Completely integrable nonlinear Schrödinger type equations on moving space curves

K. Porsezian*
Institut für Theoretische Physik, Universität Hannover, 30167 Hannover, Germany
and Centre for Laser Technology, Department of Physics, Anna University, Madras-200 025, India
(Received 27 June 1996)

Using the Lamb formalism, we show that some completely integrable homogeneous and inhomogeneous nonlinear Schrödinger (NLS) type equations such as derivative NLS, extended NLS, higher-order NLS, inhomogeneous NLS, circularly and radially symmetric NLS, and generalized inhomogeneous radially symmetric NLS equations can be related to certain types of moving helical space curves. [S1063-651X(97)06903-1]

PACS number(s): 03.40.Kf, 02.40.−k, 11.10.Lm, 68.10.−m

I. INTRODUCTION

It is well known that nonlinear science plays an important role in explaining many phenomena in science . After the discovery of solitons in 1967 by Zubusky and Kruskal [1], while solving the Korteweg–de Vries equation numerically, a continuous search has been going on to identify this exciting concept in all areas of science exhibiting nonlinear behavior. Solitons have emerged as a paradigm for nonlinear phenomena in a variety of fields such as hydrodynamics, nonlinear optics, plasmas, biological molecules, solid state physics, and certain field theories [2]. The experimental verification of solitons has also been reported from the above areas. Presently there are hundreds of nonlinear partial differential equations (NPDE) that have soliton solutions or allow soliton-type properties. The Korteweg–de Vries (KdV), the modified KdV, the nonlinear Schrödinger (NLS), and the sine-Gordon equations are some of the prototype equations that frequently appear in nonlinear science. The above-mentioned equations are classified as completely integrable equations, which have certain important properties: (i) the linear eigenvalue equation also known as Lax pair, (ii) N-soliton solutions, (iii) an infinite number of integrals of motion, and (iv) a Hamiltonian structure.

There are many nonlinear problems in physics that can be described [3] in terms of a three-dimensional vector field $\vec{t}$ of unit magnitude, normalized such that $\vec{t}\cdot\vec{t}=1$. For example (i) the propagation of light in a twisted optical fiber is studied [4] in terms of $\vec{t}(x)$, a function of one spatial variable x, the distance along the fiber. (ii) The time evolution of the (normalized) classical spin vector at a cite in the continuum version of a one-dimensional Heisenberg spin chain is described [5] by $\vec{t}(x,y)$, a function of two variables, one spatial and the other temporal; (iii) the vector field in the (2+1)-dimensional O(3) nonlinear sigma model [6] in field theory is described by $\vec{t}(x,y,z)$. There exists a connection between this sigma model and two-dimensional (2D) antiferromagnets, which in turn is relevant in the study of high-T_c superconductors.

In this Brief Report, we will consider many examples in which geometric considerations yield a NLS family of equations. Lamb [7] investigated the connection between sine-Gordon and Hirota equations with that of the motion of the moving helical space curves and also derived the linear eigenvalue problem from the Ricaati equation. This is an extension of a result obtained by Hasimoto [8], who showed that the intrinsic equation governing the curvature and torsion of an isolated thin vortex filament moving without stretching in an incompressible inviscid fluid can be reduced to the NLS equation. Recently, it has been shown that the time evolution of the space curve is associated with a geometric phase and also discussed the application of this formalism to the classical, continuous, antiferromagnetic Heisenberg spin chain [9]. In an intriguing recent paper, Goldstein and Patrich [10] related integrable evolution equations from the modified KdV hierarchy to motions of closed curves in a plane, and also the Serret-Frenet equations are shown to be equivalent [11] to the Ablowitz-Kaup-Newell-Segur (AKNS) [12] scattering problem at zero eigenvalue. Doliwa and Santini [13] have shown that the elementary geometric properties of the motion of a space curve select hierarchies of different integrable dynamical systems depending on the dimensionality (N) of the sphere. In their formulation, $N=2$ gives the modified KdV hierarchy, $N=3$ gives the NLS hierarchy, and $N>3$ gives multicomponent generalizations of the above hierarchies admitting a 3×3 eigenvalue problem. From the above investigations, it is clear that the derivative and inhomogeneous NLS-type equations have not been reported. In recent years many NLS-type equations have been derived from different physical and mathematical considerations. Then the obvious question arises of whether these equations can also be related to space curves. The main aim of this paper is to answer this question and to derive various solitons possessing NLS-type equations from the moving helical space curve. It is shown that the derivative NLS equation admitting a Kaup-Newell [14] or Wadati-Kono-Ichikawa (WKI) [15] eigenvalue problem, the extended NLS equation admitting a 3×3 eigenvalue problem, and different inhomogeneous NLS equations admitting nonisospectral eigenvalue can also be related to the moving helical space curve.

II. MOTION OF THE FILAMENT: LAMB FORMALISM

From the elementary presentation of curve theory, we know that the motion of a twisted curve may be described by

*Electronic address: ponz@itp.uni-hannover.de and annalib@sirnetm.ernet.in

specifying the curvature and torsion of each point on the curve as a function of time. At each instant the spatial variations of the unit tangent, normal and binormal vectors $\vec{t},\vec{n}$, and $\vec{b}$, respectively, are given by the Serret-Frenet equations

$$\vec{t}_s=\kappa\vec{n}, \tag{1}$$

$$\vec{b}_s=-\tau\vec{n}, \tag{2}$$

$$\vec{n}_s=\tau\vec{b}-\kappa\vec{t}. \tag{3}$$

The subscripts denote partial derivatives with respect to arc length parameter s. The curvature κ and torsion τ are now functions of time as well as s. Combining Eqs. (2) and (3), we get

$$(\vec{n}+i\vec{b})_s+i\tau(\vec{n}+i\vec{b})=-\kappa\vec{t}. \tag{4}$$

Suppose $\tau\to\tau_0$ in regions of the curve that are remote from the disturbances of interest, then Eq. (4) takes the form

$$\vec{N}_s+i\tau_0\vec{N}=-\psi\vec{t}, \tag{5}$$

where

$$\vec{N}=(\vec{n}+i\vec{b})\exp\left[i\int_{-\infty}^{s}ds'(\tau-\tau_0)\right]$$

and

$$\psi=\kappa\exp\left[i\int_{-\infty}^{s}ds'(\tau-\tau_0)\right].$$

.

As ψ is a function of both κ and τ, it provides a complete description of the twisted curve. It is for this quantity, or functions closely related to it, that various nonlinear evolution equations will be generated. Equation (1) in terms of $\vec{N}$ and ψ can be written as

$$\vec{t}_s=(1/2)(\psi^*\vec{N}+\psi\vec{N}^*), \tag{6}$$

where * indicates complex conjugate. Then the linearly independent vectors $\vec{N},\vec{N}^*$, and $\vec{t}$ satisfy the relations $\vec{N}\cdot\vec{N}^*=2$, $\vec{N}\cdot\vec{t}=\vec{N}^*\cdot\vec{t}=\vec{N}\cdot\vec{N}=0$.

It is more convenient to describe the temporal evolution of the curve in terms of $\vec{t},\vec{N}$, and $\vec{N}^*$ instead of $\vec{t},\vec{n}$, and $\vec{b}$. Following the procedure used by Hasimoto, the time variation of $\vec{N},\vec{N}^*$, and $\vec{t}$ may also be expressed as linear combinations of these vectors, i.e.,

$$\vec{N}_t=\alpha\vec{N}+\beta\vec{N}^*+\gamma\vec{t}, \tag{7}$$

$$\vec{t}_t=\lambda\vec{N}+\mu\vec{N}^*+\nu\vec{t}. \tag{8}$$

Using the above relations and multiplying Eqs. (7) and (8) by $\vec{N}$ and $\vec{t}$, yields $\alpha+\alpha^*=0$, $\beta=\nu=0$, and $\gamma=-2\mu$. So,

$$\vec{N}_t=iR\vec{N}+\gamma\vec{t}, \tag{9}$$

$$\vec{t}_t=-(1/2)(\gamma^*\vec{N}+\gamma\vec{N}^*), \tag{10}$$

where $R(s,t)$ is real. Equating the $\vec{N}_{st}$ and $\vec{N}_{ts}$ as obtained from Eqs. (5) and (9) as well as $\vec{t}_{st}$ and $\vec{t}_{ts}$ from Eqs. (6) and (10) one finds

$$\psi_t+\gamma_s+i(\tau_0\gamma-R\psi)=0, \tag{11}$$

$$R_s=(1/2)i(\gamma\psi^*-\gamma^*\psi). \tag{12}$$

The indeterminacy property of Eqs. (11) and (12), three equations for five functions, may be used to specialize the functions so as to yield various types of space curves. In particular, if the auxiliary functions R and γ can be expressed in terms of ψ and its spatial variables, then Eqs. (11) and (12) will provide an evolution equation for the spatial and time variation of the curvature and torsion of the curve expressed through ψ.

Using the above procedure, Lamb [7] derived sine-Gordon and Hirota equations and constructed the linear eigenvalue problem from Serret-Frenet equations. The main aim of this paper is to derive a completely integrable family of NLS-type equations from the moving helical space curve. The construction of the Lax pair, soliton solutions, and the equivalent spin chains will be published elsewhere. Now let us discuss the derivation of other types of equations from Eqs. (11) and (12). As the terms involving ψ or ψ_s are readily removable from the resulting equations, through a simple change of the dependent and independent variables, we will not consider the above terms in the resulting equation. Also, for simplicity, we assume that the integration constant from Eq. (12) and $\tau_0=0$ are zero for our further discussion.

III. CONNECTION WITH OTHER NLS-TYPE EQUATIONS

Mixed derivative nonlinear Schrödinger equation. This equation explains nonlinear propagation of an Alfvén wave with a small nonvanishing number [14,15]. It is also relevant in the discussion of the deformed continuous Heisenberg ferromagnet [16] and in the study of two-photon self-induced transparency and ultrashort light pulse propagation in an optical fiber [17]. The complete integrability properties of this equation have been already explained from the above investigations. It should be noted that the linear eigenvalue problem of this equation is in the form of Kaup-Newell [14] and WKI [15] type. To derive this equation, we assume $\gamma=f\psi+i\epsilon_1\psi_s$, $R=-(\epsilon_1/2)|\psi|^2$, and $f=\epsilon_2|\psi|^2$. Inserting the forms of γ,R, and f in Eqs. (11) and (12), we get

$$\psi_t+i\epsilon_1[\tfrac{1}{2}|\psi|^2\psi+\psi_{ss}]+\epsilon_2[2|\psi|^2\psi_s+\psi^2\psi_s^*]=0. \tag{13}$$

The above equation in the limit $\epsilon_2=0$ reduces to the well-known completely integrable NLS equation.

Extended NLS equation. The purpose of deriving this equation is to show that nonlinear systems that admit the 3×3 Lax pair can also be mapped onto the moving helical space curve. To get this equation, we choose $\gamma=f\psi+i\epsilon_1\psi_s+\epsilon_2\psi_{ss}$, $R=-(\epsilon_1/2)|\psi|^2+i(\epsilon_2/2)(\psi^*\psi_s-\psi\psi_s^*)$, and $f=2\epsilon_2|\psi|^2$ and obtain

$$\psi_t+i\epsilon_1[\tfrac{1}{2}|\psi|^2\psi+\psi_{ss}]+\epsilon_2[\psi_{sss}+\tfrac{9}{2}|\psi|^2\psi_s+\tfrac{3}{2}\psi^2\psi_s^*]=0. \tag{14}$$

This equation was first derived by Kodama [18] to explain the ultrashort pulse propagation in nonlinear optics and the construction of N-soliton solutions and other integrability properties were also discussed [19,20]. Instead of the above form of f, if we assume $f=(\epsilon_2/2)|\psi|^2$, one can easily recover the Hirota equation derived by Lamb [7].

Higher-order NLS equation. To derive this equation, the form of γ is assumed to be $\gamma=i\epsilon_2\psi_{sss}+i\epsilon_1\psi_s+i(3\epsilon_2/2)|\psi|^2\psi_s$, and $R=-(\epsilon_1/2)|\psi|^2-(\epsilon_2/2)(\psi^*\psi_{ss}+\psi\psi^*_{ss})+(\epsilon_2/2)|\psi_s|^2-(3\epsilon_2/8)|\psi|^2$.

With the above choices of γ and R we get

$$\psi_t+i\epsilon_1[2|\psi|^2\psi+\psi_{ss}]+i\epsilon_2 K=0, \tag{15}$$

$$K=[\psi_{ssss}+8|\psi|^2\psi_{ss}+2\psi^2\psi^*_{ss}+4|\psi_s|^2\psi+6\psi^*\psi_s^2+6|\psi|^4\psi]. \tag{16}$$

Equation (15) can be mapped from the continuum limit, up to fourth order in the lattice parameter, of the Heisenberg spin chain with biquadratic exchange interaction and also admits N-soliton solutions [21].

Inhomogeneous NLS equation. In recent years, the investigation of the nonlinear dynamics of inhomogeneous systems has attracted a lot of attention because these systems are considered to be realistic. To our knowledge, so far, inhomogeneous NPDEs from Eqs. (11) and (12) have not been reported. The form of γ for this equation is $\gamma=i\eta(s)\psi_s+i\eta_s\psi$, and $R=-(\eta/2)|\psi|^2-\int_{-\infty}^{s}\eta_{s'}|\psi|^2 ds'$.

In order to identify the integrable soliton possessing system, we assume $\eta_{ss}=0$, i.e., $\eta=\epsilon_1 s+\epsilon_2$ and obtain

$$\psi_t+i(\epsilon_1 s+\epsilon_2)[\tfrac{1}{2}|\psi|^2\psi+\psi_{ss}]+i\epsilon_1\left[2\psi_s+\psi\int_{-\infty}^{s}|\psi|^2 ds'\right]=0. \tag{17}$$

The above equation was shown to be equivalent to the site-dependent Heisenberg spin chain [22–25] with linear x dependence. The interesting nature of this equation is that it admits exploding decay type solitons due to the nonisospectral nature of the eigenvalue parameter in the linear eigenvalue problem.

Circularly symmetric NLS equation. This equation can be derived from the higher-dimensional isotropic spin chain [25] and also from the particle and gauge fields [26]. The equation is of the form

$$\psi_t+i\epsilon_1\left[\frac{1}{2}\left|\psi\right|^2\psi+\psi_{rr}+\frac{\psi_r}{r}-\frac{\psi}{r^2}+\psi\int_0^r\frac{|\psi|^2}{r'}dr'\right]=0 \tag{18}$$

and the corresponding choices of γ and R are $\gamma=i(\epsilon_1/r)\psi+i\epsilon_1\psi_r, R_r=-(\epsilon_1/r)|\psi|^2-(\epsilon_1/2)(|\psi|^2)_r$. Using simple coordinate transformation $s=r^2/4$ and $\psi=2\psi'/r$ one can establish the connection between Eqs. (17) and (18).

Generalized inhomogeneous NLS equation. This equation can be obtained from the inhomogeneous spherically symmetric Heisenberg ferromagnet in arbitrary n dimensions [27]. This equation reads as

$$\psi_t+i\left[\eta_{rr}-\eta\frac{n-1}{r^2}+\eta_r\frac{n-1}{r}\right]\psi+i\left[\eta\frac{n-1}{r}+2\eta_r\right]\psi_r+i\eta\psi_{rr}-iR\psi=0 \tag{19}$$

and choices of γ and R are $\gamma=i[\eta_r+\eta(n-1)/r]\psi+i\eta\psi_r, R_r=-\eta_r|\psi|^2-\eta[(n-1)/r](|\psi|^2)-(\eta/2)(|\psi|^2)_r$.

The above equation is integrable if

$$\eta(r)=\epsilon_1 r^{-2(n-1)}+\epsilon_2 r^{-(n-2)}. \tag{20}$$

Here also we find that the transformation $\psi'=\psi r^n/(r^n-1)$ and $s=r^n/n$ transforms formally the n-dimensional spherically symmetric system (19) with the form (20) to Eq. (17). Equation (19) is already shown to be one of the soliton possessing systems [27].

In conclusion, we have shown that the integrable homogeneous and inhomogeneous NLS family of equations can also be generated from the moving helical space curves by suitably identifying the forms of γ and R. The results presented in this Brief Report reveal that the nonlinear equations with different eigenvalue problems, other than the AKNS eigenvalue problem, and the inhomogeneous equations can also be obtained from the Lamb space curve formalism. From the soliton solutions of the above equations, one can also construct the form of κ and τ, which in turn will be very useful for the analysis of the soliton curve and other related properties of the space curve. From the above results, it is also clear that equations that are not admitting exact solitons can also put in this form. In fact, if we regard the surface as being traced out by a moving space curve, as in the work of Lamb, then it is clear that we will obtain a surface whatever the equations of motion for the curves, while only special types of equations of motion produce exact solitons. On the other hand, it is also interesting to investigate the direct connection between moving space curve and more complicated nonlinear equations like the Wadati-Konno-Ichikawa system, new derivative NLS, and so on. This work is in progress.

ACKNOWLEDGMENTS

K.P. is grateful to Professor H.J. Mikeska for his very kind hospitality and constant encouragement. He expresses his thanks to DAAD and to DST, and CSIR, Government of India, for the financial support through major research projects.

[1] N. J. Zubusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).
[2] G. L. Lamb, Jr., *Elements of Soliton Theory* (John Wiley & Sons, New York, 1980).
[3] H. D. Wahlquist and F. B. Estabrook, Phys. Rev. Lett. **31**, 1386 (1973).
[4] F. D. M. Haldane, Opt. Lett. **11**, 730 (1986).
[5] M. Lakshmanan, Phys. Lett. A **61**, 53 (1977).

[6] A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Eksp.Teor. Fiz. **22**, 503 (1975) [JETP. Lett. **22**, 245 (1975)].
[7] G. L. Lamb, Jr., J. Math. Phys. **18**, 1654 (1977).
[8] H. Hasimoto, J. Fluid Mech. **51**, 477 (1972).
[9] Radha Balakrishnan, A. R. Bishop, and R. Dandoloff, Phys. Rev. Lett. **64**, 2107 (1990); Phys. Rev. B **47**, 3108 (1993); **47**, 5438 (1993).
[10] R. E. Goldstein and D. M. Petrich, Phys. Rev. Lett. **67**, 3203 (1991); **69**, 555 (1992).
[11] K. Nakayama, H. Segur, and M. Wadati, Phys. Rev. Lett. **69**, 2603 (1992).
[12] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. Phys. **53**, 249 (1974).
[13] A. Doliwa and P. M. Santini, Phys. Lett. A **185**, 373 (1994).
[14] D. J. Kaup and A. C. Newell, J. Math. Phys. **19**, 798 (1978).
[15] M. Wadati, K. Kono, and Y. Ichikawa. J Phys. Soc. Jpn. **46**, 1965 (1979).
[16] K. Porsezian, K. M. Tamizhmani, and M. Lakshmanan, Phys. Lett. A **124**, 159 (1987).
[17] A. A. Zabolotskii, Phys. Lett. A **124**, 500 (1987).
[18] Y. Kodama, J Stat. Phys. **39** , 597 (1985).
[19] K. Porsezian and K. Nakkeeran, Phys. Rev. Lett. **76**, 3955 (1996).
[20] S. Sasa and J. Satsuma, J. Phys. Soc. Jpn. **60**, 409 (1991).
[21] K. Porsezian, M. Daniel, and M. Lakshmanan, J. Math. Phys. **33**, 1807 (1992).
[22] M. Lakshmanan and R. K. Bullough, Phys. Lett. A **80**, 287 (1980).
[23] R. Balakrishnan, Phys. Lett. A **92**, 243 (1982).
[24] A. V. Mikhailov and A. I. Yaremchuk, Pis'ma Zh. Eksp. Teor. Fiz. **36**, 78 (1982) [JETP Lett. **36**, 95 (1982)].
[25] K. Porsezian and M. Lakshmanan, J. Math. Phys. **32**, 2923 (1992).
[26] Th. W. Ruijgrok and J. Jurkiewicz, Physica A **103**, 573 (1980).
[27] M. Daniel, K. Porsezian, and M. Lakshmanan, J. Math. Phys. **35**, 6498 (1994).