Boundedness of attractors in the complex Lorenz model

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Using the properties of a principal fiber bundle associated with the complex Lorenz model phase space, we introduce a nonsingular base-space representation of the model. This representation enables us to find the surfaces bounding the attractors in the base space and reveal the interconnection between boundedness properties and peculiarities of the phase dynamics of complex variables. [S1063-651X(97)05602-X]

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The complex generalization of the Lorenz model (LM) [1,2]

$$\dot{x}=-\sigma(x-y),$$
$$\dot{y}=-(1-i\delta)y+(r-z)x, \tag{1}$$
$$\dot{z}=-bz+\frac{1}{2}(x^*y+xy^*),$$

was introduced by Gibbon and McGuinness [3]. Formally the complexity of the variables x and y (in the LM these variables are real) appears due to the presence of the parameter δ and the complexity of the parameter $r=r_1+ir_2$. The generalization of the LM by Gibbon and McGuinness is, however, much more meaningful and covers a variety of dynamic systems described by partial differential equations and possessing a dispersion instability, such as the baroclinic instability in a heated liquid [3,4] or pulsations in a laser [5]. Equations (1) were intensively studied both as a laser model and in a more general context [5–12]. However, the knowledge of the complex Lorenz model (CLM) is still far from that achieved for its real counterpart, which fills books [2]. Meanwhile, the CLM has properties that essentially distinguish it from the ''real'' LM. The most intriguing of them is the phase dynamics of the complex variables. This remarkable feature has been recognized already in the first investigations of CLM [3] and then its study was stimulated by the problem of laser field phase dynamics [5,8–12].

From the technical viewpoint, it is the phase dynamics that makes it difficult to apply to the CLM the analysis methods, which have proved their efficiency in the ''standard'' LM. The known approach to avoid these difficulties [5] is based on the variable substitution $x=x_1\exp(i\Phi)$, $y=(x_2+ix_3)\exp(i\Phi)$ and $z=x_4$, where $x_{1,2,3,4}$ are real, which yields a closed set of equations of motion for $x_{1,2,3,4}$ and a separate equation for the total phase Φ. The disadvantage of this approach is that the variable substitution is not one-to-one for $x=0$, which results in a singularity of equations for $x_{1,2,3,4}$. Since x regularly takes the zero value at $r_2=\delta=0$, this drawback makes the $x_{1,2,3,4}$ representation of the CLM to be ineffective for the analysis of the transition from the ''complex'' to ''real'' behavior.

In the present paper we propose a new representation for CLM, which contains the total phase evolution in an implicit but not in the explicit form. Being in the fiber-bundle relationship with the original one, this representation does not exhibit singularities at any parameter values, and provides an efficient and clear method for studying the properties of the CLM. In particular, it enabled us to analyze the boundedness of attractors for the CLM and to establish the interconnection between this boundedness and the dynamics of x and y phases. At the end of the paper we describe a method to derive a similar representation for the general case of a dynamic model characterized by complex variables.

Consider the real functions u, v, and w of CLM phase variables, introduced as

$$u=(|x|^2-|y|^2)/2 \tag{2}$$

and

$$v+iw=x^*y. \tag{3}$$

Note that for $R=(u^2+v^2+w^2)^{1/2}=(|x|^2+|y|^2)/2$, one can write $|x|^2=R+u$ and $|y|^2=R-u$. Being considered as the Cartesian coordinates in the Euclidean space $\mathcal{P}$, the functions u, v, w, and z provide the projection map $\Pi:\mathcal{H}\to\mathcal{P}$. This map projects all the elements of the CLM phase space $\mathcal{H}$, differing only by the common phase factor in x and y, into the same point in $\mathcal{P}$. It is to be noted that for the physical systems described by CLM, e.g., for a laser [5], such elements of $\mathcal{H}$ belong to the same physical state. Differentiating Eqs. (2) and (3) with respect to time and using Eqs. (1) one gets the equations of motion for the coordinates of the system representative vector in the space $\mathcal{P}$

$$\dot{u}=-(\sigma+1)u+(\sigma-r_1+z)v-r_2w-(\sigma-1)R,$$
$$\dot{v}=-(\sigma+1)v-\delta w-(\sigma-r_1+z)u+(\sigma+r_1-z)R, \tag{4}$$
$$\dot{w}=-(\sigma+1)w+\delta v+r_2(R+u),$$
$$\dot{z}=-bz+v.$$

Consider the subspaces $\mathcal{H}'$ and $\mathcal{P}'$ of the spaces $\mathcal{H}$ and $\mathcal{P}$, respectively, such as $(x,y)\in\mathcal{H}'$ and $(u,v,w)\in\mathcal{P}'$. Note that $\mathcal{H}'$ is identical to $\mathcal{C}^2$ and $\mathcal{P}'$ is equivalent to $\mathbb{R}^3$. One can observe that the spaces $\mathcal{H}'$ and $\mathcal{P}'$ and the map Π form a

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principal fiber bundle [13] so that $\mathcal{H}'$ is the bundle space, $\mathcal{P}'$ is the base space, and the structure group is U(1), which acts in the subspace $\mathcal{H}'$. This fiber bundle is similar to the one known from quantum mechanics [14] and formed by the Hilbert space of state vectors, the density matrix space, and the corresponding projection map. The similarity becomes clearer by identifying the ''state vector'' $|X\rangle$ for the CLM with the pair of complex numbers $(x,y)\in\mathcal{H}'$ and noting that $|x|^2=R+u$ and $|y|^2=R-u$ are just the diagonal elements and $v+iw$ is the off-diagonal one for the corresponding ''density matrix.'' Since $\mathcal{H}$ and $\mathcal{P}$ are both obtained from $\mathcal{H}'$ and $\mathcal{P}'$ by their direct product on the same set of z variable values, the triplet $(\mathcal{H},\mathcal{P},\Pi)$ also forms a fiber bundle. The remarkable property of the fiber bundles $(\mathcal{H}',\mathcal{P}',\Pi)$ and $(\mathcal{H},\mathcal{P},\Pi)$ is that the evolution in $\mathcal{H}$ may be extracted from the trajectory in $\mathcal{P}$. Indeed, it was proved [14] that if the evolution of the state vector obeys the connection

$$\mathrm{Im}(\langle X|\dot{X}\rangle)=0, \tag{5}$$

then the phase $\gamma_g(t)=\arg[\langle X(0)|X(t)\rangle]$ may be calculated as

$$\gamma_g=-\oint_{\Gamma T}A_s ds, \tag{6}$$

where A_s is

$$A_s=\mathrm{Im}\frac{\langle X(s)|d/ds|X(s)\rangle}{\langle X(s)|X(s)\rangle}, \tag{7}$$

$\langle|\rangle$ denotes the standard scalar product on $\mathcal{C}^2$ and ΓT is the closed contour in $\mathcal{H}'$ composed by the segment T of the trajectory between two states and any curve Γ, which projects onto the geodesic in $\mathcal{P}'$. Since this phase is completely determined by the geometry of the contour, it was called the geometric phase [14]. For the CLM the state vector may be made to obey Eq. (5) by means of the gauge transformation

$$|X\rangle\rightarrow|X\rangle\exp(i\gamma_d), \tag{8}$$

where the ''dynamics phase'' γ_d is given by the following equation [12]:

$$\gamma_d=\int_0^t \mathrm{Im}\left[\frac{\langle X|F\rangle}{\langle X|X\rangle}\right]_{t'}dt', \tag{9}$$

$|F\rangle$ is the right-hand side vector for the first two equations in Eq. (1).

Calculating the time derivative of the dynamic phase

$$\dot{\gamma}_d=-\frac{\delta(R-u)-\mathrm{Im}[(\sigma+r-z)(v+iw)]}{2R}, \tag{10}$$

one can see that it is the function of the point in $\mathcal{P}$. To show that γ_g may also be extracted from $\mathcal{P}$, we introduce the spherical coordinates ρ, θ and ϕ of the point in $\mathcal{P}'$

$$u=\rho\cos\theta,\quad v=\rho\sin\theta\cos\phi,\quad w=\rho\sin\theta\sin\phi.$$

Expressing A_s in Eq. (6) in terms of ρ, θ, and ϕ, one gets

$$\gamma_g=\oint_{\Gamma T}\sin^2(\theta/2)d\phi, \tag{11}$$

where the integral is taken in $\mathcal{P}'$ along the contour composed by the trajectory and the geodesic. One can see that the right-hand side of Eq. (11) is nothing but half the solid angle subtended by the contour. Thus the evolution of the complete phase of $|X\rangle$, that is $\gamma_d+\gamma_g$, may be reconstructed from the trajectory in $\mathcal{P}$, determined by Eqs. (4). So, one may use these equations instead of Eq. (1) for studying the CLM.

To demonstrate the benefit of using Eqs. (4), let us consider a surface Q in $\mathcal{P}$ given by the equation

$$Q:q(u,v,w,z)=\alpha u+\beta w+\alpha\frac{\delta}{|\delta|}R=0, \tag{12}$$

where $\alpha=-|\delta|/[(2\sigma)^2+\delta^2]^{1/2}$ and $\beta=2\sigma/[(2\sigma)^2+\delta^2]^{1/2}$. If one restricts himself by the subspace $\mathcal{P}'\in\mathcal{P}$, Q is the two-dimensional semicone with the top in the origin and the symmetry axis along the unit vector $(\alpha,0,\beta)$; the cosine of the angle between the axis and the generator of the cone is equal to $\pm\alpha$ depending on the sign of δ. For $\delta>0$ the cone spreads towards the positive values of w and negative u; at $\delta=0$ it is merely the plane $w=0$; if $\delta<0$, the surface spreads towards the negative part of the w axis and the positive one of the u axis. Note that q is positive in the interior cone for $\delta>0$ and negative for $\delta<0$.

Consider the time derivative

$$\dot{q}=(\mathbf{f},\nabla q), \tag{13}$$

where $\mathbf{f}$ is the phase velocity vector in $\mathcal{P}$. It follows from Eq. (13) that a trajectory, which goes through the given point on the surface Q, is directed ''inside'' or ''outside'' it, respectively, if $\dot{q}$ takes the positive or negative value at this point. Let us see that $\dot{q}$ has the same sign at all points of Q. Using Eqs. (4), one gets

$$\dot{q}=-(\sigma+1)q-[\alpha(\sigma-1)-\beta r_2]\left(u+\frac{\delta}{|\delta|}R\right), \tag{14}$$

(note that $q=0$ on Q). Let $\delta>0$. In this case the value $u+(\delta/|\delta|)R$ is non-negative at the surface Q. Thus, for $\alpha(\sigma-1)-\beta r_2<(>)0$ the trajectories on the surface Q are tangent to it or directed towards the region $q>(<)0$.

Note, that Eqs. (4) are invariant with respect to the transformation

$$\delta\rightarrow-\delta,\quad r_2\rightarrow-r_2,\quad w\rightarrow-w. \tag{15}$$

Therefore, for $\delta<0$ the surface Q is also a boundary. One can also prove that every trajectory in $\mathcal{P}$ once enter the region bounded by the surface Q (see the Appendix).

Under the condition $\alpha(\sigma-1)-\beta r_2=0$ the surface Q is a stable manifold as it follows from Eq. (14). This condition can be written in the form

$$r_2=r_{2c}=\delta\frac{1-\sigma}{2\sigma}. \tag{16}$$

For $\delta>0$ and $r_2>r_{2c}$, and for $\delta<0$ and $r_2<r_{2c}$ all attractors are located within the region of $\mathcal{P}$, which corresponds to the "smaller" part of $\mathcal{P}'$ bounded by the cone Q. Otherwise, all limit sets of trajectories in $\mathcal{P}$ are situated in the "exterior" of Q.

In the Appendix it is also shown that there exists another bounding surface in $\mathcal{P}$, which is a spheroid S given by the equation

$$S:2R+(z-\sigma-r_1)^2=K^2(r_1+\sigma)^2, \tag{17}$$

where

$$K^2=\frac{1}{4}+\frac{b}{4}\max(\sigma^{-1},1). \tag{18}$$

The existence of the bounding surface Q provides an important outcome related to the properties of the phase evolution of the state vector $|X\rangle$. First, it means that in the laser case ($r_2=0$) all the attractors are located in the region of $\mathcal{P}$, where $q>0$ for $\delta>0$ and in the symmetric region for $\delta<0$. If one restricts oneself by the space $\mathcal{P}'$, it is the region within the solid angle subtended by the cone $q=0$. Therefore, for the trajectory belonging to any attractor, the solid angle subtended by the contour ΓT [see Eq. (6)] is limited by the bounding cone. In the limit $\delta\rightarrow\pm 0$ the cone turns into the plane $w=0$, so that the solid angle subtended by the contour ΓT tends to the limit value $\pm 2\pi$. Consider now the behavior of the phase slope time average when the detuning δ changes near the resonance value $\delta=0$. If follows from Eq. (11) that such an average contains the mean solid angle subtended by the contour ΓT in $\mathcal{P}'$ as the contribution from the "geometric" part of the total phase. Thus at $\delta=0$ the curve of the mean phase slope (the frequency) versus detuning exhibits a jump by $2\pi/\tau=\nu$, where τ and ν are the period and the frequency of the amplitude pulsations.

It is to be noted that the CLM is only one model from a very wide class of those characterized by complex dynamic variables, for which the physical state is determined up to the phase factor. The examples are linear and nonlinear Schrödinger equations, the generalized Ginzburg-Landau equation, and space-time Maxwell-Bloch equations. In conclusion we wish to discuss the general way to realize the base-space approach for such models. First the space of "state vectors" $|X\rangle$ must be defined with the proper scalar product in it. Generally it is the subspace of the total phase space, in which the symmetry group U(1) acts (for the Schrödinger and Ginzburg-Landau equations they are the total spaces themselves). Then the connection (5) must be satisfied by means of the gauge transformation of the form (8) with

$$\gamma_d=\int_0^t \mathrm{Im}\left[\frac{\langle X|\hat{U}|X\rangle}{\langle X|X\rangle}\right]_{t'} dt', \tag{19}$$

where $\hat{U}$ is the operator (generally nonlinear) in the right-hand side of the state vector dynamic equation $\partial_t|X\rangle=\hat{U}|X\rangle$. This provides the possibility to calculate the total phase accumulation from the base space data as the sum of the dynamic and geometric phases; the last one is given by Eqs. (6) and (7). Finally, the base-space representation of the model may be realized directly along the lines of the quantum-mechanical rule for the transition from the pure-state state vector $|X\rangle$ to the density matrix $\hat{P}=|X\rangle\langle X|$. Although this circumstance is well known for the quantum-mechanical Schrödinger equation, our treatment of the complex Lorenz model shows that the base-space representation may be useful for the analysis of nonlinear models too.

To sum up, we have shown that the phase space of the CLM may be considered as a bundle space for the fiber bundle $(\mathcal{H},\mathcal{P},\Pi)$. Based on the properties of this fiber bundle we have demonstrated that the trajectory in the base space provides the information sufficient to reconstruct the trajectory in the total phase, i.e., the phase evolution. Using the equations of motion in the base space we have found the surfaces bounding the attractors in this space and concluded that for the laser the curve of the mean phase slope versus detuning must exhibit a jump at the resonance detuning value. Earlier such a jump has been observed in a numerical experiment [9] and interpreted as a geometric phase manifestation basing on the results of numerical analysis of the CLM and the formal analogy between the CLM and the Schrödinger equation [12]. Now this result is obtained in the closed analytical form based on the fundamental geometrical properties of the CLM. The method to obtain the base-space representation in the general case of a model with complex variables is discussed. Using Eqs. (4) we hope to analyze completely the CLM behavior at nonzero δ and r_2; this work is now in progress.

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APPENDIX

Let us first show that for the CLM the limit sets of trajectories in $\mathcal{H}$ are bounded by the hypersphere

$$S:|x|^2+|y|^2+(z-\sigma-r_1)^2-K^2(\sigma+r_1)^2=0, \tag{A1}$$

where K is given in Eq. (18). To prove this consider the family of spheres in $\mathcal{H}$

$$V_M\equiv|x|^2+|y|^2+(z-\sigma-r_1)^2-M^2=0, \tag{A2}$$

and the time derivatives

$$\dot{V}_M=-2\sigma|x|^2-2|y|^2-2b\left(z-\frac{\sigma+r_1}{2}\right)^2+b\frac{(\sigma+r_1)^2}{2}. \tag{A3}$$

From Eq. (A3) one can see that the function on its right-hand side does not depend on the parameters δ and r_2. Therefore, one may refer to the known result for the LM [1,2] that this function is negative at every sphere $V_M=0$ whose radius is greater than $K(\sigma+r_1)$. Since the left-hand side of Eq. (A2) is negative "inside" the hypersphere $V_M=0$, the negativeness of $\dot{V}_M$ means that all trajectories go towards the interior of S. The corresponding bounding surface for invariant sets in $\mathcal{P}$ can be obtained from Eq. (A1) by the substitution $|x|^2+|y|^2=2R$, which yields the equation for the spheroid (17).

To show that every trajectory, once and forever, comes into the region bounded by the surface Q, consider the family of surfaces

$$q_m(u,v,w,z)=\alpha u+\beta w-mR\mp\frac{m+\alpha}{2}(z^2+L)=0, \tag{A4}$$

where m is the parameter and L is a positive constant. At $m=-\alpha$ the surface (A4) coincides with that given by Eq. (12). We further restrict ourselves by the most important case $\sigma>1$, when the unstable behavior of the CLM solutions is possible [5]. We also put $\delta>0$ for certainty. For $r_2>r_{2c}$ we take $m\leqslant-\alpha$ and the sign ''minus'' at the last term of the left-hand side of Eq. (A4), so that the surfaces $q_m=0$ are situated in the region of $\mathcal{P}$ where $q\leqslant 0$. The differentiation of q_m with respect to time gives

$$\begin{aligned}\dot{q}_m=&-(\sigma+1)(\alpha u+\beta W-mR)+[m(\sigma-1)+\beta r_2]u\\&-(\alpha+m)r_2w-(m+\alpha)(\sigma+r_1)v\\&-[\alpha(\sigma-1)-\beta r_2]R+(m+\alpha)bz^2.\end{aligned}$$

At the surface $q_m=0$ this function may be replaced by

$$\begin{aligned}\dot{q}_m=&[m(\sigma-1)+\beta r_2]u-(\alpha+m)r_2w-(m+\alpha)(\sigma+r_1)v\\&-[\alpha(\sigma-1)-\beta r_2]R-\frac{m+\alpha}{2}\\&\times[(\sigma+1-2b)z^2+(\sigma+1)L]\\=&\tilde{q}_m(u,v,w)-\frac{m+\alpha}{2}[(\sigma+1-2b)z^2+(\sigma+1)L].\end{aligned} \tag{A5}$$

The cone form $\tilde{q}_m(u,v,w)$ in Eq. (A5) is a non-negative value in the entire space $\mathcal{P}$ if

$$\frac{\alpha(\sigma-1)-\beta r_2}{\{[m(\sigma-1)+\beta r_2]^2+(\alpha+m)^2[(\sigma+r_1)^2+r_2^2]\}^{1/2}}<-1.$$

This inequality is satisfied for

$$m_1<m<-\alpha, \tag{A6}$$

where

$$m_1=\frac{\alpha[(\sigma-1)^2-r_2^2-(\sigma+r_1)^2]-\beta r_2(\sigma-1)}{(\sigma-1)^2+r_2^2+(\sigma+r_1)^2}.$$

The nonempty interval (A6) exists for $r_2>r_{2c}$ and $\sigma>1$. We now choose the positive constant L in Eq. (A4) in such a way that (i) for $m\leqslant m_1$ the surface $q_m=0$ to be outside the spheroid given by Eq. (17), and (ii) to make the last term in Eq. (A5) positive within this region. Both the requirements can be simultaneously satisfied if L is sufficiently large. Indeed, because the values of u,v,w are limited at S, one can choose L so that for $m\leqslant m_1$ the inequality $q_m|_S\geqslant 0$ is satisfied; also it is easy to see that for $m_1<m<-\alpha$ the positiveness of $\dot{q}_m|_{q_m=0}$ can be preserved simultaneously by the appropriate choice of L. Thus for $r_2>r_{2c}$ we have the family of the bounding surfaces which evolves from the ''exterior'' of spheroid S at $m=m_1$ to the limiting surface Q at $m=-\alpha$. Each trajectory from the ''exterior'' of S intersects consequently each such a surface and finally occur ''within'' the surface Q.

Similar consideration can be easily done for $r_2<r_{2c}$ with the only difference that now in Eq. (A4) m changes from 1 to $-\alpha$ and the sign ''plus'' must be taken for the last term.

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