(\lambda, \mu)$ are plotted as lines in the real (λ, μ) plane, for $N=5$.

The level-spacing probability distribution $P(s)$ is related to the probability of having an empty interval with width S , $E(s)$, by

$$P(s) = \frac{d^2}{ds^2} E(s) \quad (6.20)$$

and $E(s)$ is obtained from the integration of $P_N(\lambda_1, \dots, \lambda_N)$ in which the region $-s/2 < \lambda_i < s/2$ is vacant. Thus we write

$$E(s) = \prod_i \left(\int_{-\infty}^{\infty} - \int_{-s/2}^{s/2} \right) d\lambda_i P_N(\lambda_1, \dots, \lambda_N). \quad (6.21)$$

Since the kernel $K_N(\lambda, \mu)$ has a universal form, the same as the case without external source, we conclude that $P(s)$ becomes the same as GUE.

VII. ZEROS OF THE KERNEL $K_N(\lambda, \mu)$

Since at short distance the kernel $K_N(\lambda, \mu)$ exhibits oscillations, and thus changes sign, this kernel must have lines of zeros in the (λ, μ) plane. Note that the solutions of the equation

$$K_N(\lambda, \mu) = 0 \quad (7.1)$$

are always real in the case of the one-matrix model. In the (λ, μ) plane the solutions of Eq. (7.1) lie on $2N$ lines, as shown in Fig. 1.

When μ is large enough, $K_N(\lambda, \mu)$ is approximated by the product of the Hermite polynomial $H_N(\lambda)$ multiplied by μ^N , as shown in Eq. (A4), and this Hermite polynomial has N real zeros. These N real zeros give rise to N noncrossing lines in the (λ, μ) plane for finite μ . This behavior is a

manifestation of the short-distance universality in the large- N limit. The distances between these N lines become equal when $\lambda - \mu$ is order of $1/N$. These N lines are parallel to the line $\lambda = \mu$ when λ and μ are inside the support of the density of state. At the edge, these lines bend, and universality no longer holds.

We have discussed the time-dependent matrix model, which becomes equivalent to the two-matrix model. In this two-matrix model, the lines of zeros of $K_N(\lambda, \mu)$ show a different behavior. In this case, the kernel $K_N(\lambda, \mu)$ is given by Eqs. (5.31) or (5.32). When $c \neq 1$, the lines of zeros of $K_N(\lambda, \mu)$ in the real (λ, μ) plane are not parallel to the $\lambda = \mu$ line, as shown in Fig. 2, in which one can see that some lines turn over. Consequently along the line $\lambda = -\mu$, some real solutions are missing. This behavior is related to the fact that the short-distance universality does not hold in the two-matrix model.

In the presence of an external source for the one-matrix model, the solutions of Eq. (7.1) are not modified much by the source if its eigenvalues are smoothly distributed. This is also a manifestation of the short-distance universality for the oscillatory behavior. When the support of the eigenvalues of the source are split into separate parts, this is reflected on the position of the lines of zeros, as shown in Fig. 3. The number of lines of zeros is conserved for an arbitrary distribution of the external eigenvalues.

VIII. DISCUSSION

In this paper, we applied the contour-integral representation for the kernel which characterizes the correlation functions, for the one-matrix model, the time-dependent matrix model, and in the presence of an external source. We investigated the form factor $S(\tau)$, which is the Fourier transform of the two-level correlation function, by the use of these contour-integral representations. The universality of the two-level correlation $\rho_c(\lambda, \mu)$ for $\lambda - \mu$, of order $1/N$, immediately implies the linear behavior of $S(\tau)$ in the large- N limit. We found explicit deviations from the linear behavior of $S(\tau)$, and found a surprising connection to the Laguerre ensemble for the average of $S(\tau)$.

Near the Heisenberg time $\tau = \tau_c$, a crossover behavior is observed. For the time-dependent matrix model, which we mapped into an equivalent two-matrix model, the universal behavior of the one-matrix model is no longer present, and the singularity at $\tau = \tau_c$ in $S(\tau)$ is then smeared out. This behavior indicates that near the Heisenberg time, the form factor is not universal. The nonuniversality was pointed out by the authors of [11] for the case of mesoscopic dirty metals. Our result is consistent with this nonuniversality. Finally, we investigated the zeros of the kernel $K_N(\lambda, \mu)$ for the two-matrix model, and found differences between the one- and two-matrix models.

For the matrix model with a nonzero external source, the universality of two-level correlation function holds, as shown in a previous paper [14]. With the technique of contour-integral representations, we have also obtained all the higher correlation functions.

ACKNOWLEDGMENTS

This work was supported by a joint research project between CNRS and JSPS, and by CREST of JST. S.H. is grateful for a Grant-in-Aid for Scientific Research by the Ministry of Education, Science and Culture.

APPENDIX A: INTEGRAL REPRESENTATION FOR THE KERNEL $K_N(\lambda, \mu)$

It is known that the kernel $K_N(\lambda, \mu)$ is a sum of products of orthogonal polynomial $P_l(\lambda)$ and $P_l(\mu)$ [1]. In the case of the Gaussian unitary ensemble, the orthogonal polynomials $P_l(\lambda)$ are simply Hermite polynomials. We may write these Hermite polynomials as contour integrals

$$H_l(\lambda) = \oint \frac{du}{2\pi i} \frac{l!}{u^{l+1}} e^{\lambda u - (1/2)u^2}. \tag{A1}$$

Their normalization is

$$\int_{-\infty}^{\infty} d\lambda H_l(\lambda) H_m(\lambda) e^{-(1/2)\lambda^2} = \sqrt{2\pi} l! \delta_{l,m}. \tag{A2}$$

It is convenient to use at the same time another integral representation for these Hermite polynomials obtained by introducing an auxiliary variable t ,

$$\begin{aligned} H_l(\lambda) &= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \oint \frac{du}{2\pi i} \frac{l!}{u^{l+1}} e^{\lambda u + itu - (1/2)t^2} \\ &= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} (it)^l e^{-(t^2/2) - it\lambda + (t^2/2)}. \end{aligned} \tag{A3}$$

The kernel $K_N(\lambda, \mu)$ is given by

$$K_N(\lambda, \mu) = \left(\frac{N}{2\pi}\right)^{1/2} \sum_{l=0}^{N-1} \frac{H_l(\sqrt{N}\lambda) H_l(\sqrt{N}\mu)}{l!} e^{-(N/2)\lambda^2}. \tag{A4}$$

We use two different expressions (A1) and (A3) of Hermite polynomials in the kernel (A4). The summation of the geometric series give $[1 - (it/u)^N]/1 - (it/u)$. Then we shift $t \rightarrow t/\sqrt{N}$ and $u \rightarrow -\sqrt{N}u$.

$$\begin{aligned} K_N(\lambda, \mu) &= - \int_{-\infty}^{\infty} \frac{dt}{2\pi} \oint \frac{du}{2\pi i} \left(-\frac{it}{Nu}\right)^N \\ &\quad \times \frac{1}{u + \frac{it}{N}} e^{-(N/2)u^2 - (1/2N)t^2 - it\lambda - Nu\mu}. \end{aligned} \tag{A5}$$

This expression coincides with Eq. (2.22), which has been derived earlier by Kazakov's method of a vanishing external source.

This integral representation may also be applied to the two-matrix model. The expressions for $K_N(\lambda, \mu)$ and

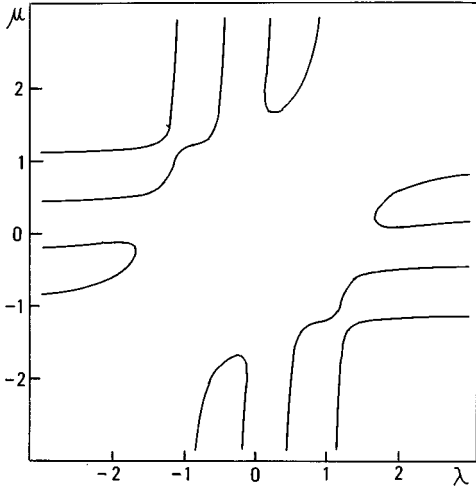


FIG. 2. The lines of zeros of $K_N(\lambda, \mu)$, $N=5$, for the two-matrix model with $c=1/2$.

$\bar{K}_N(\lambda, \mu)$ in Eq. (5.27) are obtained from Eqs. (5.31) and (5.32) by the same integral representations of Eqs. (A1) and (A3).

APPENDIX B: PROPERTIES OF THE KERNEL $K_N(\lambda, \mu)$ IN THE EXTERNAL SOURCE

We consider the proof of Eq. (6.13). By the integral representation, we write the integrand, which is a product of the kernels, by I ,

$$\begin{aligned}
 I &= K_N(\lambda, \mu) K_N(\mu, \nu) \\
 &= \int_{-\infty}^{\infty} \frac{dt_1}{2\pi} \oint \frac{du_1}{2\pi i} \int_{-\infty}^{\infty} \frac{dt_2}{2\pi} \oint \frac{du_2}{2\pi i} \prod_{\gamma} \left(\frac{a_{\gamma} + \frac{it_1}{N}}{u_1 - a_{\gamma}} \right) \\
 &\quad \times \left(\frac{a_{\gamma} + \frac{it_2}{N}}{u_2 - a_{\gamma}} \right) \frac{1}{u_1 + \frac{it_1}{N}} \frac{1}{u_2 + \frac{it_2}{N}} \\
 &\quad \times e^{-(N/2)u_1^2 - (N/2)u_2^2 - (t_1^2/2N) - (t_2^2/2N) - it_1\lambda - it_2\mu - Nu_1\mu - Nu_2\nu}.
 \end{aligned} \tag{B1}$$

Making the shift $t_2 \rightarrow t_2 + iNu_1$, and integrating I over μ , we obtain the $\delta(t_2)$ function. Thus the integral of I becomes

$$\begin{aligned}
 \int_{-\infty}^{\infty} d\mu I &= -(-1)^N \int_{-\infty}^{\infty} \frac{dt_1}{2\pi} \oint \frac{du_1}{2\pi i} \oint \frac{du_2}{2\pi i} \\
 &\quad \times \prod_{\gamma} \left(\frac{a_{\gamma} + \frac{it_1}{N}}{u_2 - a_{\gamma}} \right) \frac{1}{[u_1 + (it_1/N)](u_2 - u_1)} \\
 &\quad \times e^{-(t_1^2/2N) - it_1\lambda - (N/2)u_2^2 - Nu_2\nu}.
 \end{aligned} \tag{B2}$$

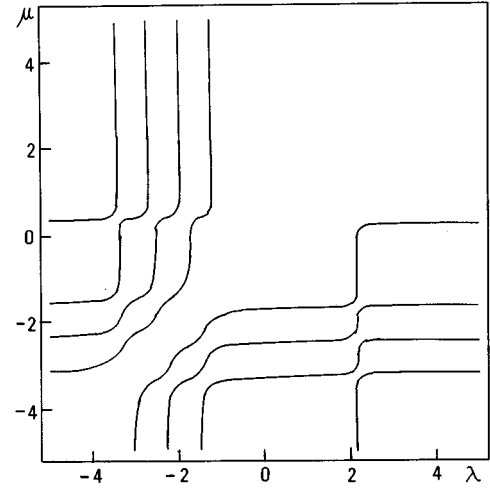


FIG. 3. The lines of zeros of $K_N(\lambda, \mu)$, $N=5$ for the external source, $a_1 = -2$, $a_2 = 2$, $a_3 = 2.25$, $a_4 = 2.5$, and $a_5 = 2.75$

The contour integral of u_1 can be performed around $u_1 = -it_1/N$, since u_1 appears only in the denominator. Then we obtain

$$\begin{aligned}
 \int d\mu I &= \int \frac{dt_1}{2\pi} \oint \frac{du_2}{2\pi i} \\
 &\quad \times \prod_{\gamma} \left(\frac{a_{\gamma} + \frac{it_1}{N}}{u_2 - a_{\gamma}} \right) \frac{1}{u_2 + \frac{it_1}{N}} \\
 &\quad \times e^{-(N/2)u_2^2 - (t_1^2/2N) - it_1\lambda - Nu_2\nu} \\
 &= K_N(\lambda, \nu).
 \end{aligned} \tag{B3}$$

The integral equation of Eq. (6.14) is verified similarly. We evaluate first the following integral involving Hermite polynomials,

$$\begin{aligned}
 &\int_{-\infty}^{\infty} H_n(\sqrt{N}\mu) e^{-(N/2)(\mu+u)^2} d\mu \\
 &= \int_{-\infty}^{\infty} H_n(\mu) e^{-(1/2)(\mu+\sqrt{N}u)^2} \frac{du}{\sqrt{N}} \\
 &= \int_{-\infty}^{\infty} H_n(\mu) \left(\sum_{l=0}^{\infty} H_l(\mu) \frac{(-\sqrt{N}u)^l}{l!} \right) e^{-(1/2)\mu^2} \frac{d\mu}{\sqrt{N}} \\
 &= \sqrt{2\pi} (-u)^n N^{(n-1)/2}.
 \end{aligned} \tag{B4}$$

Using the expression of Eq. (6.4) for $K_N(\lambda, \mu)$ with Eq. (B4), we obtain

$$\begin{aligned} & \int K_N(\lambda, \mu) H_n(\sqrt{N}\mu) e^{-(N/2)\mu^2} d\mu \\ &= -(-1)^N \sqrt{2\pi} N^{(n-1)/2} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \oint \frac{du}{2\pi i} \\ & \times \prod_{\gamma=1}^N \left(\frac{a_\gamma + \frac{it}{N}}{u - a_\gamma} \right) \frac{(-u)^n}{u + \frac{it}{N}} e^{-(t^2/2N) - it\lambda}. \end{aligned} \tag{B5}$$

When $n < N$, this contour-integration of u converges for $|u| \rightarrow \infty$; we may then take the residues of the poles outside of the contour, i.e., the pole $u = -it/N$ instead of the a_γ 's. Then, by evaluating the residue, and performing the t integration, we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} K_N(\lambda, \mu) H_n(\sqrt{N}\mu) e^{-(N/2)\mu^2} d\mu \\ &= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \left(\frac{it}{N} \right)^n N^{(n-1)/2} e^{-(1/2N)t^2 - it\lambda} \\ &= H_n(\sqrt{N}\lambda) e^{-(N/2)\lambda^2}. \end{aligned} \tag{B6}$$

Note that the contour integration around infinity does not converge for $n \geq N$; we can no longer take the poles outside of the contour, and we obtain a different result.

The kernel $K_N(\lambda, \mu)$ for a nonzero external source can be written as a determinant; this is useful for numerical calculations. Since the kernel $K_N(\lambda, \mu)$ satisfies Eq. (B6), we write an expression for $K_N(\lambda, \mu)$ as a determinant, in which the variable μ appears only in the first row of the $N \times N$ matrix. The eigenvalues of external source a_i appears only in the i th column of the matrix. This is related to the fact that the exchange between a_i and a_j in Eq. (B7) does not affect $K_N(\lambda, \mu)$. We can write the matrix element of the i th row as a polynomial of order of $(i-1)$ for λ . The first three rows, for example, are expressed by

$$K_N(\lambda, \mu) = -\frac{1}{\sqrt{2\pi}} \frac{e^{-(N/2)\lambda^2}}{\prod_{i < j} (a_i - a_j)} \det \begin{pmatrix} e^{-(N/2)a_1^2 - Na_1\mu} & e^{-(N/2)a_2^2 - Na_2\mu} & \dots & e^{-(N/2)a_N^2 - Na_N\mu} \\ \lambda + a_1 & \lambda + a_2 & \dots & \lambda + a_N \\ (\lambda + a_1)^2 + \frac{1}{N} & (\lambda + a_2)^2 + \frac{1}{N} & \dots & (\lambda + a_N)^2 + \frac{1}{N} \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{B7}$$

The matrix element m_{ij} is $(\lambda + a_j)^{i-1}$ when i is even. When i is odd, then m_{ij} is given by $(\lambda + a_j)^{i-1} + C$, where C is a constant depending upon N . This constant is determined to be consistent with Eq. (B6).

As a simple example for $N=2$, we obtain

$$\begin{aligned} K_2(\lambda, \mu) &= -\frac{1}{\sqrt{2\pi}} \frac{1}{(a_1 - a_2)} e^{-\lambda^2} \\ & \times \det \begin{pmatrix} e^{-a_1^2 - 2a_1\mu} & e^{-a_2^2 - 2a_2\mu} \\ \lambda + a_1 & \lambda + a_2 \end{pmatrix}. \end{aligned} \tag{B8}$$

It may be interesting to note that there exists values of the a_i for which the two-level correlation function $\rho_c(\lambda, \mu) = -K_N(\lambda, \mu)K_N(\mu, \lambda)$, which is normally negative, may become positive. For instance, in the case $N=2$, we consider $\mu=0, a_2=0$. In this case, from Eq. (B8), we obtain $K_2(\lambda, 0) = -1/\sqrt{2\pi}(\lambda e^{-a_1^2} - \lambda - a_1)e^{-\lambda^2}/a_1$ and $K_2(0, \lambda) = 1/\sqrt{2\pi}$. Then we have

$$\rho_c(\lambda, 0) = \frac{1}{8\pi a_1} (\lambda e^{-a_1^2} - \lambda - a_1) e^{-\lambda^2}. \tag{B9}$$

This expression may take positive values, for example when $a_1=1$ and $\lambda < -1.6$. Expression (B7) is useful for locating numerically the zeros of $K_N(\lambda, \mu)$.

Finally let us mention that the contour-integral technique which we have used here may be extended to the n -point functions. If we define

$$\begin{aligned} U(t_1, \dots, t_n) &= \oint \frac{\prod du_i}{(2\pi i)^n} \prod_{i=1}^n \prod_{\gamma=1}^N \frac{[u_i - a_\gamma - (it_i/N)]}{(u_i - a_\gamma)} \\ & \times \prod_{i < j} \frac{[u_i - u_j - (it_i/N) + (it_j/N)]}{\left(u_i - u_j - \frac{it_i}{N} \right) [u_i - u_j + (it_j/N)]} \\ & \times \prod_{i < j} (u_i - u_j) \frac{1}{\prod t_i} e^{-(1/2N)\sum t_i^2 - i\sum t_i u_i}; \end{aligned} \tag{B10}$$

then $R_n(\lambda_1, \dots, \lambda_n)$ is obtained by Fourier transform of $U(t_1, \dots, t_n)$ as

$$R_n(\lambda_1, \dots, \lambda_n) = \int U(t_1, \dots, t_n) e^{-i\sum t_i \lambda_i} dt_i. \tag{B11}$$

- [1] M. L. Mehta, *Random Matrices*, 2nd ed. (Academic, New York, 1991).
- [2] F. Dyson, *J. Math. Phys.* **13**, 90 (1972).
- [3] O. Bohigas, in *Chaos and Quantum Physics, Proceedings of the Les Houches Summer School, Session LII* (North-Holland, Amsterdam, 1991), p. 91.
- [4] E. Brézin and A. Zee, *Nucl. Phys. B* **402**, 613 (1993).
- [5] G. Hackenbroich and H. A. Weidenmuller, *Phys. Rev. Lett.* **74**, 4118 (1995).
- [6] J. Ambjorn and Yu. M. Makeenko, *Mod. Phys. Lett. A* **5**, 1753 (1990).
- [7] E. Brézin and A. Zee, *C. R. Acad. Sci.* **17**, 735 (1993).
- [8] A. M. Khorunzhy, B. Khorunzhenko, and L. Pastur, *J. Phys. A* **28**, L31 (1995).
- [9] M. V. Berry, *Proc. R. Soc. London Ser. A* **400**, 229 (1985).
- [10] V. E. Kravtsov and A. D. Mirlin, *Pis'ma Zh. Éksp. Teor. Fiz.* **60**, 645 (1994) [*JETP Lett.* **60**, 657 (1994)].
- [11] A. V. Andreev and B. L. Altshuler, *Phys. Rev. Lett.* **75**, 902 (1995).
- [12] O. Agam, B. L. Altshuler, and A. V. Andreev, *Phys. Rev. Lett.* **75**, 4389 (1995).
- [13] O. Agam and S. Fishman, *Phys. Rev. Lett.* **76**, 726 (1996).
- [14] E. Brézin and S. Hikami, *Nucl. Phys. B* **479**, 697 (1996).
- [15] V. A. Kazakov, *Nucl. Phys. B* **354**, 614 (1991).
- [16] E. Brézin, S. Hikami, and A. Zee, *Nucl. Phys. B* **464**, 411 (1996).
- [17] S. Hikami and A. Zee, *Nucl. Phys. B* **446**, 337 (1995).
- [18] K. Slevin and T. Nagao, *Phys. Rev. Lett.* **70**, 635 (1993).
- [19] A. V. Andreev, B. D. Simons, and N. Taniguchi, *Nucl. Phys. B* **432**, 487 (1994).
- [20] E. Brézin, S. Hikami, and A. Zee, *Phys. Rev. E* **51**, 5442 (1995).
- [21] T. Guhr, *J. Math. Phys.* **32**, 336 (1991).
- [22] C. Itzykson and J.-B. Zuber, *J. Math. Phys.* **21**, 411 (1980).
- [23] E. Brézin and A. Zee, *Phys. Rev. E* **49**, 2588 (1994).
- [24] C. W. J. Beenakker, *Phys. Rev. Lett.* **70**, 1155 (1993).
- [25] A. Szafer and B. L. Altshuler, *Phys. Rev. Lett.* **70**, 587 (1993).
- [26] O. Narayan and B. S. Shastry, *Phys. Rev. Lett.* **71**, 2106 (1993).
- [27] J. D'Anna, E. Brézin, and A. Zee, *Nucl. Phys.* **443**, 433 (1995).